The Structure of the Discriminant of some Space Curve Singularities

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Abstract

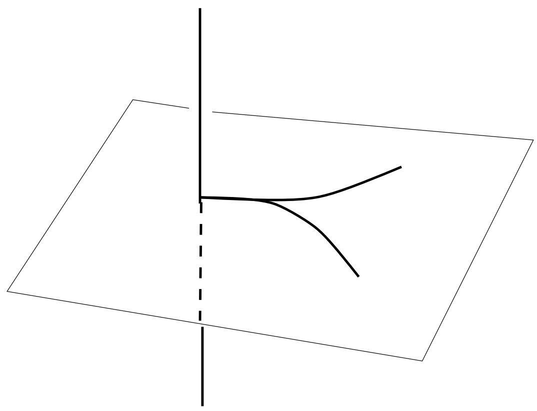
Among the space curve singularities of the simplest type are the so called *wedges* $D = C \vee L$, consisting of a plane curve singularity C together with a line L transverse to the plane of C. In this note we describe the discriminant of D in terms of C. In particular, we show that the complement of the discriminant of D is a $K(\pi, 1)$ if the complement of the discriminant of C is a $K(\pi, 1)$. We also give a formula for the multiplicity of the discriminant of $C \vee L$ and of space curves of multiplicity three.

1 Introduction

Let $D \subset \mathbb{C}^3, 0$ be a (reduced) space curve singularity and let $\pi : \mathscr{D} \longrightarrow B$ be its semi-universal deformation. As D is a Cohen-Macaulay subspace of codimension two, B is a smooth space of dimension $\tau := \dim T^1_D$ ([9]). Let $\Delta \subset B$ be the discriminant of π, that is, the locus over which the fibres are singular. Apart from the fact that Δ is a *free divisor* ([10]), not much seems to be known about its structure. At least for the list of simple space curve singularities ([5]), one would like to have answers to the following basic questions:

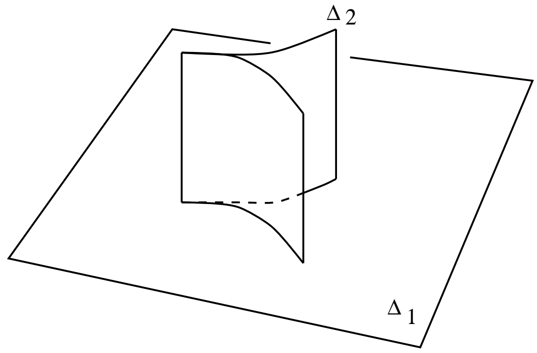
1. How many components does Δ have, and what are their multiplicities?
2. What can one say about the fundamental group of $B \setminus \Delta$ and its monodromy action on the cohomology $H^1(F)$ of the Milnor fibre F?
3. Is $B \setminus \Delta$ a $K(\pi, 1)$-space? Surprisingly often (see for example [3], [4], [13]), the complement of the discriminant in the base space of a versal deformation has this very special property, although little is known in general.
4. Is there a natural geometrical description of Δ for the simplest space-curve singularities? For simple hypersurface singularities, the classical description of the discriminant in terms of Coxeter groups, due to Arnold and Brieskorn, provides the basis for the proof of the $K(\pi, 1)$ property.

The simplest type of space curve which is not a complete intersection is obtained from a plane curve singularity C by *wedging* it with a line L transverse to the plane of C. We write $D = C \vee L$. For curves of this type we are able to reduce questions 1, 2, 3 and 4 above to questions about the plane curve C.



Let $\mathscr{C} \longrightarrow B_C$ be the semi-universal deformation of C, and let $\mathscr{C}_\Delta$ be the union of all singular fibres. Let $\mathscr{D} \longrightarrow B_D$ be the semi-universal deformation of D and let $\Delta_D \subset B_D$ be its discriminant. Our results are based on the following theorem, which is proved in Section 3.

