

Linear free divisors and Frobenius manifolds

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Abstract

We study linear functions on fibrations whose central fibre is a linear free divisor. We analyse the Gauß-Manin system associated to these functions, and prove the existence of a primitive and homogenous form. As a consequence, we show that the base space of the semi-universal unfolding of such a function carries a natural Frobenius manifold structure.

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1 Introduction

In this paper we study Frobenius manifolds arising as deformation spaces of linear functions on certain non-isolated singularities, the so-called linear free divisors. It is a nowadays classical result that the semi-universal unfolding space of an isolated hypersurface singularity can be equipped with a Frobenius structure. One of the main motivations to study Frobenius manifolds comes from the fact that they also arise in a very different area: the total cohomology space of a projective manifold carries such a structure, defined by the quantum multiplication. Mirror symmetry postulates an equivalence between these two types of Frobenius structures. In order to carry this program out, one is forced to study not only local singularities (which are in fact never the mirror of a quantum cohomology ring) but polynomial functions on affine manifolds. It has been shown in [DS03] (and later, with a somewhat different strategy in [Dou05]) that given a convenient and non-degenerate Laurent polynomial $\tilde{f} : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$, the base space M of a semi-universal unfolding $\tilde{F} : (\mathbb{C}^*)^n \times M \rightarrow \mathbb{C}$ can be equipped with a (canonical) Frobenius structure. An important example is the function $\tilde{f} = x_1 + \dots + x_{n-1} + \frac{t}{x_1 \dots x_{n-1}}$ for some fixed $t \in \mathbb{C}^*$: the Frobenius structure obtained on its unfolding space is known (see [Giv95], [Giv98] and [Bar00]) to be isomorphic to the full quantum cohomology of the projective space \mathbb{P}^{n-1} . More generally, one can consider the Laurent polynomial $\tilde{f} = x_1 + \dots + x_{n-1} + \frac{t}{x_1^{w_1} \dots x_{n-1}^{w_{n-1}}}$ for some weights $(w_1, \dots, w_{n-1}) \in \mathbb{N}^{n-1}$, here the Frobenius structure corresponds to the (orbifold) quantum cohomology of the weighted projective space $\mathbb{P}(1, w_1, \dots, w_{n-1})$ (see [Man08] and [CCLT06]). A detailed analysis on how to construct the Frobenius structure

40 for these functions is given in [DS04], some of the techniques in this paper are similar to those used here. Notice
 41 that the mirror of the ordinary projective space might be interpreted in a slightly different way, namely, as the
 42 restriction of the linear polynomial $f = x_1 + \dots + x_n : \mathbb{C}^n \rightarrow \mathbb{C}$ to the non-singular fibre $h(x_1, \dots, x_n) - t = 0$
 43 of the torus fibration defined by the homogeneous polynomial $h = x_1 \cdot \dots \cdot x_n$.

44 In the present work, we construct Frobenius structures on the unfolding spaces of a class of functions generalizing
 45 this basic example, namely, we consider homogenous functions h such that its zero fibre is a *linear free divisor*.
 46 Linear free divisor have been recently introduced by R.-O. Buchweitz and the second author in [BM06] (see
 47 also [GMNS06]), but are closely related to the more classical *prehomogeneous spaces* of T. Kimura and M. Sato
 48 ([SK77]). They are defined as free divisors $D = h^{-1}(0)$ in some vector space V whose sheaf of derivations can be
 49 generated by vector fields having only linear coefficients. The classical example is of course the normal crossing
 50 divisor. Following the analogy with the mirror of \mathbb{P}^n , we are interested in characterising when a linear function
 51 f has isolated singularities on the Milnor fibre $D_t = h^{-1}(t)$, $t \neq 0$. As it turns out, not all linear free divisors
 52 support such functions, but the large class of *reductive* ones do, and for these the set of linear functions having
 53 only isolated singularities can be characterised as the complement of the *dual divisor*.

54 Let us give a short overview on the paper. In section 2 we state and prove some general results on linear free
 55 divisors. In particular, we introduce the notion of *special linear free divisors*, and show that reductive ones are
 56 always special. This is proved by studying the relative logarithmic de Rham complex (subsection 2.2) which is
 57 also important in the later discussion of the Gauß-Manin-system. The cohomology of this complex is computed
 58 in reductive case, thanks to a classical theorem of Hochschild and Serre.

59 Section 3 discusses linear functions f on linear free divisors D , as well as on their Milnor fibres D_t . We show (in
 60 an even more general situation where D is not a linear free divisor) that $f|_{D_t}$ is a Morse function if the restriction
 61 $f|_D$ is right-left stable. This implies in particular that the Frobenius structures associated to the functions $f|_{D_t}$
 62 are all semi-simple. Subsection 3.2 discusses deformation problems associated to the two functions (f, h) .
 63 In particular, we show that linear forms in the complement of the dual divisor have the necessary finiteness
 64 properties. In order to exhibit Frobenius structures, the fibration defined by $f|_{D_t}$ is required to have a good
 65 behaviour at infinity, comprised in the notion of *tameness*. In subsection 3.3 it is shown that this property
 66 indeed hold for these functions.

67 In section 4 we study the (algebraic) Gauß-Manin system and the (algebraic) Brieskorn lattice of $f|_{D_t}$. We
 68 actually define both as a family over the parameter space of h , and using logarithmic forms along D (more
 69 precisely, the relative logarithmic de Rham complex mentioned above) we get very specific extensions of these
 70 families over D . The fact that D is a linear free divisor allows us to construct explicitly a basis of this family of
 71 Brieskorn lattice, hence showing its freeness. Next we give an explicit solution to the so called Birkhoff problem.
 72 Although this solution is not a good basis in the sense of M. Saito [Sai89], that is, it might not compute the
 73 spectrum at infinity of $f|_{D_t}$, we give an algorithmic procedure to turn it into one. This allows us in particular to
 74 compute the monodromy of $f|_{D_t}$. We finish this section by showing that this solution to the Birkhoff problem is
 75 also compatible with a natural pairing defined on the Brieskorn lattice, at least under an additional hypothesis
 76 (which is satisfied in many examples) on the spectral numbers.

