Functors on the category of factorisations of a function

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A la memoria de Jairo Charris, mi profesor

1 Introduction: a determinantal factorisation of D_4

Let us begin with a simple example. Consider the 2-variable family of matrices

$$M(x_1, x_2) = \begin{pmatrix} x_2 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & x_1 \end{pmatrix}$$

Its determinant is the well known D_4 singularity, $f(x_1, x_2) = x_1^2 x_2 - x_2^3$. So

$$(\mathbf{C}^2, 0) \xrightarrow{M} \mathrm{Mat}_3(\mathbf{C}) \xrightarrow{\det} \mathbf{C}$$

is a *factorisation* of D_4. A deformation of M induces a deformation of its determinant. The deformation theory of D_4 is very well known: a miniversal deformation is given by

$$F(x_1, x_2, u_1, u_2, u_3, u_4) = x_1^2 x_2 - x_2^3 + u_1 x_2^2 + u_2 x_2 + u_3 x_1 + u_4.$$

However, deformations of M induce only a subset of the possible deformations of D_4; it turns out that every deformation of D_4 obtained by deforming M is parametrised-right-equivalent to one induced from the deformation

$$F_1(x_1, x_2, v_1, \ldots, v_7) = x_1^2 x_2 - x_2^3 +$$

$$v_1 x_1^2 - (v_1 + v_6) x_2^2 + v_5 x_1 x_2 + (v_1 v_5 - v_3 v_7 - v_2 v_4) x_1 + (v_2 v_7 - v_1 v_6 + v_3 v_4) x_2 + (v_2 v_6 - v_3 v_5) v_7 \quad (1)$$

Deformation of M within the (smaller) class of *symmetric* matrices leads to an even more restricted class of deformations of its determinant, all of which are parametrised-right-equivalent to deformations induced from the deformation

$$F_2(x_1, x_2, w_1, \ldots, w_4) = x_1^2 x_2 - x_2^3 +$$

$$w_1(x_1^2 - x_2^2) + (w_1 w_4 - w_2^2 - w_3^2) x_1 + 2 w_2 w_3 x_2 + w_4 x_1 x_2 - w_3^2 w_4. \quad (2)$$

These assertions are proved by introducing suitable notions of equivalence, $\mathcal{K}_{\det}$ and $\mathcal{K}_{\det}^{\mathrm{s}}$, for matrix and symmetric matrix families like M - both of which are, in fact, liftings, of right equivalence of determinants. These notions, due principally to Jim Damon (though see [4] for the precise version given below), are recalled in Section 3 below. The deformations F_1 and F_2 of det M are the determinants of versal deformations of M — versal, that is, with respect to $\mathcal{K}_{\mathrm{det}}$- and $\mathcal{K}^{\mathrm{s}}_{\mathrm{det}}$-equivalence. As these two matrix deformations we may take, respectively,

$$M_1(x,v) = \begin{pmatrix} x_2+v_1 & v_4 & v_7 \\ v_2 & x_1+v_5 & x_2 \\ v_3 & x_2+v_6 & x_1 \end{pmatrix} \quad \text{and} \quad M_2(x,w) = \begin{pmatrix} x_2+w_1 & w_2 & w_3 \\ w_2 & x_1+w_4 & x_2 \\ w_3 & x_2 & x_1 \end{pmatrix}$$

(the construction of these deformations is explained in Section 3). By definition of the notion of versal deformation for right-equivalence, F_1 and F_2 are parametrised-right-equivalent to deformations induced from F. That is, denoting by B, B_1 and B_2 the base-spaces $(\mathbf{C}^4,0)$, $(\mathbf{C}^7,0)$ and $(\mathbf{C}^4,0)$ of F, M_1 and M_2 respectively, there are maps $\rho_1 : B_1 \to B$ and $\rho_2 : B_2 \to B$ such that F_1 and F_2 are parametrised-right-equivalent, respectively, to the deformations $\rho_1^*(F)$ and $\rho_2^*(F)$ given by

$$\rho_1^*(F)(x,v) = F(x,\rho_1(v))$$

and

$$\rho_2^*(F)(x,w) = F(x,\rho_2(w)).$$

By inspection of the formulae for M_1 and M_2, one can see that there is a map $\rho_{21} : B_2 \to B_1$ such that

$$M_2(x,w) = M_1(x,\rho_{21}(w));$$

in fact the definition of $\mathcal{K}_{\mathrm{det}}$-versality guarantees the existence of a map ρ_{21} such that M_2 is parametrised $\mathcal{K}_{\mathrm{det}}$-equivalent to $\rho_{21}^*(M_1)$.

So our factorisations of D_4, and their asociated deformation theories, have given rise to a triangle of base spaces and inducing maps

$$\begin{array}{ccc} B_2 & \xrightarrow{\rho_{21}} & B_1 \\ \rho_1 \searrow & & \swarrow \rho_2 \\ & B & \end{array} \tag{3}$$

There is also a dual class of diagrams. Suppose M_3 is a symmetric deformation of M, with base-space B_3. Then M_3 is equivalent to a deformation induced from M_2 by a map $\sigma_2 : B_3 \to B_2$, since M_2 is a versal deformation in the space of symmetric matrices. But M_3 can also be considered as an unrestricted matrix, ignoring its symmetry. As such, it is equivalent to a deformation induced from M_1, by some map $B_3 \to B_1$. So there is another triangle of base-spaces and inducing maps,

$$\begin{array}{ccc} & B_3 & \\ \sigma_1 \swarrow & & \searrow \sigma_2 \\ B_2 & \xrightarrow{\rho_{21}} & B_1 \end{array} \tag{4}$$

where ρ_{21} is same map as in the diagram (3).

The existence of these diagrams is due to the compound factorisation of D_4,

$$(\mathbf{C}^2, 0) \xrightarrow{M} \mathrm{Symm}_3(\mathbf{C}) \xrightarrow{i} \mathrm{Mat}_3(\mathbf{C}) \xrightarrow{\det} \mathbf{C}$$

where i is the inclusion of the space of symmetric 3×3 matrices in the space of all 3×3 matrices.

A compound factorisation of *any* function g gives rise to diagrams like (3) and (4).

Question: When are such triangles commutative?

This paper is concerned with this and related questions. Our main result, in Section 6, is an infinitesimal condition for the commutativity of such diagrams. We begin by placing these questions in a more general context, in the *category of factorisations of a function.* The questions about commutativity of these diagrams become questions about the functoriality of certain maps from this category to the categories of $\mathcal{O}_X/J_g$-modules and algebras.

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2 The category of factorisations

Fix a map $g : X \to Z$. The set of all factorisations $X \xrightarrow{F} Y \xrightarrow{f} Z$ of g (i.e., for which $f \circ F = g$) forms a category, in which a morphism

$$\alpha : (X \xrightarrow{F_1} Y_1 \xrightarrow{f_1} Z) \to (X \xrightarrow{F_2} Y_2 \xrightarrow{f_2} Z)$$

is a map $\alpha : Y_1 \to Y_2$ giving rise to a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ Y_1 \arrow[rr, "\alpha"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & Z & \end{tikzcd}$$

Since in this case $F_2 = \alpha \circ F_1$ and $f_1 = f_2 \circ \alpha$, morphisms correspond to the re-bracketing of compound factorisations of g: the morphism above can be written

$$\alpha : (f_2 \circ \alpha, F_1) \to (f_2, \alpha \circ F_1),$$

where for brevity we are representing each factorisation as an ordered pair of maps.

We will denote the category of factorisations of g by $\mathscr{F}(g)$.

Example 2.1. The previous section is concerned with the morphism

$$\begin{array}{ccccc} & & (\mathbf{C}^2,0) & & \\ & M \swarrow & & \searrow M & \\ \mathrm{Symm}_n(\mathbf{C}) & & \xrightarrow{i} & & \mathrm{Mat}_3(\mathbf{C}) \\ & \det \searrow & & \swarrow \det & \\ & & \mathbf{C} & & \end{array}$$

where i is inclusion.