Theorem 1.1. *Let $D = C \vee L$. The discriminant Δ_D is the union of a smooth hypersurface Δ_1 and a hypersurface Δ_2 isomorphic to $\mathscr{C}_\Delta \times (\mathbb{C}, 0)$. Δ_1 and Δ_2 meet transversely.*



As corollaries we obtain

Corollary 1.2. *Δ_D has two irreducible components, unless C is an A_1 singularity, in which case Δ_D is the normal crossing of three smooth components.*

Corollary 1.3. *The multiplicity of the discriminant Δ_D is*

$$mult(\Delta_D) = \mu(C) + mult(C).$$

Let $\mathscr{C}^* = \mathscr{C} \setminus \mathscr{C}_\Delta$ be the union of all smooth fibres.

Corollary 1.4. *1. $B_D \setminus \Delta_D \simeq \mathbb{C}^* \times \mathscr{C}^*$.*

2. $\pi_1(B_D \setminus \Delta_D) = \mathbb{Z} \times \pi_1(\mathscr{C}^*)$ and $\pi_k(B_D \setminus \Delta_D) = \pi_k(\mathscr{C}^*)$ for $k \geq 2$.

3. In particular, if $B_{\mathbb{C}} \setminus \Delta_C$ is a $K(\pi, 1)$-space, then also $B_D \setminus \Delta_D$ is a $K(\pi, 1)$-space.

Remark 1.5. Another consequence of 1.1 is that $\mathscr{C}_\Delta$ is a free divisor, since it is known ([10]) that Δ_D itself is free. In fact it turns out (and is not hard to show) that the same goes for the semi-universal deformation of *any* ICIS curve singularity: the part of the total space lying over the discriminant is a free divisor.

2 Preliminaries

If C is given by $f(x, y) = 0$, then $D := C \vee L$ is described by the ideal $(x, y) \cap (f(x, y), z) \subset \mathbb{C}[[x, y, z]]$. This intersection is readily seen to be equal to $(f(x, y), zx, zy)$. When we write f in the form $f = Ax - By, A, B \in \mathbb{C}[[x, y]]$, then one gets these equations as 2 x 2 minors of the matrix

$$M := \begin{pmatrix} z & A & B \\ 0 & y & x \end{pmatrix}$$

The total space $\mathscr{D}$ of the miniversal deformation of D is defined by the minors of a matrix

$$\tilde{M}(s_1, \cdots, s_\tau) := \begin{pmatrix} z & \tilde{A} & \tilde{B} \\ s_1 & y & x \end{pmatrix}$$

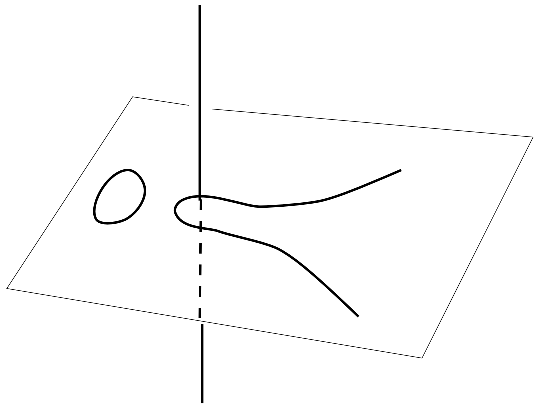
which reduces to the matrix M when $s_1 = \cdots = s_\tau = 0$ (see [9]). Let $B_0 \subset B_D$ be the subspace defined by $s_1 = 0$, and consider the restriction of the miniversal family to B_0:

$$\begin{array}{ccc} \mathscr{D}_0 & \hookrightarrow & \mathscr{D} \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B_D \end{array}$$

For each point $s \in B_0$, the curve D_s consists of the line $L = \{(0,0,z) | z \in \mathbb{C}\}$ together with the plane curve C_s defined by the determinant of the matrix

$$\begin{pmatrix} A_s & B_s \\ y & x \end{pmatrix}$$

(where $A_s(x,y) = \tilde{A}(x, y, s_2, \dots, s_\tau)$ and similarly B) which meets L at $(0,0,0)$.



A typical curve D_s, for $s \in B_0$.