77 In section 5 we finally apply all these results to construct Frobenius structures on the unfolding spaces of the
 78 functions $f|_{D_t}$ (subsection 5.1) and on $f|_D$ (subsection 5.2). Whereas the former exists in all cases, the latter
 79 depends on a conjecture concerning a natural pairing on the Gauss-Manin-system. Similarly, assuming this
 80 conjecture, we give some partial results concerning *logarithmic Frobenius structures* as defined in [Rei08] in
 81 subsection 5.3.

82 We end the paper with some examples (section 6). On the one hand, they illustrate the different phenomena
 83 that can occur, as for instance, the fact that there might not be a *canonical* choice (as in [DS03]) of a primitive
 84 form. On the other hand, they support the conjecture concerning the pairing used in the discussion of the limit
 85 Frobenius structure.

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89 2 Reductive and special linear free divisors

90 2.1 Definition and examples

91 A hypersurface D in a complex manifold X is a *free divisor* if the \mathcal{O}_X -module $\text{Der}(-\log D)$ is locally free. If
 92 $X = \mathbb{C}^n$ then D is furthermore a *linear free divisor* if $\text{Der}(-\log D)$ has an $\mathcal{O}_{\mathbb{C}^n}$ -basis consisting of weight-zero

93 vector fields – vector fields whose coefficients, with respect to a standard linear coordinate system, are linear
94 functions (see [GMNS06, Section 1]). By Serre’s conjecture, if $D \subset \mathbb{C}^n$ is a free divisor then $\text{Der}(-\log D)$ is
95 globally free. If $D \subset \mathbb{C}^n$ is a linear free divisor then the group $G_D := \{A \in \text{Gl}_n(\mathbb{C}) : AD = D\}$ of its linear
96 automorphisms is algebraic of dimension n . We denote by G_D^0 the connected component of G_D containing
97 the identity, and by Sl_D the intersection of G_D^0 with $\text{Sl}_n(\mathbb{C})$. The infinitesimal action of the Lie algebra
98 \mathfrak{g}_D of G_D^0 generates $\text{Der}(-\log D)$ over $\mathcal{O}_{\mathbb{C}^n}$, and it follows that the complement of D is a single G_D^0 -orbit
99 ([GMNS06, Section 2]). G_D^0 is a *prehomogeneous vector space* (cf [SK77]). Thus, \mathbb{C}^n , with this action of G_D^0 ,
100 is a *prehomogeneous vector space* (cf [SK77]) i.e. a representation ρ of a group G on a vector space V in which
101 the group has an open orbit. The complement of the open orbit in a prehomogeneous vector space is known as
102 the *discriminant*. The (reduced) discriminant in a prehomogeneous vector space is a linear free divisor if and
103 only if the dimensions of G and V and the degree of the discriminant are all equal.

104 By Saito’s criterion ([Sai80]), the determinant of the matrix of coefficients of a set of generators of $\text{Der}(-\log D)$
105 is a reduced equation for D , which is therefore homogeneous of degree n . Throughout the paper we will denote
106 the reduced homogeneous equation of the linear free divisor D by h .

107 If the group G acts on the vector space V , then a rational function $f \in \mathbb{C}(V)$ is a *semi-invariant* (or *relative*
108 *invariant*) if there is a character $\chi_f : G \rightarrow \mathbb{C}^*$ such that for all $g \in G$, $f \circ g = \chi_f(g)f$. In this case χ_f
109 is the *character associated to f* . Sato and Kimura prove ([SK77, §4 Lemma 4]) that semi-invariants with
110 multiplicatively independent associated characters are algebraically independent. If D is a linear free divisor
111 with equation h , then h is a semi-invariant ([SK77, §4]) (for the action of G_D^0). For it is clear that g must leave
112 D invariant, and thus $h \circ g$ is some complex multiple of h . This multiple is easily seen to define a character,
113 which we call χ_h .

114 **Definition 2.1.** We call the linear free divisor D *special* if χ_h is equal to the determinant of the representation,
115 and *reductive* if the group G_D^0 is reductive.

116 We show in 2.9 below that every reductive linear free divisor is special. We do not know if the converse holds.
117 The term “special” is used here because the condition means that the elements of G_D which fix h lie in $\text{Sl}_n(\mathbb{C})$.
118 Not all linear free divisors are special. Consider the example of the group B_k of upper triangular complex
119 matrices acting on the space $V = \text{Sym}_k(\mathbb{C})$ of symmetric $k \times k$ matrices by transpose conjugation,

$$B \cdot S = {}^t B S B \tag{2.1}$$

120 The discriminant here is a linear free divisor ([GMNS06, Example 5.1]). Its equation is the product of the
121 determinants of the top left-hand $l \times l$ submatrices of the generic $k \times k$ symmetric matrix, for $l = 1, \dots, k$. It
122 follows that if $B = \text{diag}(\lambda_1, \dots, \lambda_k) \in B_k$ then

$$h \circ \rho(B) = \lambda_1^{2k} \lambda_2^{2k-2} \dots \lambda_k^2 h,$$

123 and D is not special. The simplest example is the case $k = 2$, here the divisor has the equation

$$h = x(xz - y^2) \tag{2.2}$$

124 Irreducible prehomogeneous vector spaces are classified in [SK77]. However, irreducible representations account
125 for very few of the linear free divisors known. For more examples we turn to the representation spaces of quivers:

126 **Proposition 2.2** ([BM06]). (i) Let Q be a quiver without oriented loops and let \mathbf{d} be (a dimension vector
127 which is) a real Schur root of Q . Then the triple $(\text{Gl}_{Q,\mathbf{d}}, \rho, \text{Rep}(Q, \mathbf{d}))$ is a prehomogeneous vector space
128 and the complement of the open orbit is a divisor D (the “discriminant” of the representation ρ of the
129 quiver group $\text{Gl}_{Q,\mathbf{d}}$ on the representation space $\text{Rep}(Q, \mathbf{d})$).