Remark 2.2. Let M be an $n \times n$ matrix whose entries are functions and let $g = \det M$. Let M^* be the adjugate matrix of signed submaximal minors of M. Then $MM^* = M^*M = gI_n$. Any pair of matrices (M, N) such that $MN = NM = gI_n$ is known in commutative algebra and representation theory as a *matrix factorisation* of g. Matrix factorisations correspond to maximal Cohen-Macaulay modules on the hypersurface $\{g = 0\}$, and are extensively studied, e.g. in [5],[10]. However, the morphisms of matrix factorisations referred to by algebraists in this context are not morphisms in our sense. For this reason, here we avoid the term 'matrix factorisation'.

Example 2.3. If there is a morphism $\alpha : (f_1, F_1) \to (f_2, F_2)$ then $F_2 = \alpha \circ F_1$ and $f_1 = f_2 \circ \alpha$. It follows that $F_1^*(J_{f_1}) \subset F_2^*(J_{f_2})$. For

$$\begin{aligned} F_1^*(\partial f_1/\partial x_j) &= \left(\textstyle\sum_k((\partial f_2/\partial y_k) \circ \alpha)\partial \alpha_k/\partial x_j\right) \circ F_1 \\ &= \left(\textstyle\sum_k(\partial f_2/\partial y_k) \circ \alpha \circ F_1\right)\left(\partial \alpha_k/\partial x_j\right) \circ F_1 \\ &= \left(\textstyle\sum_k(\partial f_2/\partial y_k) \circ F_2\right)\left(\partial \alpha_k/\partial x_j\right) \circ F_1 \qquad \in \quad F_2^*(J_{f_2}) \end{aligned}$$

We will refer to the natural algebra-homomorphism $\mathcal{O}/F_1^*(J_{f_1}) \to \mathcal{O}/F_2^*(J_{f_2})$ which results from this inclusion as the *comparison map*.

Proposition 2.4. *The map $C : \mathscr{F}(g) \to \mathcal{O}/J_g$-algebras taking (f, F) to $\mathcal{O}/F^*(J_f)$, and taking a morphism $\alpha : (f_1, F_1) \to (f_2, F_2)$ to the comparison map $\mathcal{O}/F_1^*(J_{f_1}) \to \mathcal{O}/F_2^*(J_{f_2})$, is a covariant functor $\mathscr{F}(g) \to \mathcal{O}/J_g$-algebras.* □

Here and where possible in the rest of the paper, we use $\mathcal{O}$ in place of $\mathcal{O}_X$, where X is the domain of the map g whose factorisations we are considering.

Example 2.5. As initial and final objects in $\mathscr{F}(g)$ we have the factorisations

$$(\mathbf{C}^m,0) \xrightarrow{\mathrm{id}_{\mathbf{C}^m}} (\mathbf{C}^m,0) \xrightarrow{g} (\mathbf{C},0)$$

and

$$(\mathbf{C}^m,0) \xrightarrow{g} (\mathbf{C}^n,0) \xrightarrow{\mathrm{id}_{\mathbf{C}}} (\mathbf{C},0),$$

since for any other factorisation $f \circ F = g$ there are unique diagrams

$$\begin{array}{ccccc} & & \mathbf{C}^m & & \\ & \mathrm{id}_{\mathbf{C}^m} \swarrow & & \searrow F & \\ \mathbf{C}^m & & \xrightarrow{\alpha_0} & & \mathbf{C}^n \\ & g \searrow & & \swarrow f & \\ & & \mathbf{C} & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \mathbf{C}^m & & \\ & F \swarrow & & \searrow g & \\ \mathbf{C}^m & & \xrightarrow{\alpha_\infty} & & \mathbf{C} \\ & f \searrow & & \swarrow \mathrm{id}_{\mathbf{C}} & \\ & & \mathbf{C} & & \end{array}$$

in which $\alpha_0 = F$ and $\alpha_\infty = f$. We have $C(g, \mathrm{id}_{\mathbf{C}^m}) = \mathcal{O}/J_g$ and $C(\mathrm{id}_{\mathbf{C}}, g) = 0$. Thus C takes the initial and final objects of $\mathscr{F}(g)$ to the initial and final objects of the category of $\mathcal{O}/J_g$-algebras.

3 $\mathcal{K}_V$ and $\mathcal{K}_f$-equivalence

In [2], Jim Damon generalised the notion of contact equivalence of map-germs $(X, x_0) \to (Y, y_0)$, as follows.

Definition 3.1. *Given the germ of a subvariety $(V, y_0) \subset (Y, y_0)$ the map-germs $F, G : (X, x_0) \to (Y, y_0)$, are $\mathcal{K}_V$ -equivalent if there are diffeomorphisms Φ of $(X \times Y, (x_0, y_0))$ and φ of (X, x_0) such that*

1. *$\pi_X \circ \Phi = \varphi \circ \pi_X$, (i.e. Φ lifts φ)*
2. *Φ takes $X \times V$ to itself, and*
3. *Φ induces a diffeomorphism graph$(F) \to$ graph(G).*

Here $\pi_X : X \times Y \to X$ is projection. In the analytic category, "diffeomorphism" means bianalytic automorphism. Ordinary contact equivalence becomes $\mathcal{K}_V$-equivalence with $V = \{y_0\}$.

In the case that V is a hypersurface in Y with equation f, the closely related group of equivalences $\mathcal{K}_f$ was introduced in [4]; the definition is that of $\mathcal{K}_V$ except that (2) is replaced by the condition

$2'$. $f \circ \pi_Y \circ \Phi = f \circ \pi_Y$.

Lemma 3.2. *(i) If F and F' are $\mathcal{K}_f$-equivalent then $f \circ F$ and $f \circ F'$ are right-equivalent. (ii) If F and F' are $\mathcal{K}_V$-equivalent then $f \circ F$ and $f \circ F'$ are contact-equivalent.*

Proof By (1) and (3) of the definition,

$$\Phi(x, F(x)) = (\varphi(x), F'(\varphi(x))). \tag{8}$$

Now (2') implies

$$f(F(x)) = f \circ \pi_Y(\Phi(x, F(x)) = f \circ \pi_Y(\varphi(x), F'(\varphi(x))) = f(F'(\varphi(x))),$$

i.e. $f \circ F$ and $f \circ F'$ are right-equivalent, while (2) implies

$$F^{-1}(V) = (\mathrm{gr}(F))^{-1}(X \times V) = (\mathrm{gr}(F))^{-1}(\Phi^{-1}(X \times V))$$

which is equal to $(F' \circ \varphi)^{-1}(V)$ by (8). Thus $(f \circ F)^{-1}(0) = (f \circ F' \circ \varphi)^{-1}(0)$, from which contact-equivalence follows. □

In 4.4 below we show that the converse to both parts of this Lemma is in general false.

Our concern with these equivalence relations and deformation theories is motivated by the fact that many naturally occuring maps and functions arise, and are naturally deformed, as composites. The discriminant hypersurface D_f (set of critical values) of a map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ with $n \geq p$, is a 'non-linear section' of the discriminant D_F of a stable unfolding F, and as equation for D_f we can take the composite of an equation h for D_F with a suitable 'base-change' map i. In fact Damon shows in [3] that the $\mathcal{A}$-codimension of f is equal to the $\mathcal{K}_{D_F}$-codimension of i, and in [4] Damon and the author showed that the discriminant of a stable perturbation of f has homotopy type of a wedge of $(p-1)$-spheres, whose number is equal to the $\mathcal{K}_h$-codimension of i.

When X and Y are smooth, the tangent space to the $\mathcal{K}_V$-orbit of a germ F is

$$T\mathcal{K}_V F = tF(\theta_{X,x_0}) + F^*(\mathrm{Der}(-\log V)),$$

where $\mathrm{Der}(-\log V)$ is the $\mathcal{O}_{Y,y_0}$-module of germs of vector fields on Y which are tangent to V at its smooth points. The expression for $T\mathcal{K}_f F$ is the same as that for $T\mathcal{K}_V F$ except that $\mathrm{Der}(-\log V)$ is replaced by $\mathrm{Der}(-\log f)$, the $\mathcal{O}_Y$-module of vector fields tangent to all of the level sets of f.