So there is a decomposition

$$\mathscr{D}_0 = (B_0 \times L) \cup \mathscr{E}_0$$

where the fibre of $\mathscr{E}_0$ over $s \in B_0$ is C_s. The intersection of the two components of $\mathscr{D}_0$ projects isomorphically to the base B_0, and hence the diagram

(1)
$$\begin{tikzcd} (B_0 \times L) \cap \mathscr{E}_0 \arrow[r, hook] \arrow[dr, "\simeq"'] & \mathscr{E}_0 \arrow[d] \\ & B_0 \end{tikzcd}$$

is a *deformation with section* of the plane curve C. We will show that as such it is miniversal.
Let us clarify these terms. Let X be a germ of analytic space, and let $\pi : \mathscr{X} \to B$ be a deformation of X with section $d : B \to \mathscr{X}$. Then $\pi : \mathscr{X} \to B$ with its section d is *versal as a deformation with section* if for every deformation $\mathscr{X}_S \xrightarrow{\pi_S} S$ with section $d_S : S \to \mathscr{X}_S$, there exists a map $k : S \to \mathscr{X}$ and a fibre square

$$\begin{tikzcd} \mathscr{X}_S \arrow[r, "K"] \arrow[d, "\pi_S"] & \mathscr{X} \arrow[d, "\pi"] \\ S \arrow[r, "k"] & B \end{tikzcd}$$

with the additional property that $K \circ d_S = d \circ k$.
A deformation of the plane curve germ C with section can be obtained as follows. Start with a miniversal deformation $\mathscr{C} \xrightarrow{p} B_C$ of C, and pull it back over itself:

$$\begin{tikzcd} \mathscr{C} \times_{B_C} \mathscr{C} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathscr{C} \arrow[d, "p"] \\ \mathscr{C} \arrow[r, "p"] & B_C \end{tikzcd}$$

(where p_1 and p_2 are the cartesian projections). The deformation $\mathscr{C} \times_{B_C} \mathscr{C} \longrightarrow \mathscr{C}$ has a section $\mathscr{C} \xrightarrow{d} \mathscr{C} \times_{B_C} \mathscr{C}$ given by the diagonal embedding.

Lemma 2.1. *$\mathscr{C} \times_{B_C} \mathscr{C} \longrightarrow \mathscr{C}$ with its section d is miniversal as a deformation with section.*

Proof For simplicity of notation we write B_C as B. Let $\mathscr{C}_S \xrightarrow{\pi_S} S$ be a deformation of C with section $d_S : S \to \mathscr{C}_S$. As $\mathscr{C} \to B$ is versal, the deformation π_S is induced by pulling back $\mathscr{C} \to B$ over some map $j : S \to B$. Thus we may assume that $\mathscr{C}_S = \mathscr{C} \times_B S$. The section d_S now has the form $d_S(s) = (c(s), s)$. Let J denote the (cartesian) projection $\mathscr{C}_S \to \mathscr{C}$. By pulling back $\mathscr{C} \times_B \mathscr{C} \to \mathscr{C}$ over $J \circ d_S$, we obtain the following diagram.

$$\begin{tikzcd}
 & \mathscr{C} \times_B \mathscr{C} \arrow[r, "p_2"] \arrow[dd, "p_1"] & \mathscr{C} \arrow[dd, "p"] \\
(\mathscr{C} \times_B \mathscr{C}) \times_{\mathscr{C}} S \arrow[ur] \arrow[dd] & & \\
 & \mathscr{C} \arrow[r, "p"] & B \\
S \arrow[r, "d_S"'] \arrow[ur] & \mathscr{C}_S \arrow[r, "\pi_S"'] \arrow[u, "J"] & S \arrow[u, "j"]
\end{tikzcd}$$

Now that we have identified $\mathscr{C}_S$ with $\mathscr{C} \times_B S$, it is straightforward to check that we can identify $\mathscr{C} \times_B \mathscr{C} \times_{\mathscr{C}} S$ with $\mathscr{C}_S$, and the induced section $d \circ J \circ d_S$ (which lands naturally in $\mathscr{C} \times_B \mathscr{C} \times_{\mathscr{C}} S$) with d_S. We advise the reader to make the necessary tautological calculation.
This proves versality; thus, we have a versal deformation with section, whose base space is smooth and has dimension one greater than the dimension of the miniversal base-space B of C without section. The space of first order deformations with section is $m/((f) + mJ_f)$ (where m is the maximal ideal and f the defining equation), which has dimension $\tau + 1$. Miniversality follows. $\diamond$

Lemma 2.2. *The deformation with section (1) is miniversal as deformation of C with section.*