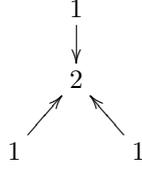
130 (ii) If in each irreducible component of D there is an open orbit, then D is a linear free divisor.

131 (iii) If Q is a Dynkin quiver then the condition of (ii) holds for all real Schur roots \mathbf{d} .

132 We note that the normal crossing divisor appears as the discriminant in the representation space $\text{Rep}(Q, \mathbf{1})$ for
133 every quiver Q whose underlying graph is a tree. Here $\mathbf{1}$ is the dimension vector which takes the value 1 at
134 every node.

135 All of the linear free divisors constructed in Proposition 2.2 are reductive. For if D is the discriminant in
136 $\text{Rep}(Q, \mathbf{d})$ then G_D^0 is the quotient of $\text{Gl}_{Q,\mathbf{d}} = \prod_i \text{Gl}_{d_i}(\mathbb{C})$ by a 1-dimensional central subgroup.

Example 2.3. (i) Consider the quiver of type D_4 with real Schur root

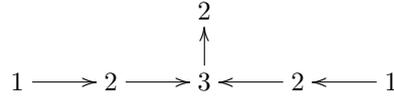


137 By choosing a basis for each vector space we can identify the representation space $\text{Rep}(Q, \mathbf{d})$ with the
 138 space of 2×3 matrices, with each of the three morphisms corresponding to a column. The open orbit in
 139 $\text{Rep}(Q, \mathbf{d})$ consists of matrices whose columns are pairwise linearly independent. The discriminant thus
 140 has equation

$$h = (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{12}a_{23} - a_{22}a_{13}). \quad (2.3)$$

141 This example generalises: instead of three arrows converging to the central node, we take n , and set the
 142 dimension of the space at the central node to $n - 1$. The representation space can now be identified with
 143 the space of $(n - 1) \times n$ matrices, and the discriminant is once again defined by the vanishing of the
 144 product of maximal minors. Again it is a linear free divisor ([GMNS06, Example 5.3]), even though for
 145 $n > 3$ the quiver is no longer a Dynkin quiver. We refer to it as the *star quiver*, and denote it by \star_n .

(ii) The linear free divisor arising by the construction of Proposition 2.2 from the quiver of type E_6 with real Schur root



146 has five irreducible components. In the 22-dimensional representation space $\text{Rep}(Q, \mathbf{d})$, we take coordinates
 147 a, b, \dots, v . Then

$$h = F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5 \quad (2.4)$$

where four of the components have the equations

$F_1 = dfpq - cgpq - dfor + cgor + efps - chps + egrs - dhrr - efot + chot - egqt + dhqt$
$F_2 = jlpq - impq - jlor + imor + klps - inps + kmrs - jnrs - klot + inot - kmqt + jnqt$
$F_3 = -aejl - bhjl + adkl + bgkl + aeim + bhim - ackm - bfgm - adin - bgin + acjn + bfj$
$F_4 = egju - dhiu - efju + chju + dfku - cgku + eglv - dhlv - efmv + chmv + dfnv - cgnv$

148 and the fifth has the equation $F_5 = 0$, which is of degree 6, with 48 monomials. This example is discussed
 149 in detail in [BM06, Example 7.3]

2.2 The relative logarithmic de Rham complex

151 Let D be a linear free divisor with equation h . We set $\text{Der}(-\log h) = \{\chi \in \text{Der}(-\log D) : \chi \cdot h = 0\}$. Under
 152 the infinitesimal action of G_D^0 , the Lie algebra of $\ker(\chi_h)$, which we denote by \mathfrak{g}_h , is identified with the weight
 153 zero part of $\text{Der}(-\log h)$, which we denote by $\text{Der}(-\log h)_0$. $\text{Der}(-\log h)$ is a summand of $\text{Der}(-\log D)$, as is
 154 shown by the equality

$$\xi = \frac{\xi \cdot h}{E \cdot h} E + \left(\xi - \frac{\xi \cdot h}{E \cdot h} E \right)$$

155 in which E is the Euler vector field and the second summand on the right is easily seen to annihilate h .
 156 The quotient complex

$$\Omega^\bullet(\log h) := \frac{\Omega^\bullet(\log D)}{dh/h \wedge \Omega^{\bullet-1}(\log D)} = \frac{\Omega^\bullet(\log D)}{h^* (\Omega_{\mathbb{C}}^1(\log\{0\})) \wedge \Omega^{\bullet-1}(\log D)}$$

157 is the ‘‘relative logarithmic complex’’ associated with the function $h : \mathbb{C}^n \rightarrow \mathbb{C}$. Each module $\Omega^k(\log h)$ is
 158 isomorphic to the submodule

$$\Omega^k(\log h)' := \{\omega \in \Omega^k(\log D) : \iota_E \omega = 0\} \subset \Omega^k(\log D).$$

159 For the natural map $i : \omega \mapsto \omega + (dh/h) \wedge \Omega^{\bullet-1}(\log D)$ gives an injection $\Omega^k(\log h)' \rightarrow \Omega^k(\log h)$, since for $\omega \in$
 160 $\Omega^k(\log h)'$, if $\omega = (dh/h) \wedge \omega_1$ for some ω_1 then

$$0 = \iota_E(\omega) = \iota_E\left(\frac{dh}{h} \wedge \omega_1\right) = n\omega_1 - \frac{dh}{h} \wedge \iota_E(\omega_1)$$

161 and thus $\omega_1 = (dh/nh) \wedge \iota_E(\omega_1)$ and $\omega = (dh/h) \wedge (dh/nh) \wedge \iota_E(\omega_1) = 0$. Because

$$\frac{1}{n} \iota_E\left(\frac{dh}{h} \wedge \omega\right) \in \Omega^k(\log h)'$$

162 and

$$\omega - \frac{1}{n} \iota_E\left(\frac{dh}{h} \wedge \omega\right) \in \frac{dh}{h} \wedge \Omega^{k-1}(\log D) \quad (2.5)$$