The T^1's for the deformation theory of the germ F under $\mathcal{K}_f$ and $\mathcal{K}_V$-equivalence have usually been denoted by $T^1_{\mathcal{K}_f}F$ and $T^1_{\mathcal{K}_V}F$. In keeping with our change of viewpoint in this paper, we will call them $T^1(f, F)$ and $\widetilde{T}^1(f, F)$ instead. That is,

$$T^1(f,F) = \frac{\theta(F)}{T\mathcal{K}_f F}, \quad \widetilde{T}^1(f,F) = \frac{\theta(F)}{T\mathcal{K}_V F}.$$

Both $\mathcal{K}_V$ and $\mathcal{K}_f$ are *geometric* groups of diffeomorphisms, in the sense of Damon in [1]. This means that standard results such as finite determinacy of a map with finite-dimensional T^1, and the versality theorem, that infinitesimal versality implies versality, hold. The latter theorem is what justifies the statement in Section 1 that the deformations M_1 and M_2 are $\mathcal{K}_{\det}$ and $\mathcal{K}^s_{\det}$-versal deformations of M. For example, M_1 is constructed by finding a vector-space basis $m_1, \ldots, m_7$ for $T^1(M, \det)$, and taking $M_1(x, v_1, \ldots, v_7) = M(x) + \sum_j v_j m_j$.

Example 3.3.

$$T^1(g, id_{\mathbf{C}^m}) = 0 = \widetilde{T}^1(g, id_{\mathbf{C}^m}).$$

$$T^1(id_{\mathbf{C}}, g) = \mathcal{O}_X/J_g, \quad \widetilde{T}^1(id_{\mathbf{C}}, g) = \mathcal{O}_X/(g + J_g).$$

Proposition 3.4. *For any factorisation* $X \xrightarrow{F} Y \xrightarrow{f} \mathbf{C}$ *of* g, $T^1(f, F)$ *and* $\widetilde{T}^1(f, F)$ *are* $\mathcal{O}/J_g$*-modules.*

Proof Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be variables on X and Y respectively. We have to show that for each i and j,

$$\frac{\partial g}{\partial x_i}\frac{\partial}{\partial y_j} \in T\mathcal{K}_f F.$$

This is easy: since $\mathrm{Der}(-\log f)$ contains the Hamiltonian vector fields

$$\chi_{k,\ell}(f) := \frac{\partial f}{\partial y_k}\frac{\partial}{\partial y_\ell} - \frac{\partial f}{\partial y_\ell}\frac{\partial}{\partial y_k},$$

we have

$$\frac{\partial g}{\partial x_i}\frac{\partial}{\partial y_j} = \sum_k \left(\frac{\partial f}{\partial y_k}\circ F\right)\frac{\partial F_k}{\partial x_i}\frac{\partial}{\partial y_j} = \sum_{k\neq j}\frac{\partial F_k}{\partial x_i}\,\chi_{k,j}(f)\circ F + tF\left(\left(\frac{\partial f}{\partial y_k}\circ F\right)\frac{\partial}{\partial x_i}\right)$$

□

In [6], the relation between $\mathcal{K}_f$ equivalence and right-equivalence of the composites, and $\mathcal{K}_V$-equivalence and contact-equivalence of the composites, was further explored. We summarise the main result: let $\mathbf{L}_\bullet(f)$ be a free $\mathcal{O}_Y$-resolution of $\mathcal{O}_Y/J_f$, and let $\mathbf{K}_\bullet(f\circ F)$ be the Koszul complex on the first order partials of $f\circ F$. There is a morphism of complexes $\mathbf{K}_\bullet(f)\to\mathbf{L}_\bullet(f)$, and composing the pull-back of this by F, with the canonical morphism $\mathbf{K}_\bullet(f\circ F)\to F^*(\mathbf{K}_\bullet(f))$, we get a morphism $\phi_f : \mathbf{K}_\bullet(f\circ F)\to F^*(\mathbf{L}_\bullet(f))$ lifting the comparison morphism $\mathcal{O}_X/J_{f\circ F}\to\mathcal{O}_X/F^*(J_f)$. Let $C_\bullet(\phi_f)$ denote the cone of the morphism of complexes ϕ_f. There is an analogous construction of a morphism of complexes ϕ_V : $\widetilde{\mathbf{K}}_\bullet(f)\to F^*(\widetilde{\mathbf{L}}_\bullet)$ lifting the comparison morphism $\mathcal{O}_X/(f\circ F+J_{f\circ F})\to\mathcal{O}_X/F^*((f)+J_f)$, and of its cone, $C_\bullet(\phi_V)$.

Theorem 3.5. ([6])

$$T^1_{\mathcal{K}_f}F = H_1(C_\bullet(\phi_f)),\quad T^1_{\mathcal{K}_V}F = H_1(C_\bullet(\phi_V)).$$

□

If $\phi:\mathbf{A}_\bullet\to\mathbf{B}_\bullet$ is a morphism of complexes, there is a long exact sequence

$$\cdots\to H_k(\mathbf{A}_\bullet)\to H_k(\mathbf{B}_\bullet)\to H_k(C_\bullet(\phi))\to H_{k-1}(\mathbf{A}_\bullet)\to\cdots,$$

and thus we have

Corollary 3.6. *There are canonical long exact sequences*

$$\cdots\to H_1(\mathbf{K}_\bullet(g))\to H_1(F^*(\mathbf{L}_\bullet))\to T^1(f,F)\to\frac{\mathcal{O}_X}{J_g}\to\frac{\mathcal{O}_X}{F^*(J_f)}\to 0 \tag{9}$$

and

$$\cdots\to H_1(\widetilde{\mathbf{K}}_\bullet(g))\to H_1(F^*(\widetilde{\mathbf{L}}_\bullet))\to\widetilde{T}^1(f,F)\to\frac{\mathcal{O}_X}{(g+J_g)}\to\frac{\mathcal{O}_X}{F^*((f)+J_f)}\to 0 \tag{10}$$

in which the maps $T^1(f,F)\to\mathcal{O}_X/J_g$ and $\widetilde{T}^1(f,F)\to\mathcal{O}_X/(g+J_g)$ are induced by $F^*(tf)$: $\theta(F)\to\mathcal{O}_X$. □

Note that $H_1(F^*(\mathbf{L}_\bullet)) = \mathrm{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_X)$ and $H_1(F^*(\widetilde{\mathbf{L}}_\bullet)) = \mathrm{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/(f+J_f), \mathcal{O}_X)$.

Remark 3.7. (i) Let $g = f \circ F$. Then $\mathcal{O}_{\mathbf{C}^m}/J_g$ is the T^1 of g for right-equivalence. The morphism $T^1_{\mathcal{K}_f}F \to T^1 g$ in (9) is a map between deformation functors, telling us which of the first order deformations of g we can get by deforming F alone. A similar statement holds for $T^1_{\mathcal{K}_V}F \to \mathcal{O}_{\mathbf{C}^m}/(g)+J_g)$, with contact-equivalence in place of right-equivalence.

(ii) In fact we can say more: if $\mathscr{F} : (\mathbf{C}^m \times B, 0) \to (\mathbf{C}^n, 0)$ is a $\mathcal{K}_f$-miniversal deformation of F, then $f \circ \mathscr{F}$ is isomorphic to a deformation induced from an $\mathscr{R}_e$-miniversal deformation $G : (\mathbf{C}^m \times S, 0) \to (\mathbf{C}, 0)$ of g by an *inducing map* $\Lambda : B \to S$. Because of the freeness of $T^1_{\mathscr{R}/S}G := \mathcal{O}_{\mathbf{C}^m \times \mathbf{C}^\mu}/J_{\mathrm{rel}}G$ over $\mathcal{O}_S$, the Kodaira-Spencer map gives an isomorphism

$$\mathscr{T}S_s \simeq \pi_*(T^1_{\mathscr{R}/S}G)_s = \bigoplus_{x \in \mathbf{C}^m} T^1_{\mathscr{R}/S}G_{(x,s)}$$

where $\mathscr{T}S$ is the tangent sheaf on S. By coherence, there is a neighbourhood of $0 \in B$ on which there is an epimorphism

$$\mathscr{T}B_b \to \pi_*(T^1_{\mathcal{K}_f/B}\mathscr{F})_b = \bigoplus_{x \in \mathbf{C}^m} T^1_{\mathcal{K}_f/B}\mathscr{F}_b.$$