Proof: By lemma 2.1, (1) is isomorphic to a deformation induced from

(2)
$$\begin{tikzcd}
\mathscr{C} \arrow[r, hook, "d"] \arrow[dr, "\simeq"'] & \mathscr{C} \times_{B_C} \mathscr{C} \arrow[d] \\
 & \mathscr{C}
\end{tikzcd}$$

by a map $\phi_1 : B_0 \to \mathscr{C}$. To any deformation of C with section we associate a canonical deformation of $C \vee L$: to each curve with marked point we associate the same curve wedged with a parallel translate of L passing through the marked point. In particular, we can apply this to the family (2). Let us call the total space of this family $\mathscr{D}_1$. As a deformation of $D = C \vee L$, $\mathscr{D}_1 \longrightarrow \mathscr{C}$ is equivalent to one induced from the miniversal deformation $\mathscr{D} \longrightarrow B_D$, and thus we have an inducing map of base spaces $\phi_2 : \mathscr{C} \longrightarrow B_D$. We summarise this situation with a diagram:

$$\begin{tikzcd}
\mathscr{D}_0 \arrow[r] \arrow[d] & \mathscr{D}_1 \arrow[r] \arrow[d] & \mathscr{D} \arrow[d] \\
B_0 \arrow[r, "\phi_1"] & \mathscr{C} \arrow[r, "\phi_2"] & B_D
\end{tikzcd}$$

Both squares are pull-back diagrams, and thus the outer rectangle is also. There is another pull back diagram with the same four corners:

$$\begin{tikzcd}
\mathscr{D}_0 \arrow[r, hook] \arrow[d] & \mathscr{D} \arrow[d] \\
B_0 \arrow[r, hook] & B_D
\end{tikzcd}$$

By miniversality of $\mathscr{D} \longrightarrow B_D$, these two diagrams must be isomorphic. That is, there is a diagram

$$\begin{tikzcd}
\mathscr{D}_0 \arrow[rr, hook] \arrow[dr] \arrow[dd] & & \mathscr{D} \arrow[dd] \\
& \mathscr{D} \arrow[ur] \arrow[dd] & \\
B_0 \arrow[rr, hook] \arrow[dr, "\phi_2\circ\phi_1"'] & & B_D \\
& B_D \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where the arrows $\psi : B_D \longrightarrow B_D$ and $\mathscr{D} \longrightarrow \mathscr{D}$ are isomorphisms. Replacing ϕ_2 by $\psi \circ \phi_2 \circ \phi_1$, we may therefore assume that ϕ_2 maps $\mathscr{C}$ to B_0. This implies that $\mathscr{D}_1 \to \mathscr{C}$ is induced from $\mathscr{D}_0 \to B_0$ by ϕ_2, as deformations of $C \vee L$. Regarding these as deformations of C with section, we see that the versal family (2) is induced from (1). It follows that (1) is versal as a deformation with section. However, it must even be miniversal as such: if not, then over some smooth curve in B_0 we have a trivial deformation of C with section, which amounts to a trivial deformation of $C \vee L$. This contradicts minimality of the deformation $\mathscr{D} \longrightarrow B_D$. $\diamond$

Let $\Delta(B_0)$ be the set of point $s \in B_0$ such that the plane curve $C_s \subset D_s$ is singular. Also, let $\mathscr{C}_\Delta$ be the part of the total space of the deformation $\mathscr{C} \longrightarrow B_C$ lying over the discriminant Δ_C. Because the families (1) and (2) are isomorphic, one concludes

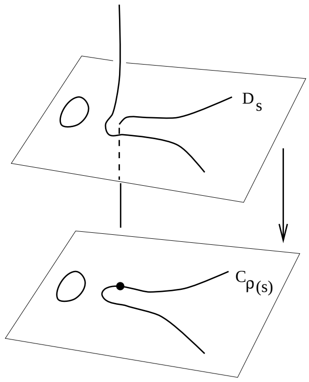
Corollary 2.3. *B_0 is isomorphic to $\mathscr{C}$ by an isomorphism taking $\Delta(B_0)$ to $\mathscr{C}_\Delta$.* $\diamond$