163 i is surjective. However, the collection of $\Omega^k(\log h)'$ is not a subcomplex of $\Omega^\bullet(\log D)$: $\iota_E(d\omega)$ may not be
 164 zero even when $\iota_E(\omega) = 0$. We define $d' : \Omega^k(\log h)' \rightarrow \Omega^{k+1}(\log h)'$ by composing the usual exterior derivative
 165 $\Omega^k(\log h)' \rightarrow \Omega^{k+1}(\log D)$ with the projection operator $P : \Omega^\bullet(\log D) \rightarrow \Omega^\bullet(\log h)'$ defined by

$$P(\omega) = \frac{1}{n} \iota_E\left(\frac{dh}{h} \wedge \omega\right) = \omega - \frac{1}{n} \frac{dh}{h} \wedge \iota_E(\omega). \quad (2.6)$$

166 **Lemma 2.4.** (i) $d \circ i = i \circ d'$.

167 (ii) $(d')^2 = 0$.

168 (iii) $i : (\Omega^\bullet(\log h)', d') \rightarrow (\Omega^\bullet(\log h), d)$ is an isomorphism of complexes.

169 *Proof.* The first statement is an obvious consequence of the second equality in (2.6). The second follows because
 170 $d^2 = 0$ and i is an injection. The third is a consequence of (i) and (ii). \square

171 **Lemma 2.5.** The weight zero part of $(\Omega^\bullet(\log h)', d')$ is a subcomplex of $(\Omega^\bullet(\log D), d)$.

172 *Proof.* Let $\omega \in \Omega^k(\log h)'$. We have

$$P_{k+1}(d\omega) = \frac{1}{n} \iota_E\left(\frac{dh}{h} \wedge d\omega\right) = d\omega - \frac{1}{n} \frac{dh}{h} \wedge \iota_E(d\omega) = d\omega - \frac{1}{n} \frac{dh}{h} \wedge (L_E(\omega) - d\iota_E(\omega))$$

173 where L_E is the Lie derivative with respect to E . By assumption, $\iota_E(\omega) = 0$, and since $L_E(\sigma) = \text{weight}(\sigma)\sigma$
 174 for any homogeneous form, it follows that if $\text{weight}(\omega) = 0$ then $d'\omega = d\omega$. \square

175 Let

$$\alpha = \iota_E\left(\frac{dx_1 \wedge \cdots \wedge dx_n}{h}\right) \quad (2.7)$$

176 Evidently $\alpha \in \Omega^{n-1}(\log h)'$, and moreover

$$\alpha = n \frac{dx_1 \wedge \cdots \wedge dx_n}{dh}.$$

177 For $\xi \in \text{Der}(-\log h)$, we define the form $\lambda_\xi = \iota_\xi \alpha$. Notice that α generates the rank one $\mathbb{C}[V]$ -module
 178 $\Omega^{n-1}(\log h)$: $\alpha \wedge dh/h$ is a constant multiple of $dx_1 \wedge \cdots \wedge dx_n/h$, which is a generator of $\Omega^n(\log D)$ (remember
 179 that dh/h is the element of $\Omega^1(\log D)$ dual to $E \in \text{Der}(-\log D)$).

180 **Lemma 2.6.** The linear free divisor $D \subset \mathbb{C}^n$ is special if and only if $d\lambda_\xi = 0$ for all $\xi \in \text{Der}(-\log h)_0$.

181 *Proof.* Let $\xi \in \text{Der}(-\log h)_0$ and let $\lambda_\xi = \iota_\xi \alpha = \iota_\xi \iota_E \text{vol}$. Since α generates $\Omega^{n-1}(\log h)$ and λ_ξ has weight
 182 zero, $d'\lambda_\xi = c\alpha$ for some scalar c . By the previous lemma, the same is true for $d\lambda_\xi$. Since $dh \wedge \alpha = \text{vol}$, it
 183 follows that $dh \wedge d\lambda_\xi = c \text{vol}$. Now $dh \wedge d\lambda_\xi = -d(dh \wedge \lambda_\xi) = d\iota_\xi(\text{vol}) = L_\xi(\text{vol})$. An easy calculation shows that
 184 $L_\xi(\text{vol}) = \text{trace}(A)\text{vol}$, where A is the $n \times n$ matrix such that $A \cdot x = \xi(x)$. Hence

$$d\lambda_\xi = 0 \Leftrightarrow \text{trace}(A) = 0.$$

185 Thus $d\lambda_\xi = 0$ for all $\xi \in \text{Der}(-\log D)$ if and only if $\text{trace}(A) = 0$ for all matrices $A \in \ker d\chi_h$, i.e. if and only
 186 if $\ker d\chi_h \subseteq \ker d \det$. Since both kernels have codimension 1, the inclusion holds if and only if equality holds,
 187 and this is equivalent to χ_h being a power of \det . On the other hand, regarding G_D^0 as a subgroup of $\text{Gl}_n(\mathbb{C})$,
 188 both \det and χ_h are polynomials of degree n , so they must be equal. \square