Then there is a commutative diagram

$$\begin{array}{ccc} \pi_*(T^1_{\mathcal{K}_f/B}\mathscr{F})_b & \longrightarrow & \pi_*\big(T^1_{\mathscr{R}/S}G\big)_{\Lambda(b)} \\ \uparrow & & \uparrow\simeq \\ \mathscr{T}B_b & \xrightarrow{d_b\Lambda} & \mathscr{T}S_{\Lambda(b)} \end{array} \tag{11}$$

in which the vertical maps are the Kodaira-Spencer morphisms and the map at the top is the relative version of the morphism $T^1_{\mathcal{K}_f}F \to \mathcal{O}_{\mathbf{C}^m}/J_g$ in the long exact sequence (9). It follows from commutativity that Λ *is a submersion at precisely those points b over which $\mathcal{O}_{\mathbf{C}^m}/F_u^*(J_f)$ vanishes.*

(iii) Where the left hand Kodaira Spencer map is an isomorphism also (e.g. for symmetric matrix families in two and three variables, with $f = \det$, as shown in [6]), *the relative version of the morphism $T^1_{\mathcal{K}_f}F \to \mathcal{O}_{\mathbf{C}^n}/J_g$ can be identified with the derivative of the inducing map Λ.* As a consequence of this, Λ is an immersion at those points b over which $H^1(\mathbf{L}_\bullet(f))$ (which is of course just $\mathrm{Tor}_1^{\mathcal{O}_{\mathbf{C}^n}}(\mathcal{O}_{\mathbf{C}^n}/J_f, \mathcal{O}_{\mathbf{C}^m})$) vanishes.

(iv) In [7], V. V. Goryunov and V. Zakalyukin observe that for simple symmetric matrix families M in two variables, and with $f = \det$, the $\mathcal{K}_{\det}$-codimension of M is equal to the $\mathscr{R}_e$-codimension (= Milnor number) of $\det M$, and that morover the discriminant of the inducing map Λ is the image under Λ of the set of points u such that for some $x \in \mathbf{C}^2$, $M_u(x)$ has corank ≥ 2. They conjecture that the assumption of simplicity can be omitted in these statements. The first part of this conjecture was proved in [6]. That the second part also holds follows from (ii) above, since $V(J_{\det})$ is precisely the set of matrices of corank ≥ 2.

4 Is T^1 a functor?

From now on we consider only $T^1 = T^1\mathcal{K}_f$. An analogous theory can be developed for $\widetilde{T}^1$.

Suppose we are given a morphism

$$\begin{array}{ccccc} & & \mathbf{C}^m & & \\ & F_1 \swarrow & & \searrow F_2 & \\ \mathbf{C}^n & & \xrightarrow{\alpha} & & \mathbf{C}^p \\ & f_1 \searrow & & \swarrow f_2 & \\ & & \mathbf{C} & & \end{array} \quad . \tag{12}$$

The map α gives rise to an $\mathcal{O}$-linear map $F_1^*(t\alpha) : \theta(F_1) \to \theta(F_2)$, sending $\frac{d(F_{1t}}{dt}_{|t=0}$ to $\frac{d(\alpha \circ F_{1,t})}{dt}_{|t=0}$. The spaces $T^1(f_j, F_j)$ and $\widetilde{T}^1(f_j, F_j)$ are quotients of $\theta(F_j)$ and one might hope that $F_1^*(t\alpha)$ would gives pass to the quotient and give rise to morphisms of $\mathcal{O}$-modules (and therefore of $\mathcal{O}/J_g$-modules, in view of 3.4)

$$T^1(\alpha) : T^1(f_1, F_1) \to T^1(f_2, F_2)$$

and

$$\widetilde{T}^1(\alpha) : \widetilde{T}^1(F_1, F_1) \to \widetilde{T}^1(f_2, F_2).$$

Since the existence of the morphism α means that $F_2 = \alpha \circ F_1$ and $f_1 = f_2 \circ \alpha$, we change our notation slightly, and write F for F_1 and f for f_2. For the existence of $\mathcal{O}$-linear maps $T^1(\alpha)$ and $\widetilde{T}^1(\alpha)$, what is required is that

$$F^*(t\alpha)(T\mathcal{K}_{f\circ\alpha}F) \subseteq T\mathcal{K}_f(\alpha \circ F)$$

and

$$F^*(t\alpha)(T\mathcal{K}_{\alpha^{-1}(V)}F) \subseteq T\mathcal{K}_V(\alpha \circ F).$$

Clearly $F^*(t\alpha)(tF(\theta_{\mathbf{C}^m})) = t(\alpha \circ F)(\theta_{\mathbf{C}^m}) \subseteq T\mathcal{K}_f(\alpha \circ F)$; however, as we will see in Example 4.2, $F^*(t\alpha)(F^*(\mathrm{Der}(-\log f \circ \alpha))$ and $F^*(t\alpha)(F^*(\mathrm{Der}(-\log \alpha^{-1}(V)))$ are not always contained in $T\mathcal{K}_f(\alpha \circ F)$ and $T\mathcal{K}_V(\alpha \circ F)$.

Example 4.1. For any factorisation (F, f) of g, let $\alpha_\infty : (f, F) \to (\mathrm{id}_{\mathbf{C}}, g)$ be the unique morphism to the final object $(g, \mathrm{id}_{\mathbf{C}})$. Then $T^1(\alpha_\infty)$ and $\widetilde{T}^1(\alpha_\infty)$ are just the maps $T^1(f, F) \to \mathcal{O}/J_g$ and $\widetilde{T}^1(f, F) \to \mathcal{O}/(g + J_g)$ in the exact sequences (9) and (10), and therefore are $\mathcal{O}_X$-linear. Of course, it is easy to check this directly.

Example 4.2. Define $f : \mathbf{C}^4 \to \mathbf{C}$ and $\alpha : \mathbf{C}^3 \to \mathbf{C}^4$ by

$$f(w, x, y, z) = (z + w)x + (z - w)y + w^2 + z^2,$$

$$\alpha(x, y, z) = (0, x, y, z),$$

so that

$$f \circ \alpha(x, y, z) = xz + yz + z^2 = z(x + y + z).$$

Both $\mathrm{Der}(-\log f \circ \alpha)$ and $\mathrm{Der}(-\log \alpha^{-1}(V))$ contain the constant vector field $\partial/\partial x - \partial/\partial y$, while f itself has an isolated (indeed non-degenerate) critical point at 0, so that $\mathrm{Der}(-\log f)$ and $\mathrm{Der}(-\log V)$ are contained in $m_{\mathbf{C}^4,0}\theta(f)$. Since $t\alpha = \alpha$ is a linear injection, $t\alpha(\mathrm{Der}(-\log f \circ \alpha)) \not\subset \alpha^*(\mathrm{Der}(-\log f))$ and $t\alpha(\mathrm{Der}(-\log \alpha^{-1}(V))) \not\subset \alpha^*(\mathrm{Der}(-\log V))$.

Now define $F : \mathbf{C}^2 \to \mathbf{C}^3$ by $F(u,v) = (u^2, u^2, v)$, and let $g = f \circ \alpha \circ F = 2u^2v + v^2$. Then $T^1(f \circ \alpha, F)$ is the quotient of $\theta(F)$ by the $\mathcal{O}_{\mathbf{C}^2}$-module $T\mathcal{K}_{f\circ\alpha}F$, which contains the two linearly independent constant vector fields $tF(\partial/\partial v) = \partial/\partial z$ and $\partial/\partial x - \partial/\partial y$. On the other hand, up to scalar multiples $T\mathcal{K}_f(\alpha \circ F)$ and $T\mathcal{K}_V(\alpha \circ F)$ contain only one constant vector field, namely $t(\alpha \circ F)(\partial/\partial v)$. As $F^*(t\alpha)$ is a linear injection with constant matrix,

$$F^*(t\alpha)(T\mathcal{K}_{f\circ\alpha}F) \not\subset T\mathcal{K}_V f \circ F,$$

and $T^1(\alpha) : T^1(f \circ \alpha, F) \to T^1(f, \alpha \circ F)$ and $\widetilde{T}^1(\alpha) : \widetilde{T}^1(f \circ \alpha, F) \to \widetilde{T}^1(f, \alpha \circ F)$ cannot be $\mathcal{O}$-linear.

A straightforward calculation shows that $\dim_{\mathbf{C}} T^1(f \circ \alpha) = 1$ and $\dim_{\mathbf{C}} T^1(f, \alpha \circ F) = 5$.