3 Projecting the miniversal deformation

We define a map $\rho : B_D \longrightarrow B_0$ by $\rho(s_1, \cdots, s_\tau) = (0, s_2, \ldots, s_\tau)$. This is covered by the map $\bar{\rho} : \mathbb{C}^3 \times B_D \longrightarrow \mathbb{C}^2 \times B_0$ defined by $\bar{\rho}(x, y, z, s_1, \ldots, s_\tau) = (x, y, s_2, \ldots, s_\tau)$. Clearly $\bar{\rho}(D_s)$ is the plane curve $C_{\rho(s)}$ defined by the determinant of the matrix

$$\begin{pmatrix} A_s & B_s \\ y & x \end{pmatrix}$$

Geometrically, we can see the map ρ by projecting a fibre of the deformation $\mathscr{D} \to B_D$ from (x, y, z) space to (x, y)-space. The image is the fibre $C_{\rho(s)}$ of the deformation of C, *minus a closed disc containing the image of the asymptote.* The size of the disc depends on the choice of representatives of the Milnor fibration, but does not affect the topology of the image. In the ideal case where the germ D and its versal deformation are weighted homogeneneous, and we take all of $\mathbb{C}^3$ as Milnor ball, then $\bar{\rho}(D_s)$ is precisely equal to $C_{\rho(s)} \setminus x_\infty$. For the purpose of describing the monodromy, it is convenient to imagine ourselves in this ideal situation.



Proof of Theorem 1.1 If $s_1 = 0$, the curve D_s is singular; thus the hyperplane $B_0 = \{s_1 = 0\}$ is a component of the discriminant.
If $s_1 \neq 0$, D_s is singular if and only if $C_{\rho(s)}$ is singular. For suppose that $C_{\rho(s)}$ is non-singular. Then $A(0,0,\rho(s))$ and $B(0,0,\rho(s))$ do not both vanish; if they did, the equation of $C_{\rho(s)}$ would lie in the square of the maximal ideal at $(0,0)$, and $C_{\rho(s)}$ would be singular. It now follows that $\bar{\rho}$ induces an isomorphism $D_s \to C_{\rho(s)} \setminus \{(0,0)\}$; the inverse to $\bar{\rho}$ on $C_{\rho(s)} \setminus \{(0,0)\}$ is given by $(x,y) \mapsto (x,y,z)$ with $z = s_1 A(x,y,\rho(s))/y = s_1 B(x,y,\rho(s))/x$, and there are no points in D_s lying over $(0,0)$. Hence D_s is non-singular. Conversely, if $C_{\rho(s)}$ is singular at some point $(a,b) \neq (0,0)$ then by the above isomorphism, D_s is singular at the unique point lying over it. Finally, if $C_{\rho(s)}$ is singular at $(0,0)$ then $A(0,0,\rho(s)) = B(0,0,\rho(s)) = 0$, and it follows that D_s contains the line L as well as the lift of the curve $C_{\rho(s)}$, and is thus singular where they meet. We have shown that $\Delta_D = B_0 \cup \rho^{-1}(\Delta(B_0))$. By Corollary 2.3, $\Delta(B_0) \simeq \mathscr{C}_\Delta$, and this completes the proof. $\diamond$
We note that Theorem 1.1 implies in particular that

$$\tau(D) = \tau(C) + 2$$

where τ is the dimension of the miniversal base. (This is in accordance with a general formula for $\tau(C_1 \vee C_2)$, due to Jan Stevens, [1].) The interpretation is as follows. The base space of C has dimension $\tau(C)$. The miniversal base of deformations with section was $\mathscr{C}$, so has one dimension more. The last dimension comes from the parameter s_1, which smoothes out the intersection point of the line and the plane curve.