189 If D is a linear free divisor with reductive group G_D^0 and reduced homogeneous equation h then by Mather's
190 lemma ([Mat69, lemma 3.1]) the fibre $D_t := h^{-1}(t)$, $t \neq 0$, is a single orbit of the group $\ker(\chi_h)$. It follows
191 that D_t is a finite quotient of $\ker(\chi_h)$ since $\dim(D_t) = \dim(\ker(\chi_h))$ and the action is algebraic. Hence
192 D_t has cohomology isomorphic to $H^*(\ker(\chi_h), \mathbb{C})$. Now $\ker(\chi_h)$ is reductive - its Lie algebra \mathfrak{g}_h has the same
193 semi-simple part as \mathfrak{g}_D , and a centre one dimension smaller than that of \mathfrak{g}_D . Thus, $\ker(\chi_h)$ has a compact $n-1$ -
194 dimensional Lie group K_h as deformation retract. Poincaré duality for K_h implies a duality on the cohomology
195 of $\ker(\chi_h)$, and this duality carries over to $H^*(D_t; \mathbb{C})$. How is this reflected in the cohomology of the complex
196 $\Gamma(V, \Omega^\bullet(\log h))$ of global algebraic logarithmic forms? Notice that evidently $H^0(\Gamma(V, \Omega^\bullet(\log h))) = \mathbb{C}[h]$, since
197 the kernel of d_h consists precisely of functions constant along the fibres of h . It is considerably less obvious that
198 $H^{n-1}(\Gamma(V, \Omega^\bullet(\log h)))$ should be isomorphic to $\mathbb{C}[h]$, for this cohomology group is naturally a quotient, rather
199 than a subspace, of $\mathbb{C}[V]$. We prove it (in Theorem 2.7 below) by showing that thanks to the reductiveness of
200 G_D^0 , it follows from a classical theorem of Hochschild and Serre ([HS53, Theorem 10]) on the cohomology of Lie
201 algebras. From Theorem 2.7 we then deduce that every reductive linear free divisor is special.
202 We write $\Omega_\bullet(\log h)_m$ for the graded part of $\Gamma(V, \Omega^\bullet(\log h))$ of weight m .

203 **Theorem 2.7.** *Let $D \subset \mathbb{C}^n$ be a reductive linear free divisor with homogeneous equation h . There is a natural*
204 *graded isomorphism*

$$H^*(\Omega^\bullet(\log h)_0) \otimes_{\mathbb{C}} \mathbb{C}[h] \rightarrow H^*(\Gamma(V, \Omega^\bullet(\log h))).$$

205 *In particular, $H^*(\Gamma(V, \Omega^\bullet(\log h)))$ is a free $\mathbb{C}[h]$ -module.*

206 *Proof.* The complex $\Omega^\bullet(\log h)_m$ is naturally identified with the complex $\bigwedge^\bullet(\mathfrak{g}_h; \text{Sym}^m(V^\vee))$ whose cohomology
207 is the Lie algebra cohomology of \mathfrak{g}_h with coefficients in the representation $\text{Sym}^m(V^\vee)$, $H^*(\mathfrak{g}_h; \text{Sym}^m(V^\vee))$. For
208 (as vector spaces)

$$\Omega^k(\log h)_m = \Omega^k(\log h)_0 \otimes_{\mathbb{C}} \text{Sym}^m(V^\vee) = \left(\bigwedge^k \mathfrak{g}_h^\vee \right) \otimes_{\mathbb{C}} \text{Sym}^m(V^\vee) = \bigwedge^k (\mathfrak{g}_h^\vee \otimes_{\mathbb{C}} \text{Sym}^m(V^\vee)),$$

209 and inspection of the formulae for the differentials in the two complexes shows that they are the same under
210 this identification¹. The representation of \mathfrak{g}_h in $\text{Sym}^k(V^\vee)$ is semi-simple (completely reducible), since \mathfrak{g}_h
211 is a reductive Lie algebra and every finite dimensional complex representation of a reductive Lie algebra is
212 semisimple. By a classical theorem of Hochschild and Serre ([HS53, Theorem 10]), if M is a semi-simple
213 representation of a finite-dimensional complex reductive Lie algebra \mathfrak{g} , then

$$H^*(\mathfrak{g}; M) = H^*(\mathfrak{g}; M^0),$$

214 where M^0 is the submodule of M on which \mathfrak{g} acts trivially. Evidently we have $H^*(\mathfrak{g}; M^0) = H^*(\mathfrak{g}; \mathbb{C}) \otimes_{\mathbb{C}} M^0$.
215 Now

$$\text{Sym}^m(V^\vee)^0 = \begin{cases} \mathbb{C} \cdot h^\ell & \text{if } m = \ell n \\ 0 & \text{otherwise} \end{cases}$$

by the uniqueness, up to scalar multiple, of the semi-invariant with a given character on a prehomogeneous
vector space, referred to in the proof of 3.8. It follows that

$$\begin{aligned} H^k(\Gamma(V, \Omega^\bullet(\log h))) &= \bigoplus_m H^k(\Omega^\bullet(\log h)_m) = \bigoplus_m H^k(\mathfrak{g}_h; \text{Sym}^m(V^\vee)) \\ &= \bigoplus_\ell H^*(\mathfrak{g}_h; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C} \cdot h^\ell = H^*(\Omega^\bullet(\log h)_0) \otimes_{\mathbb{C}} \mathbb{C}[h]. \end{aligned}$$

216

□

Corollary 2.8. *There is a $\mathbb{C}[h]$ -perfect pairing*

$$\begin{aligned} H^k((\Gamma(V, \Omega^\bullet(\log h))) \times H^{n-k-1}((\Gamma(V, \Omega^\bullet(\log h)))) &\longrightarrow H^{n-1}((\Gamma(V, \Omega^\bullet(\log h)))) \simeq \mathbb{C}[h] \\ ([\omega_1], [\omega_2]) &\longmapsto [\omega_1 \wedge \omega_2]. \end{aligned}$$

¹This identification for the case $m = 0$ was already made in [GMNS06], where it gave a proof of the global logarithmic comparison theorem for reductive linear free divisors.