Example 4.3. Let $f(x_1, x_2) = x_2 x_2$ and let $\alpha : \mathbf{C} \to \mathbf{C}^2$ be the inclusion $\alpha(x_1) = (x_1, 0)$. Then $f \circ \alpha = 0$, so $\mathrm{Der}(-\log(f \circ \alpha)) = \theta_{\mathbf{C}}$. But there is no extension of $\partial/\partial x_1 \in \theta_{\mathbf{C}}$ to a vector field in $\mathrm{Der}(-\log f)$. In other words, $t\alpha(\mathrm{Der}(-\log f \circ \alpha)) \not\subset \alpha^*(\mathrm{Der}(-\log f))$.

Remark 4.4. The previous examples make clear that the converse to Lemma 3.2 does not in general hold. In the case of Example 4.2, the family $F_t(u,v) = (u^2 + t, u^2 - t, v)$ is $\mathcal{K}_{f\circ\alpha}$-trivial, from which it follows, by 3.2, that $f \circ \alpha \circ F_t$ is $\mathscr{R}$-trivial. Nevertheless $\alpha \circ F_t$ is not $\mathcal{K}_f$-trivial, nor even $\mathcal{K}_V$-trivial (where $V = \{f = 0\}$).

4.1 $T^1(\alpha)$ as a C-linear map

Suppose that for $j = 1, 2$, $T^1(f_j, F_j)$ has finite length, and let $\mathscr{F}_j$ be a $\mathcal{K}_{f_j}$-miniversal deformation of F_j, with base B_{F_j}. Since $\alpha \circ \mathscr{F}_1$ is a deformation of F_2, it is isomorphic to a deformation induced from $\mathscr{F}_2$ by a base change map $\rho_{1,2} : B_{F_1} \to B_{F_2}$. Although the map $\rho_{1,2}$ is not unique, its derivative at $0 \in B_{F_1}$ is. Modulo the identification of $T_0 B_{F_j}$ with $T^1(f_j, F_j)$, it is the map

$$\frac{dF_{1,t}}{dt}_{|t=0} \mapsto \frac{d(\alpha \circ F_{1,t})}{dt}_{|t=0}.$$

In other words, it is induced by $F_1^*(t\alpha) : \theta(F_1) \to \theta(F_2)$.

Now suppose also that g has isolated singularity, and let $\mathscr{G} : X \times B_g \to \mathbf{C}$ be an $\mathscr{R}_e$-miniversal deformation of g. By versality of $\mathscr{G}$ there are base-change maps $\rho_j : B_{F_j} \to B_g$, such that $f_j \circ \mathscr{F}_j \sim_{\mathscr{R}\,\mathrm{un}} \rho_j^*(\mathscr{G})$. Once again, these are not unique, but their derivatives at 0 are; after the requisite identifications, they are given by

$$\frac{dF_{1,t}}{dt}_{|t=0} \mapsto \frac{df_j \circ F_{1,t}}{dt}_{|t=0}.$$

In other words, this map is induced by $F_j^*(tf_j) : \theta(F) \to \theta(g)$, and coincides with the map $T^1(f, F) \to \mathcal{O}_X/J_g$ in the long exact sequence 9).

Lemma 4.5. *We can always choose* $\rho_1 = \rho_2 \circ \rho_{1,2}$ *and thus ensure that the diagram*

$$\begin{tikzcd} B_{F_1} \arrow[rr, "\rho_{1,2}"] \arrow[dr, "\rho_1"'] & & B_{F_2} \arrow[dl, "\rho_2"] \\ & B_g & \end{tikzcd} \tag{13}$$

commutes.

Proof Given a group of equivalences $\mathscr{G}$ acting on a space of map-germs, let $\mathscr{G}_{\mathrm{un}}$ denote the corresponding group of parametrised equivalences acting on deformations.

By definition of $\rho_{1,2}$, we have

$$\alpha \circ \mathscr{F}_1 \ \sim_{\mathcal{K}_{f_2,\mathrm{un}}} \ \rho_{1,2}^*(\mathscr{F}_2).$$

From this it follows, by the parametrised version of Lemma 3.2, that

$$f_2 \circ \alpha \circ \mathscr{F}_1 \ \sim_{\mathscr{R}_{\mathrm{un}}} \ f_2 \circ \rho_{1,2}^*(\mathscr{F}_2) = \rho_{1,2}^*(f_2 \circ \mathscr{F}_2).$$

But $f_2 \circ \alpha = f_1$, so

$$f_1 \circ \mathscr{F}_1 \ \sim_{\mathscr{R}_{\mathrm{un}}} \ \rho_{1,2}^*(f_2 \circ \mathscr{F}_2).$$

Moreover,

$$f_2 \circ \mathscr{F}_2 \sim_{\mathscr{R}_{\mathrm{un}}} \ \rho_2^* \mathscr{G},$$

so

$$f_1 \circ \mathscr{F}_1 \ \sim_{\mathscr{R}_{\mathrm{un}}} \rho_{1,2}^*(\rho^*(\mathscr{G})) = (\rho \circ \rho_{1,2})^*(\mathscr{G}).$$

Therefore we can take, as ρ_1, the composite $\rho_2 \circ \rho_{1,2}$. □

5 Coherent morphisms

Continuing with the notation and assumptions of the previous section, let $H : (\mathbf{C}^m \times B) \to (\mathbf{C}^n, 0)$ be any deformation of F_1. Then H also induces a deformation $\alpha \circ H$ of F_2. The deformations H and $\alpha \circ H$ are, respectively, $\mathcal{K}_{f_1,\mathrm{un}}$-equivalent and $\mathcal{K}_{f_2,\mathrm{un}}$-equivalent to the deformations of F_1 and F_2 induced by base-change maps $B \xrightarrow{i_1} B_{F_1}$ and $B \xrightarrow{i_2} B_{F_2}$, whose derivatives at 0 are uniquely determined. Thus, there is a diagram

$$\begin{tikzcd} & B \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_{F_1} \arrow[rr, "\rho_{1,2}"] & & B_{F_2} \end{tikzcd} \tag{14}$$

Example 4.2 shows that the maps i_1 and i_2 cannot always be chosen so as to make this diagram commute. For example, in 4.2 the 1-parameter deformation $H(u, v, t) = (u^2 + tu, u^2, v)$ of F_1 is infinitesimally $\mathcal{K}_{f_1}$-trivial, so that $i_1 : B \to B_{F_1}$ has vanishing derivative at 0. On the other hand, the deformation $\alpha \circ H$ of F_2 is not infinitesimally $\mathcal{K}_{f_2}$-trivial, so that $i_2 : B \to B_{F_2}$ is an immersion.

Definition 5.1. The morphism $\alpha : (f_1, F_1) \to (f_2, F_2)$ is *coherent* if whenever H_1 and H_2 are deformations of F_1 on the same parameter space, then

$$H_1 \sim_{\mathcal{K}_{f_1,\mathrm{un}}} H_2 \quad \Rightarrow \quad \alpha_*(H_1) \sim_{\mathcal{K}_{f_2,\mathrm{un}}} \alpha_*(H_2) \tag{15}$$

Proposition 5.2. *(1) Let α be a coherent morphism. Then*
(i) we can always choose the base-change map i_2 in diagram (14) to be equal to $\rho_{1,2} \circ i_1$, and thus ensure that (14) commutes, and
(ii) $T^1(\alpha) : T^1(f_1, F_1) \to T^1(f_2, F_2)$ is $\mathcal{O}$-linear.
(2) The composite of coherent morphisms is coherent.

Proof (1)(i) The proof is the same as the deduction of 4.5 from the parametrised version of 3.2.

(ii) All that is needed for $\mathcal{O}$-linearity is that whenever the 1-parameter deformation H of F_1 is infinitesimally trivial, then so should $\alpha \circ H$ be infinitesimally trivial also. This is immediate from the definition of coherence.

Statement (2) is obvious. □

Example 4.2 shows that immersions are not necessariy coherent. Projection, however, always defines a coherent morphism.

Proposition 5.3. *The morphism π in the diagram*

$$\begin{array}{ccccc} & & \mathbf{C}^m & & \\ & (F,G) \swarrow & & \searrow F & \\ \mathbf{C}^n \times \mathbf{C}^p & & \xrightarrow{\pi} & & \mathbf{C}^n \\ & f \circ \pi \searrow & & \swarrow f & \\ & & \mathbf{C} & & \end{array},$$

is coherent.