4 The complement of the discriminant

The first statement of Corollary 1.4 is an obvious consequence of Theorem 1.1, and the second is then immediate also.
For the third statement we use the long exact homotopy sequence associated to the fibration $\mathscr{C}^* \to B_C \setminus \Delta_C$. This gives isomorphisms $\pi_k(\mathscr{C}^*) \simeq \pi_k(B_C \setminus \Delta)$ for all $k \geq 3$ and a 5-term exact sequence:

$$0 \longrightarrow \pi_2(\mathscr{C}^*) \longrightarrow \pi_2(B_C \setminus \Delta) \longrightarrow \pi_1(F) \longrightarrow \pi_1(\mathscr{C}^*) \longrightarrow \pi_1(B_C \setminus \Delta) \longrightarrow 1.$$

In particular, if $B_C \setminus \Delta$ is a $K(\pi,1)$-space, then also $B_D \setminus \Delta_D$ is a $K(\pi,1)$-space. $\diamond$
From the picture in the previous section, it is also clear that the Milnor fibre D_s of D is homeomorphic, via $\overline{\rho}$, to the Milnor fibre $C_{\rho(s)}$ of C, with the marked point removed. It follows that

$$\mu(D) = \mu(C) + 1.$$

Over the set $B_D \setminus \Delta_D$ we have the Milnor fibration $\mathscr{D}^* \longrightarrow B_D \setminus \Delta_D$. Our description of the discriminant allows us to give a geometrical description of the monodromy.
Let s_0 be a base point in $B_D \setminus \Delta_D$, and let $b_0 = \rho(s_0) \in \mathscr{C}$. The factor $\mathbb{Z}$ of the fundamental group $\pi_1(B_D \setminus \Delta_D, s_0) = \mathbb{Z} \times \pi_1(\mathscr{C}^*)$ is generated by a loop σ_1 which winds once around B_0 while holding $s_2, \ldots, s_\tau$ constant. Join s_0 to B_0 by a line segment ℓ in which only the first coordinate varies. Along this segment, the fibre degenerates to a wedge of a line and a plane curve, thus acquiring an A_1 singularity. We transport the local Milnor fibre of this singularity into D_s by lifting ℓ to the Milnor fibration. The loop σ_1 acts by

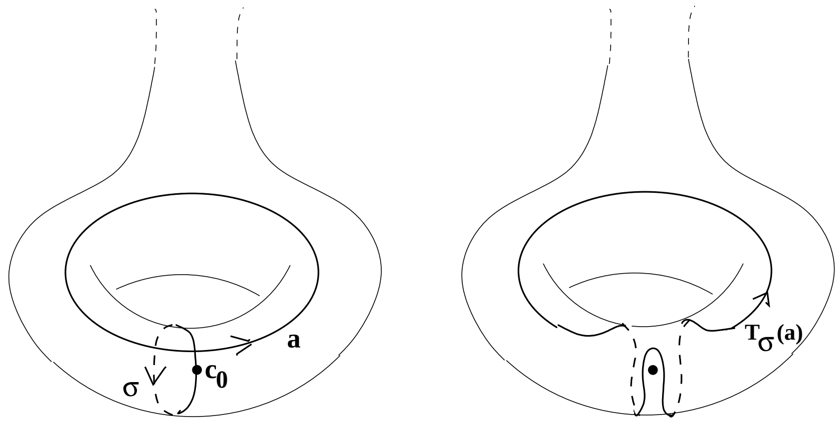
monodromy on D_{s_0}, imparting the usual *Dehn twist* to the local Milnor fibre of the A_1 singularity. That is, a neighbourhood of the puncture in D_s is diffeomorphic to a half-open cylinder; the geometric monodromy induced by σ_1 twists the outer (open) end of the cyclinder through 2π while leaving the closed end fixed.

We can identify $\{s_0\} \times \mathbb{C}^{\tau-1}$ with $\mathscr{C}$ and the complement of Δ_D in $\{s_0\} \times \mathbb{C}^{\tau-1}$ with $\mathscr{C}^*$, and this identification extends to the respective fibrations; thus the monodromy action of the second factor of $\pi_1(B_D \setminus \Delta_D, d_0)$ is the same as the monodromy action of $\pi_1(\mathscr{C}^*, c_0)$ on the punctured curve $C_{b_0} \setminus c_0$.