217 *Proof.* The pairing is evidently well defined. Poincaré Duality on the compact deformation retract K_h of $\ker(\chi_h)$
 218 gives rise to a perfect pairing

$$H^k(D_t) \times H^{n-k-1}(D_t) \rightarrow H^{n-1}(D_t),$$

219 Now

$$H^k(D_t) = H^k(\Gamma(V, \Omega^\bullet(\log h) \otimes_{\mathbb{C}[h]} \mathbb{C}[h]/(h-t)))$$

220 by the affine de Rham theorem, since $\Omega^k(\log h)/(h-t) = \Omega_{D_t}^k$. In view of 2.7, the perfect pairing on $H^*(D_t)$
 221 lifts to a $\mathbb{C}[h]$ -perfect pairing on $H^*(\Gamma(V, \Omega^\bullet(\log h)))$. \square

222 **Corollary 2.9.** *A linear free divisor with reductive group is special.*

223 *Proof.* By what was said before, $H^{n-1}(\ker(\chi_h), \mathbb{C})$ is isomorphic to $H^{n-1}(\Omega^\bullet(\log h)_0)$, so Poincaré duality for
 224 $\ker \chi_h$ implies that the class of α in $H^{n-1}(\Omega^\bullet(\log h)_0)$ is non-zero. Recall from the proof of lemma 2.6 that if
 225 $\lambda = \iota_\xi \alpha = \iota_{\xi \iota_E}(\text{vol}/h)$ with $\xi \in \text{Der}(-\log h)_0$, then $d\lambda = c\alpha$ for some $c \in \mathbb{C}$ in $\Omega^\bullet(\log h)_0$. As the class of α is
 226 non-zero, this forces $d\lambda$ to be zero. The conclusion follows from 2.6. \square

227 3 Functions on Linear Free Divisors and their Milnor Fibrations

228 3.1 Right-left stable functions on divisors

229 Let h and f be homogenous polynomials in n variables, where the degree of h is n . As before, we write
 230 $D = h^{-1}(0)$ and $D_t = h^{-1}(t)$ for $t \neq 0$. However, we do not assume in this subsection that D is a free divisor.
 231 We call $f|_{D_t}$ a *Morse function* if all its critical points are non-degenerate and all its critical values are distinct.

232 **Lemma 3.1.** *$f|_{D_t}$ is a Morse function if and only if $\mathbb{C}[D_t]/J_f$ is generated over \mathbb{C} by the powers of f .*

233 *Proof.* Suppose $f|_{D_t}$ is a Morse function, with critical points p_1, \dots, p_N . Since any quotient of $\mathbb{C}[D_t]$ with finite
 234 support is a product of its localisations, we have

$$\mathbb{C}[D_t]/J_f \simeq \bigoplus_{j=1}^N \mathcal{O}_{D_t, p_j} / J_f = \bigoplus_{j=1}^N \mathbb{C}_{p_j}.$$

235 The image in $\bigoplus_{j=1}^N \mathbb{C}_{p_j}$ of f^k is the vector $(f(p_1)^k, \dots, f(p_N)^k)$. These vectors, for $0 \leq k \leq N-1$, make up
 236 the Vandermonde determinant, which is non-zero because the $f(p_j)$ are pairwise distinct. Hence they span
 237 $\bigoplus_{j=1}^N \mathbb{C}_{p_j}$.

238 Conversely, if $1, f, \dots, f^N$ span $\mathbb{C}[D_t]/J_f$ then the powers of f span each local ring $\mathcal{O}_{D_t, p_j} / J_f$. This implies that
 239 there is an \mathcal{R}_e -versal deformation of the singularity of $f|_{D_t}$ at p_j of the form $F(x, u) = g_u \circ f(x)$. In particular,
 240 the critical point of $f|_{D_t}$ at p_j does not split, and so must be non-degenerate. Now choose a minimal R such
 241 that $1, f, \dots, f^{R-1}$ span $\mathbb{C}[D_t]/J_f$. Since all the critical points are non-degenerate, projection of $\mathbb{C}[D_t]/J_f$ to the
 242 product of its local rings shows that the matrix $M := [f^{k-1}(p_j)]_{1 \leq k \leq R, 1 \leq j \leq N}$ has rank N . But if $f(p_i) = f(p_j)$
 243 for some $i \neq j$ then M has two equal columns. So the critical values of f must be pairwise distinct. \square

244 If (X, x) is a germ of complex variety, an analytic map-germ $f : (X, x) \rightarrow (\mathbb{C}^p, 0)$ is *right-left stable* if every
 245 germ of deformation $F : (X \times \mathbb{C}, (x, 0)) \rightarrow (\mathbb{C}^p \times \mathbb{C}, (0, 0))$ can be trivialised by suitable parametrised families of
 246 bi-analytic diffeomorphisms of source and target. A necessary and sufficient condition for right-left stability is
 247 infinitesimal right-left stability: $df(\theta_{X,0}) + f^{-1}(\theta_{\mathbb{C}^p,0}) = \theta(f)$, where $\theta_{X,0}$ is the space of germs of vector fields
 248 on X and $\theta(f)$ is the space of infinitesimal deformations of f (freely generated over $\mathcal{O}_{X,0}$ by $\partial/\partial y_1, \dots, \partial/\partial y_p$,
 249 where y_1, \dots, y_p are coordinates on \mathbb{C}^p). When $p = 1$, $\theta(f) \simeq \mathcal{O}_{X,0}$ and $f^{-1}(\theta_{\mathbb{C},0}) \simeq \mathbb{C}\{f\}$. Note also that if
 250 $X \subset \mathbb{C}^n$ then $\theta_{X,0}$ is the image of $\text{Der}(-\log X)_0$ under the restriction of $\theta_{\mathbb{C}^n}$ to X .

251 **Proposition 3.2.** *If $f|_D : D \rightarrow \mathbb{C}$ has a right-left stable singularity at 0 then $f|_{D_t}$ is a Morse function, or
 252 non-singular.*

253 *Proof.* $f|_D$ has a stable singularity at 0 if and only if the image in $\mathcal{O}_{D,0}$ of $df(\text{Der}(-\log D)) + \mathbb{C}\{f\}$ is all of
 254 $\mathcal{O}_{D,0}$. Write $\mathfrak{m} := \mathfrak{m}_{\mathbb{C}^n,0}$. Since $df(\chi_E) = f$, stability implies

$$df(\text{Der}(-\log h)) + (f) + (h) = \mathfrak{m}. \quad (3.1)$$

As $\deg(h) > 1$, we deduce that

$$\mathfrak{m}/(df(\text{Der}(-\log h)) + \mathfrak{m}^2) = \langle f \rangle_{\mathbb{C}}.$$