Proof Suppose that $(\mathscr{F}_i, \mathscr{G}_i)$ for $i = 1, 2$ are $\mathcal{K}_{f\circ\pi\,\mathrm{un}}$-equivalent deformations of (F, G) on base B. Let $\Phi : B \times X \times Y_1 \times Y_2 \to B \times X \times Y_1 \times Y_2$ be a diffeomorphism in $\mathcal{K}_{f\circ\pi\,\mathrm{un}}$ which transforms the graph of $(\mathscr{F}_1, \mathscr{G}_1)$ to that of $(\mathscr{F}_2, \mathscr{G}_2)$. Write $\Phi(u, x, y_1, y_2) = (u, \Phi_X(u, x), \Phi_{Y_1}(u, x, y_1, y_2), \Phi_{Y_2}(u, x, y_1, y_2)$ (using the subindices merely as labels). Note that by definition of $\mathcal{K}_{f\circ\pi\,\mathrm{un}}$-equivalence, we have

1. $\Phi(0, x, y_1, y_2) = (0, x, y_1, y_2)$ for all (x, y_1, y_2) (Φ is an unfolding of the identity on $X \times Y_1 \times Y_2$), and
2. $f(\Phi_{Y_1}(u, x, y_1, y_2)) = f(y_1)$ for all (u, x, y_1, y_2).

Define $\Psi : B \times X \times Y_1 \to B \times X \times Y_1$ by

$$\Psi(u, x, y_1) = (u, \Phi_X(u, x), \Phi_{Y_1}(u, x, y_1, \mathscr{G}_1(u, x)).$$

By (1) and the inverse function theorem, Ψ is a diffeomorphism, and unfolds the identity on $X \times Y_1$, by (2) it preserves the value of f, and evidently it transforms the graph of $\mathscr{F}_1$ to that of $\mathscr{F}_2$. Hence $\mathscr{F}_1$ and $\mathscr{F}_2$ are $\mathcal{K}_{f\,\mathrm{un}}$-equivalent. $\square$

This example of coherence, though trivial, will be useful in the proof of the main theorem of the next section.

Proposition 5.4. *The canonical morphisms α_0 and α_∞ associated to the initial and terminal object,*

$$\begin{array}{ccccccc} & \mathbf{C}^m & & & & \mathbf{C}^m & \\ id_{\mathbf{C}^m} \swarrow & & \searrow F & & F \swarrow & & \searrow g \\ \mathbf{C}^m & \xrightarrow{\alpha_0} & \mathbf{C}^n & \quad and \quad & \mathbf{C}^m & \xrightarrow{\alpha_\infty} & \mathbf{C} \\ g \searrow & & \swarrow f & & f \searrow & & \swarrow id_{\mathbf{C}} \\ & \mathbf{C} & & & & \mathbf{C} & \end{array},$$

(where $\alpha_0 = F$, $\alpha_\infty = f$) are coherent.

Proof If $\mathscr{F} : \mathbf{C}^m \times B \to \mathbf{C}^m$ is any deformation of $\mathrm{id}_{\mathbf{C}^m}$, then there is an unfolding of $\mathrm{id}_{\mathbf{C}^m}$, $\Phi : \mathbf{C}^m \times B \to \mathbf{C}^m \times B$, such that $\mathscr{F} \circ \Phi = \pi$, where $\pi : \mathbf{C}^m \times B \to \mathbf{C}^m$ is projection. It follows that $\alpha_0 \circ \mathscr{F} \circ \Phi = F \circ \pi$. Thus every deformation of F arising as the composite with α of a deformation of $\mathrm{id}_{\mathbf{C}^m}$ is trivial. Hence any two such are equivalent to one another, and *a fortiori* α_0 is coherent.

Coherence of α_∞, is essentially a parametrised version of 3.2(i). For $\mathcal{K}_{\mathrm{id}_{\mathbf{C}}}$ is just right-equivalence $\mathscr{R}$; so it is necessary to show that if $\mathscr{F}_1 \sim_{\mathcal{K}_{f\,\mathrm{un}}} \mathscr{F}_2$ then $f \circ \mathscr{F}_1$ and $f \circ \mathscr{F}_2$ are $\mathscr{R}_{\mathrm{un}}$-equivalent deformations of g. Suppose $\Phi(u,x,y) = (u, \phi_X(u,x), \Phi_Y(u,x,y))$ is a $\mathcal{K}_{f\,\mathrm{un}}$-equivalence transforming the graph of $\mathscr{F}_1$ to that of $\mathscr{F}_2$. Then

$$\Phi_Y(u,x,\mathscr{F}_1(u,x)) = \mathscr{F}_2(u, \Phi_X(u,x)).$$

But by definition of $\mathcal{K}_{f\,\mathrm{un}}$-equivalence, $f(\Phi_Y(u,x,y)) = f(y)$ for every u, x, y, and in particular

$$f(\mathscr{F}_1(u,x)) = f(\mathscr{F}_2(u, \Phi_X(u,x)).$$

In other words $f \circ \mathscr{F}_1$ and $f \circ \mathscr{F}_2$ are $\mathscr{R}_{\mathrm{un}}$-equivalent. $\square$

6 An infinitesimal condition for coherence

The simplest condition to guarantee $\mathcal{O}$-linearity of the map $T^1(\alpha)$ associated with a morphism α in $\mathscr{C}_g$ is

$$t\alpha\big(\mathrm{Der}(-\log(f \circ \alpha))\big) \subset \alpha^* Der(-\log f)). \tag{16}$$

For $\mathcal{O}$-linearity is just the inclusion

$$F^*(t\alpha)(tF(\theta_{\mathbf{C}^m}) + F^*(\mathrm{Der}(-\log f \circ \alpha)) \subset t(\alpha \circ F)(\theta_{\mathbf{C}^m}) + (\alpha \circ F)^*(\mathrm{Der}(-\log f)). \tag{17}$$

Clearly $F^*(t\alpha)(tF(\theta_{\mathbf{C}^m})) \subset t(\alpha \circ F)(\theta_{\mathbf{C}^m})$. By applying F^* to the inclusion (16) we deduce the inclusion of the second summand on the left of (17) in the second summand on the right.

Condition (16) is a coherent version of the linear algebraic statement

$$d_x\alpha(\ker d_x(f\circ\alpha)) \subset \ker d_{\alpha(x)}f. \tag{18}$$

However, whereas (18) always holds, (16) does not, as demonstrated by Example 4.2.

Example 6.1. 1. In fact (16) holds for any *flat* morphism (see Remark 6.2 below). In Section 7 we give examples of non-flat morphisms satisfying (16).

2. The canonical morphism α_∞ to the final object in $\mathscr{C}_g$ described in Example 2.5 satisfies (16). The canonical morphism from the initial object does not always satisfy (16).

Remark 6.2. Since

$$\mathrm{Der}(-\log(f\circ\alpha)) = Z_1(\mathbf{K}_\bullet(f\circ\alpha))$$

and

$$\alpha^*(\mathrm{Der}(-\log f)) = B_1\big(\alpha^*(\mathbf{L}_\bullet(f))\big),$$

(16) is equivalent to the vanishing of the morphism

$$H_1(\mathbf{K}_\bullet(f\circ\alpha)) = \frac{Z_1(\mathbf{K}_\bullet(f\circ\alpha))}{B_1(\mathbf{K}_\bullet(f\circ\alpha))} \to \frac{Z_1\big(\alpha^*(\mathbf{L}_\bullet(f))\big)}{B_1\big(\alpha^*(\mathbf{L}_\bullet(f))\big)} = \mathrm{Tor}_1(\frac{\mathcal{O}_{\mathbf{C}^p}}{J_f}, \mathcal{O}_{\mathbf{C}^n}),$$

induced by $t\alpha$ in the long exact sequence $S(\alpha, F)$.