Elements of $\pi_1(\mathscr{C}^*, c_0)$ can be seen as lifts of elements in $\pi_1(B_C \setminus \Delta, b_0)$ where $b_0 = p(c_0)$. It follows from the construction of the miniversal family in Lemma 2.1 that any lift of $\sigma \in \pi_1(B_C \setminus \Delta_C, b_0)$ acts in the same way on the homology of the fibre of $\mathscr{C}^* \times_B \mathscr{C}^*$ over c_0 as does σ on C_{b_0} (the two fibres are canonically the same). However, the action on the *punctured* curve $C_{b_0} \setminus c_0$ (which is diffeomorphic to the Milnor fibre D_s) is more complicated. In particular, let σ be a loop in $\pi_1(C_{b_0}, c_0)$ and $i_*(\sigma)$ its image in $\pi_1(\mathscr{C}^*, c_0)$. The fibre of $\mathscr{C}^* \times_B \mathscr{C}^*$ over each point $\sigma(t)$ is the same curve, C_{b_0}, but the puncture moves: over $\sigma(t)$ it is precisely $\sigma(t)$. Thus the geometric monodromy at time t is a diffeomorphism of C_{b_0} fixing the boundary and mapping $c_0 = \sigma(0)$ to $\sigma(t)$. This diffeomorphism can be chosen to be the identity outside an arbitrarily small neighbourhood of the curve σ. The homological monodromy T_σ of $i_*(\sigma)$ thus acts on $a \in H_1(C_{b_0} \setminus c_0; \mathbb{Z})$ by

$$a \mapsto a + (a \cdot \sigma) r,$$

where r is the class of a small positively oriented loop around the puncture in C_{b_0}.



5 Components of the Discriminant

If D is the union of the three coordinate axes (so its ideal is (xy, yz, zx)), then it has a 3-dimensional deformation space. Its discriminant is the union of the three coordinate planes. This is in accordance with our theorem: the component Δ_1 is one of these planes, the part Δ_2 consists of the union of singular fibres of the miniversal deformation of the A_1-singularity (so: two lines) crossed with a trivial factor. We show that this case is exceptional.

Proposition 5.1. *If C is* not *the A_1-singularity, then Δ_2 is irreducible.*

proof: As $\Delta_2 = \mathscr{C}_\Delta \times (\mathbb{C}, 0)$, we have to show that $\mathscr{C}_\Delta$ is irreducible. Let $F(x, y, s) = 0$ be an equation for $\mathscr{C}$, and let $h(s) = 0$ be an equation for Δ. As h is irreducible, $R := \mathbb{C}[[x, y, x]]/(h)$ is a domain. If $\mathscr{C}_\Delta$ is reducible, then (F) is reducible in R. That is, we can write $F = F_1 F_2 + \alpha \cdot h$. Let $\gamma(t)$ be a parametrised curve lying in Δ_{reg} for $t \neq 0$ and with $\gamma(0) = 0$. Then $F(x, y, \gamma(t)) = F_1(x, y, \gamma(t)) F_2(x, y, \gamma(t))$ describes, for $t \neq 0$, a family of reducible plane curves with a single node. By conservation of intersection multiplicity,

the two curves ($\{F_1(x,y,0)=0\}$ and $\{F_2(x,y,0)=0\}$ have intersection multiplicity 1 at $x=y=0$. It follows that both these curves are smooth, and that they meet transversely. This proves the proposition. $\diamond$
Corollary 1.2 is an immediate consequence of the proposition.

6 Multiplicity of the Discriminant

Proof of Corollary 1.3 The multiplicity of the discriminant is the intersection multiplicity of Δ_D with a general line $S \subset B_D$. The restriction of $D \longrightarrow B_D$ to S is a surface singularity X, mapping to S. Moving S will result in a line S' that intersects the discriminant Δ_D in $\mathrm{mult}(\Delta_D)$ distinct points. The surface X' over S' will be the union of Milnor fibres F_D of D, together with $\mathrm{mult}(\Delta_D)$ fibres with a node. On the other hand, X' is a smoothing of X (since S' meets Δ_D transversely at smooth points). A simple computation of Euler characteristics gives the relation

$$\mathrm{mult}(\Delta_D) = \chi(X') - 1 + \beta_1(F_D)$$

As the Milnor fibre F_D is isomorphic to a Milnor fibre F of C minus one point, one has $\beta_1(F_D) = \mu(C)+1$. The generic perturbation with parameter t of the matrix M will give a matrix