It follows that for all $k \in \mathbb{N}$,

$$(\mathfrak{m}^k + df(\text{Der}(-\log h)))/(df(\text{Der}(-\log h)) + \mathfrak{m}^{k+1}) = \langle f^k \rangle_{\mathbb{C}}$$

and thus that

$$df(\text{Der}(-\log h)) + \mathbb{C}\{f\} = \mathcal{O}_{\mathbb{C}^n,0}. \quad (3.2)$$

Now (3.1) implies that $V(df(\text{Der}(-\log h)))$ is either a line or a point. Call it L_f . If $L_f \not\subseteq D$, then the sheaf $h_*(\mathcal{O}_{\mathbb{C}^n}/df(\text{Der}(-\log h)))$ is finite over $\mathcal{O}_{\mathbb{C}}$, and (3.2) shows that its stalk at 0 is generated by $1, f, \dots, f^R$ for some finite R . Hence these same sections generate $h_*(\mathcal{O}_{\mathbb{C}^n}/df(\text{Der}(-\log h)))_t$ for t near 0, and therefore for all t , by homogeneity. As $h_*(\mathcal{O}_{\mathbb{C}^n}/df(\text{Der}(-\log h)))_t = \mathbb{C}[D_t]/J_f$, by 3.1 $f|_{D_t}$ is a Morse function.

On the other hand, if $L_f \subset D$, then $f : D_t \rightarrow \mathbb{C}$ is non-singular. \square

We do not know of any example where the latter alternative holds.

Proposition 3.3. *If $f : (D, 0) \rightarrow (\mathbb{C}, 0)$ is right-left stable then f is linear and $\text{Der}(-\log D)_0$ must contain at least n linearly independent weight zero vector fields. In particular, the only free divisors supporting right-left stable functions are linear free divisors.*

Proof. From equation (3.1) it is obvious that f must be linear, and that $\text{Der}(-\log h)$ must contain at least $n - 1$ independent weight zero vector fields; these, together with the Euler field, make n in $\text{Der}(-\log D)$. \square

We note that the hypothesis of the proposition is fulfilled by a generic linear function on the hypersurface defined by $\sum_j x_j^2 = 0$, which is not a free divisor.

3.2 \mathcal{R}_D and \mathcal{R}_h -equivalence of functions on divisors

Let $D \subset \mathbb{C}^n$ be a weighted homogeneous free divisor and let h be its weighted homogeneous equation. We consider functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$ and their restrictions to the fibres of h . The natural equivalence relation to impose on functions on D is \mathcal{R}_D -equivalence: right-equivalence with respect to the group of bianalytic diffeomorphisms of \mathbb{C}^n which preserve D . However, as we are interested also in the behaviour of f on the other fibres of h we consider also *fibred right-equivalence* with respect to the function $h : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$. That is, right-equivalence under the action of the group \mathcal{R}_h consisting of germs of bianalytic diffeomorphisms $\varphi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $h \circ \varphi = h$. A standard calculation shows that the tangent spaces to the \mathcal{R}_D and \mathcal{R}_h -orbits of f are equal to $df(\text{Der}(-\log D))$ and $df(\text{Der}(-\log h))$ respectively. We define

$$\begin{aligned} T_{\mathcal{R}_D}^1 f &:= \frac{\mathcal{O}_{\mathbb{C}^n,0}}{df(\text{Der}(-\log D))} \\ T_{\mathcal{R}_h}^1 f &:= \frac{\mathcal{O}_{\mathbb{C}^n,0}}{df(\text{Der}(-\log h)) + (h)} \\ T_{\mathcal{R}_h/\mathbb{C}}^1 f &:= \frac{\mathcal{O}_{\mathbb{C}^n,0}}{df(\text{Der}(-\log h))} \end{aligned}$$

and say that f is \mathcal{R}_D -finite or \mathcal{R}_h -finite if $\dim_{\mathbb{C}} T_{\mathcal{R}_D}^1 f < \infty$ or $\dim_{\mathbb{C}} T_{\mathcal{R}_h}^1 f < \infty$ respectively. Note that it is only in the definition of $T_{\mathcal{R}_h}^1 f$ that we explicitly restrict to the hypersurface D .

We remark that a closely related notion called D - \mathcal{K} -equivalence is studied by Damon in [Dam06].

Proposition 3.4. *If the germ $f \in \mathcal{O}_{\mathbb{C}^n,0}$ is \mathcal{R}_h -finite then there exist $\varepsilon > 0$ and $\eta > 0$ such that for $t \in \mathbb{C}$ with $|t| < \eta$,*

$$\sum_{x \in D_t \cap B_\varepsilon} \mu(f|_{D_t}; x) = \dim_{\mathbb{C}} T_{\mathcal{R}_h}^1 f.$$

If f is weighted homogeneous (with respect to the same weights as h) then ε and η may be taken to be infinite.

Proof. Let ξ_1, \dots, ξ_{n-1} be an $\mathcal{O}_{\mathbb{C}^n,0}$ -basis for $\text{Der}(-\log h)$. The \mathcal{R}_h -finiteness of f implies that the functions $df(\xi_1), \dots, df(\xi_{n-1})$ is a regular sequence in $\mathcal{O}_{\mathbb{C}^n,0}$, so that $T_{\mathcal{R}_h/\mathbb{C}}^1 f$ is a complete intersection ring, and in particular Cohen-Macaulay, of dimension 1. The condition of \mathcal{R}_h -finiteness is equivalent to $T_{\mathcal{R}_h/\mathbb{C}}^1 f$ being finite over $\mathcal{O}_{\mathbb{C},0}$. It follows that it is locally free over $\mathcal{O}_{\mathbb{C},0}$. \square