Theorem 6.3. *If (16) holds then the morphism α is coherent.*

Proof Let A denote the graph-embedding of α: $A(y) = (y, \alpha(y))$. We can recover α from A by composition with the projection $\pi_2 : Y_1 \times Y_2 \to Y_2$. As projection gives rise to a coherent morphism, and the composite of coherent morphism is coherent, it will be enough to prove the result for the immersion A, provided, that is, that we can show that if (16) holds for α then it holds for A. That is, we have first to show that if

$$t\alpha(\mathrm{Der}(-\log f\circ\alpha)) \subset \alpha^*(\mathrm{Der}(-\log f))$$

then

$$tA(\mathrm{Der}(-\log f\circ\pi_2\circ A)) \subset A^*(\mathrm{Der}(-\log f\circ\pi_2)).$$

In fact this is straightforward: $\mathrm{Der}(-\log f\circ\pi) = (\mathrm{Der}(-\log f)\otimes\mathcal{O}_{Y_1\times Y_2})\oplus\theta_{Y_1\times Y_2|Y_2}$, while $f\circ\pi_2\circ A = f\circ\alpha$, so

$$tA(\mathrm{Der}(-\log f\circ\pi_2\circ A)) = tA(\mathrm{Der}(-\log f\circ\alpha)) \subseteq t\alpha(\mathrm{Der}(-\log f\circ\alpha))\oplus\theta(A|Y_2)$$

$$\subseteq \alpha^*(\mathrm{Der}(-\log f))\oplus\theta(A|Y_2) = A^*(\mathrm{Der}(-\log f\circ\pi_2)).$$

Now we prove the theorem under the assumption that α is an immersion. By this assumption, after a diffeomorphism we can write $Y_2 = Y_1\times N$ for some smooth manifold N, and take $\alpha : Y_1 \to Y_2$ to be the inclusion of Y_1 as $Y_1\times\{0\}\subset Y_1\times N$. To ease our notation, we rename Y_1 simply Y. The operator $t\alpha$ converts a vector field on Y to "the same" vector field on $Y\times\{0\}\subset Y\times N$, and can thus be suppressed in our notation.

Let $\mathscr{F}_i : B \times X \to Y$, for $i = 1, 2$, be $\mathcal{K}_{f\circ\alpha\,\mathrm{un}}$-equivalent deformations of $F : X \to Y$, and let Φ be a $\mathcal{K}_{f\circ\alpha\, un}$-equivalence transforming the graph of $\mathscr{F}_1$ to that of $\mathscr{F}_2$. It has the form

$$\Phi(u, x, y) = (u, \phi_X(u, x), \phi_Y(u, x, y))$$

and satisfies $f \circ \alpha(\phi_Y(u, x, y)) = f \circ \alpha(y)$, $\Phi(0, x, y) = (0, x, y)$. We have to extend Φ to a diffeomorphism $\tilde{\Phi}$ of $B \times X \times Y_1 \times N$, $\tilde{\Phi}(u, x, y, n) = (u, \tilde{\Phi}_X(u, x), \tilde{\Phi}_Y(u, x, y, n), \tilde{\Phi}_N(u, x, y, n))$, such that $f(\tilde{\Phi}_Y(u, x, y, n), \tilde{\Phi}_N(u, x, y, n)) = f(y, u)$. Any such extension will transform the graph of $\alpha \circ \mathscr{F}_1$ to that of $\alpha \circ \mathscr{F}_2$, since both graphs are contained in $B \times X \times Y \times \{0\}$.

Suppose first that $\dim B = 1$, and let t be a coordinate on B. Since Φ is an unfolding of the identity on $X \times Y$, $\Phi(t, x, y)$ is the image of $(0, x, y)$ under the integral flow of a time-dependent vector field

$$\frac{\partial}{\partial t} + \sum_i \xi_i(t, x)\frac{\partial}{\partial x_i} + \sum_j \eta_j(t, x, y)\frac{\partial}{\partial y_j}.$$

The vector field $\eta := \sum_j \eta_j \frac{\partial}{\partial y_j}$ lies in $\mathrm{Der}(-\log f \circ \alpha) \otimes \mathcal{O}_{B\times X\times Y}$, since the value of $f \circ \alpha$ is preserved along the flow lines. The hypothesis, (16), implies that η extends to a vector field $\tilde{\eta} \in \mathrm{Der}(-\log f) \otimes \mathcal{O}_{B\times X\times Y\times N}$. The map $\tilde{\Phi}$ sending $(0, x, y, n)$ to its image under the time t integral flow of the extended vector field

$$\frac{\partial}{\partial t} + \chi(u, x, y) + \tilde{\eta}(u, x, y, n)$$

is a $\mathcal{K}_{f\,\mathrm{un}}$-equivalence extending Φ.

To deal with the general case where B is smooth of dimension d, we identify B with $\mathbf{C}^d$ and let $B_1 = \mathbf{C} \times \{0\}, B_2 = \mathbf{C}^2 \times \{0\}, \ldots, B_d = B$. Suppose inductively that the diffeomorphism $\Phi_j := \Phi_{|B_{j-1}\times X\times Y}$ has been extended to a diffeomorphism $\tilde{\Phi}_{j-1}$ of $B_{j-1} \times X \times Y \times N$ which preserves the value of f as required. Let $t_1, \ldots, t_j$ be coordinates on B_j. Once again, $\Phi((t_1, \ldots, t_j, 0, \ldots, 0)), x, y)$ is the image of $\Phi((t_1, \ldots, t_{j-1}, 0, \ldots, 0), x, y)$ under the integral flow of a t_j- dependent vector field

$$\frac{\partial}{\partial t_j} + \xi(t_1, \ldots, t_j, x, y) + \eta(t_1, \ldots, t_j, x, y)$$

with $\eta \in \mathrm{Der}(-\log f \circ \alpha) \otimes \mathcal{O}_{B_j\times X\times Y}$. By (16), η extends to $\tilde{\eta} \in \mathrm{Der}(-\log f) \otimes \mathcal{O}_{B_j\times X\times Y\times N}$, and now we extend Φ_j to $\tilde{\Phi}_j$ of $B_j \times X \times Y \times N$ by the now standard procedure. This completes the proof. $\square$

The converse to 6.3 does not hold. For example, (16) does not always hold for the morphism $\alpha_0 : (g, \mathrm{id}_{\mathbf{C}^m}) \to (f, F)$, but α_0 is always coherent.

It is useful to keep in mind the diagram

$$\begin{tikzcd}
 & \ker\{\alpha^*(tf) : \theta(\alpha) \to \theta(f \circ \alpha)\} & \\
t\alpha(\mathrm{Der}(-\log f \circ \alpha)) \arrow[ur] \arrow[rr, dashed] & & \alpha^*(\mathrm{Der}(-\log f)) \arrow[ul] \\
t\alpha(\mathrm{Ham}(f \circ \alpha)) \arrow[u] \arrow[rr] & & \alpha^*(\mathrm{Ham}(f)) \arrow[u]
\end{tikzcd} ,$$

where $\mathrm{Ham}(f)$ denotes the module generated by the Hamiltonian vector fields $\eta_{ij}(f)$. The inclusion (16) is the broken arrow.

Two cases in which, in view of 6.2, (16) obviously holds, are

1. where $\mathrm{Tor}_1^{\mathcal{O}_{\mathbf{C}^p}}(\mathcal{O}_{\mathbf{C}^p}/J_f, \mathcal{O}_{\mathbf{C}^n}) = 0$, in which case the right upper arrow is an equality, and

2. where $f \circ \alpha$ has isolated singularity, in which case the left lower arrow is an equality.

The first of these two occurs, of course, if α is flat. So there are two subcategories of $\mathscr{C}_g$ in which all morphisms are coherent:

1. The subcategory whose objects are those of $\mathscr{C}_g$, and whose morphisms are the flat morphisms of $\mathscr{C}_g$, and

2. The full subcategory of $\mathscr{C}_g$ consisting of factorisations (f, F) where f has isolated singularity.

Since (2) excludes, for example, determinantal factorisations, it is rather restrictive. More interesting than (1) is the subcategory whose objects are *Cohen-Macaulay factorisations* (f, F), that is, where $\mathcal{O}_{\mathbf{C}^n}/J_f$ is Cohen-Macaulay, and whose morphisms are those morphisms α of $\mathscr{C}_g$ such that the codimension of $V(\alpha^*(J_f))$ in $\mathbf{C}^n$ is equal to the codimension of J_f in $\mathbf{C}^p$. For such morphisms, $\alpha^*(\mathbf{L}_\bullet(f))$ is acyclic, condition (1) above holds, and thus α is coherent.