$$\begin{pmatrix} z & A+\alpha t & B+\beta t \\ t & y & x \end{pmatrix}$$

This matrix defines the surface X in x,y,z,t-space. Blowing up X at the origin introduces an exceptional divisor isomorphic to the projectivised tangent cone of X. An easy calculation using the matrix just given shows that this consists of a non-singular plane quadric together with a line (and thus, the union of two rational curves). On the blown-up surface we find one singular point of type A_{m-3}, where $m = \mathrm{mult}(C)$. The minimal resolution of X thus has $m-3+2$ components. As X is rational, it has simultaneous resolution over the Artin component ([12]). But X is Cohen-Macaulay of embedding codimension 2, and thus has smooth base space. That is, the Artin component is the whole base space of X. It follows that any smoothing of X is homotopy-equivalent to the exceptional divisor of its minimal resolution, and thus has $\beta_1 = 0$ and β_2 equal to the number of components in the exceptional divisor. We have seen that this number is $\mathrm{mult}(C)-1$. The theorem follows. $\diamond$

Remark: The corollary just proved is equivalent to the statement that

$$\mathrm{mult}(\mathscr{C}_\Delta) = \mathrm{mult}(\Delta_C) + \mathrm{mult}(C) - 1$$

because $\mathrm{mult}(\Delta_2) = \mathrm{mult}(\mathscr{C}_\Delta)$ and $\mathrm{mult}(\Delta_C) = \mu(C)$, as C is a hypersurface.

7 Acknowledgement

This work was begun during a visit of the second author to Warwick in November 1996 as part of the project *Space Curve Singularities*, sponsored by the British Council and the DAAD. We thank Christian Alpert for pointing out a mistake in an earlier version of this paper. We also thank the referee for a number of helpful suggestions which have improved the structure of the paper.

References

[1] J. Stevens, *Infinitesimal deformations of decomposable curves*, Appendix to K. Behnke and J. Christophersen, *Hypersurface sections and obstructions*, Compositio Math. 77, 233-268, 1991

[2] W. Ebeling, *The monodromy groups of isolated singularities of complete intersections*, Lecture Notes in Math. 1293, Springer-Verlag, 1987

[3] V.I.Arnold, V.V.Goryunov, O.V.Lyashko and V.A.Vassiliev, Singularities I. Local and Global Theory, *Encyclopaedia of Mathematical Sciences Volume 6, Dynamical Systems VI*, Springer Verlag, Berlin, 1993

[4] V.I.Arnold, V.V.Goryunov, O.V.Lyashko and V.A.Vassiliev, Singularities II. Classification and Applications, *Encyclopaedia of Mathematical Sciences Volume 39, Dynamical Systems VIII*, Springer Verlag, Berlin, 1993

[5] A. Frühbis, *Klassification der einfache Raumkurven singularitäten, Diplomarbeit*, Kaiserslautern, 1996

[6] S. Gussein-Zade,*The monodromy groups of isolated singularities of hypersurfaces*, Usp. Math. Nauk **32**, 3, 23-65 (1977) *translated in* Russian Math. Surveys **32**, 2, 23-69, (1977)

[7] E.J.N. Looijenga, *Isolated Singular points on Complete Intersections*, L.M.S. Lecture Notes in Math. 77, Cambridge University Press, 1984

[8] K. Saito, Theory of Logarithmic Differential Forms and Logarithmic Vector Fields, J. Fac. Sci. Univ. Tokyo Sect. Math. **27** (1980), 265-291

[9] M. Schaps, *Deformations of Cohen-Macaulay schemes of codimension 2 and non-singular deformations of space curves*, Amer. J. Math. **99**, 1977, 669-685

[10] D. van Straten,*A note on the discriminant of a space curve*, Manus. Math. **87**, 167-177 , (1995).

[11] B. Teissier, *The hunting of invariants in the geometry of the discriminant*, *Real and Complex Singularities*, P.Holm (ed.), Sijthoff and Noordhoff Int. Publ., Alphen a/d Rijn, 1978, 565-677

[12] J. Wahl, *Simultaneous Resolution of Rational Singularities*, Comp. Math, Vol. **38**, Fasc. 1, 43-54, (1979).

[13] Roberta Wik, On the classification of multi-germs of maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ under $\mathcal{A}$-equivalence, in J.W.Bruce and F.Tari(eds.) *Real and Complex Singularities*, Research Notes in Maths, Chapman & Hall / CRC, 119-133.