Given a factorisation (f, F) of g, call $\operatorname{codim} V(J_f) - \operatorname{codim} V(F^*(J_f))$ the *dimensional defect* of (f, F) and denote it by $\mathrm{defect}(f, F)$. The dimensional defect plays an important role in the relation bewteen $\tau := \dim_{\mathbf{C}} T^1\mathcal{K}_f F$ and the rank of the vanishing homology of $g = f \circ F$ under perturbations of F — see [6].

We will say a morphism $\alpha : (f_1, F_1) \to (f_2, F_2)$ has *finite length* if $T^1(f, \alpha)$ has finite length.

Lemma 6.4. *Suppose that α has finite length. Then a necessary and sufficient condition for equality of $\operatorname{codim} V(\alpha^*(J_f))$ and $\operatorname{codim} V(J_f)$ is that $\mathrm{defect}(f, \alpha \circ F) = \mathrm{defect}(f \circ \alpha, F)$.*

Proof By exactness of $S(f, \alpha)$, the finite length of $T^1(f, \alpha)$ implies that $\operatorname{codim} V(J_{f\circ\alpha}) = \operatorname{codim} V(\alpha^*(J_f))$. So

$$\begin{aligned} \mathrm{defect}(f, \alpha \circ F) = \mathrm{defect}(f \circ \alpha, F) \quad &\Leftrightarrow \quad \operatorname{codim} V(J_f) = \operatorname{codim} V(J_{f\circ\alpha}) \\ &\Leftrightarrow \quad \operatorname{codim} V(J_f) = \operatorname{codim} V(\alpha^*(J_f)). \end{aligned}$$

□

For each integer k, let $\mathcal{C}_g(k)$ denote the full subcategory of $\mathcal{C}_g$ consisting of factorisations with defect k.

Theorem 6.5. *Let (f_1, F_1) and (f_2, F_2) be Cohen-Macaulay factorisations in $\mathscr{C}_g(k)$, and let $\alpha : (f_1, F_1) \to (f_2, F_2)$ be a morphism of finite length. Then α is coherent.*

Proof Let $\mathbf{L}_\bullet(f)$ and $\mathbf{L}_\bullet(f \circ \alpha)$ be free resolutions of $\mathcal{O}_{\mathbf{C}^p}/J_f$ and $\mathcal{O}_{\mathbf{C}^n}/J_{f\circ\alpha}$, over $\mathcal{O}_{\mathbf{C}^p}$ and $\mathcal{O}_{\mathbf{C}^n}$ respectively. As $\mathcal{O}_{\mathbf{C}^p}/J_f$ is Cohen Macaulay, and, by the lemma, $\operatorname{codim} V(\alpha^*(J_f)) = \operatorname{codim} V(J_f)$, it follows that $\alpha^*(\mathbf{L}_\bullet(f))$ is acyclic. This means in particular that $\mathrm{Tor}_1^{\mathcal{O}_{\mathbf{C}^p}}(\mathcal{O}_{\mathbf{C}^p}/J_f, \mathcal{O}_{\mathbf{C}^n}) = 0$. □

7 Symmetric determinantal factorisations revisited

A symmetric matrix family $S : (\mathbf{C}^m, 0) \to \mathrm{Symm}_n(\mathbf{C})$ can also be viewed as a (unrestricted) matrix family. Thus there is a morphism of factorisations

$$\begin{array}{ccccc} & & \mathbf{C}^m & & \\ & S_s \swarrow & & \searrow S_g & \\ \mathrm{Symm}_n(\mathbf{C}) & & \stackrel{i}{\hookrightarrow} & & \mathrm{Mat}_n(\mathbf{C}) \\ & \det_s \searrow & & \swarrow \det_g & \\ & & \mathbf{C} & & \end{array} \tag{19}$$

of $\det S$ (we have added subscripts to S and to det in order to indicate their target and source). This morphism is coherent, *although* $i^*(\mathbf{L}_\bullet(\det_g))$ *is not acyclic.* We take, as $\mathbf{L}_\bullet(\det)$ and $\mathbf{L}_\bullet(\det_s)$, the well known Gulliksen-Negård and Jozefiak resolutions $\mathbf{J}_\bullet$ and $\mathbf{GN}_\bullet$ (see [8] and [9] respectively), which are shown below. It turns out that there is a split morphism of complexes $p_\bullet : \mathbf{J}_\bullet \to i^*(\mathbf{GN}_\bullet)$; in particular (16) holds, and coherence follows by Theorem 6.3. The splitting has the rather remarkable form

$$0 \to \mathbf{J}_\bullet \xrightarrow{p_\bullet} i^*(\mathbf{GN}_\bullet) \xrightarrow{r_\bullet} \mathrm{Hom}(\mathbf{J}_{4-\bullet}, \mathcal{O}) \to 0; \tag{20}$$

in some sense it explains the fact that $\mathbf{J}_1$ = symmetric matrices over $\mathcal{O} = \mathcal{O}_{\mathrm{Sym}_n(\mathbf{C})}$ and $\mathbf{J}_3$ = skew-symmetric matrices over $\mathcal{O}$ are complements in $\mathrm{mat}_n(\mathcal{O})$. The split short exact sequence of complexes (20) is shown vertically in the following diagram.

$$\begin{array}{ccccccccccccc} & & 0 & \to & \mathrm{sk}_n(\mathcal{O}) & \xrightarrow{d_3^s} & \mathrm{sl}_n(\mathcal{O}) & \xrightarrow{d_2^s} & \mathrm{sym}_n(\mathcal{O}) & \xrightarrow{d_1^s} & \mathcal{O} & \to & 0 \\ & & & & p_3 \downarrow & & p_2 \downarrow & & p_1 \downarrow & & p_0 \downarrow & & \\ 0 & \to & \mathcal{O} & \xrightarrow{d_4^g} & \mathrm{mat}_n(\mathcal{O}) & \xrightarrow{d_3^g} & \mathrm{sl}_n(\mathcal{O}) \oplus \mathrm{sl}_n(\mathcal{O}) & \xrightarrow{d_2^g} & \mathrm{mat}_n(\mathcal{O}) & \xrightarrow{d_1^g} & \mathcal{O} & \to & 0 \\ & & & & r_3 \downarrow & & r_2 \downarrow & & r_1 \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O} & \xrightarrow{(d_1^s)^t} & \mathrm{sym}_n(\mathcal{O}) & \xrightarrow{(d_2^s)^t} & \mathrm{sl}_n(\mathcal{O}) & \xrightarrow{(d_3^s)^t} & \mathrm{sk}_n(\mathcal{O}) & \to & 0 & & \end{array} \tag{21}$$

where

p_0 is the identity	p_1 is inclusion	$p_2(Y) = (Y, -Y^t)$	p_3 is inclusion
$r_1(X) = \frac{1}{2}(X - X^t)$	$r_2(Y_1, Y_2) = Y_1 + Y_2^t$	$r_3(Z) = \frac{1}{2}(Z + Z^t)$	$r_4 = \mathrm{id}$

The morphism $p_\bullet$ exists despite the fact that $i^*(\mathbf{GN}_\bullet)$ is not acyclic. The diagram above shows that $H_1(i^*(GN_\bullet)) \simeq \mathrm{Ext}^3(\mathcal{O}/J_{\det}, \mathcal{O})$. The fact that $p_1 = ti$ lifts to a morphism $J_2 \to i^*(GN_2)$ is equivalent to the image of p_1 not meeting the homology of $i^*(\mathbf{GN}_\bullet)$.

It is of course well-known that $\mathcal{O}_{\mathrm{Mat}_n(\mathbf{C})}/J_{\det}$ is Gorenstein and that $\mathbf{GN}_\bullet$ is self-dual. Does this, together with the $\mathbf{Z}_2$-action generated by transposition, of which $\mathrm{Symm}_n(\mathbf{C})$ is the fixed-point set, explain the remarkable split extension (20)?

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Personal note: Jairo Charris was the first person I spoke to when I went to enquire about doing a Master's degree in the National University in 1975. He opened his office door as I walked nervously along the corridor, dripping from a recent downpour, and asked me, in English, if he could help. He could, and did. Enormously. He invited me into mathematics, and taught me that it was worth doing and that I could do it. His combination of informality, friendliness and passion for mathematics was incredibly attractive. To him, I owe my career.