280 Now suppose that $D \subset \mathbb{C}^n = V$ is a linear free divisor. We denote the dual space $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by V^\vee . The
 281 group G_D^0 acts on V^\vee by the contragredient action ρ^\vee in which

$$g \cdot f = f \circ \rho(g)^{-1}.$$

282 If we write the elements of $V^\vee \simeq \mathbb{C}^n$ as column vectors, then the representation ρ^\vee takes the form $\rho^\vee(g) = {}^t$
 283 $\rho(g)^{-1}$. and the infinitesimal action takes the form $d\rho^\vee(A) = -{}^tA$. Let A_1, \dots, A_n be a basis for \mathfrak{g}_D . Then the
 284 vector fields

$$\xi_i(x) = (\partial/\partial x_1, \dots, \partial/\partial x_n)A_i x, \quad \text{for } i = 1, \dots, n \quad (3.3)$$

285 form an $\mathcal{O}_{\mathbb{C}^n}$ basis for $\text{Der}(-\log D)$, and the determinant of the $n \times n$ matrix of their coefficients is a non-zero
 286 scalar multiple of h , by Saito's criterion. The vector fields

$$\xi_i(y) = (\partial/\partial y_1, \dots, \partial/\partial y_n)({}^tA_i)y, \quad \text{for } i = 1, \dots, n \quad (3.4)$$

287 generate the infinitesimal action of \mathfrak{g}_D on V^\vee . We denote by h^\vee the determinant of the $n \times n$ matrix of their
 288 coefficients. Its zero-locus is the complement of the open orbit of G_D^0 on V^\vee . In general ρ^\vee and ρ are not
 289 equivalent representations. Indeed, it is not always the case that $(G_D^0, \rho^\vee, V^\vee)$ is a prehomogeneous vector
 290 space. We describe an example where this occurs in 3.6 below.

291 Suppose $f \in V^\vee$. Let $L_f = \text{supp } T_{\mathcal{R}_h/\mathbb{C}}^1 f$. Since $\text{Der}(-\log h)$ is generated by weight zero vector fields, L_f is a
 292 linear subspace of V .

293 **Proposition 3.5.** *Let $f \in V^\vee$. Then*

294 (i) *L_f is a line transverse to $f^{-1}(0) \Leftrightarrow f$ is \mathcal{R}_D -finite \Leftrightarrow the G_D^0 -orbit of f in the representation ρ^\vee is open.*

295 (ii) *Set $T_p D_t = \{f = 0\}$. Then*

$$H(p) \neq 0 \quad \Longrightarrow \quad \mu(f|_{D_t}; p) = 1 \quad (3.5)$$

296 *where H is the Hessian determinant of h .*

297 (iii) *If f is \mathcal{R}_h -finite then*

298 (a) *f is \mathcal{R}_D -finite;*

299 (b) *the classes of $1, f, \dots, f^{n-1}$ form a \mathbb{C} -basis for $T_{\mathcal{R}_h}^1 f$;*

300 (c) *on each Milnor fibre $D_t := h^{-1}(t), t \neq 0$, f has n non-degenerate critical points, which form an orbit
 301 under the diagonal action of the group of n -th roots of unity on \mathbb{C}^n .*

302 *Proof.* (i) The first equivalence holds simply because

$$df(\text{Der}(-\log D)) = df(\text{Der}(-\log h)) + (df(E)) = df(\text{Der}(-\log h)) + (f).$$

303 For the second equivalence, observe that the tangent space to the G_D^0 -orbit of f is naturally identified with
 304 $df(\text{Der}(-\log D)) \subset m_{V,0}/m_{V,0}^2 = V^\vee$. For given $A \in \mathfrak{g}_D$, we have

$$\left(\frac{d}{dt} \exp(tA) \cdot f \right)_{|t=0} (x) = df \left(\frac{d}{dt} \exp(-tA) \cdot x \right)_{|t=0} = -df(\xi_A) \quad (3.6)$$

305 where ξ_A is the vector field on V arising from A under the the infinitesimal action of ρ . Because $\text{Der}(-\log D)$
 306 is generated by vector fields of weight zero, $df(\text{Der}(-\log D))$ is generated by linear forms, and so f is \mathcal{R}_D -finite
 307 if and only if $df(\text{Der}(-\log D)) \supset m_{V,0}$.

308 (ii) is well-known. To prove it, parametrise D_t around p by $\varphi : (\mathbb{C}^{n-1}, 0) \rightarrow (D_t, p)$. Then because f is linear we
 309 have

$$\frac{\partial^2(f \circ \varphi)}{\partial u_i \partial u_j} = \sum_s \frac{\partial f}{\partial x_s} \frac{\partial^2 \varphi_s}{\partial u_i \partial u_j}. \quad (3.7)$$

310 Because $h \circ \varphi$ is constant, we find that

$$0 = \sum_{s,t} \frac{\partial^2 h}{\partial x_s \partial x_t} \frac{\partial \varphi_s}{\partial u_i} \frac{\partial \varphi_t}{\partial u_j} + \sum_s \frac{\partial h}{\partial x_s} \frac{\partial^2 \varphi_s}{\partial u_i \partial u_j}. \quad (3.8)$$

311 Because $T_p D_t = \{f = 0\}$, $d_p h$ is a scalar multiple of $d_p f = f$. From this, equations (3.7) and (3.8) give an
 312 equality (up to non-zero scalar multiple) of $n \times n$ matrices,

$$\left[\frac{\partial^2(f \circ \varphi)}{\partial u_i \partial u_j} \right] = {}^t \left[\frac{\partial \varphi_s}{\partial u_i} \right] \left[\frac{\partial^2 h}{\partial x_s \partial x_t} \circ \varphi \right] \left[\frac{\partial \varphi_t}{\partial u_j} \right] \quad (3.9)$$

313 It follows that if $H \neq 0$ then the restriction of f to D_t has a non-degenerate critical point at p .

314 (iii)(a) If f is \mathcal{R}_h -finite then L_f must be a line intersecting D only at 0. If \mathcal{R}_D finiteness of f fails, then
 315 $L_f \subset \{f = 0\}$, and f is constant along L_f . But at all points $p \in L_f$, $\ker d_p f \subset \ker d_p h$, so h also is constant
 316 along L_f .

317 (iii)(b) As L_f is a line and $\mathcal{O}_V / df(\text{Der}(-\log h)) = \mathcal{O}_{L_f}$, $h|_{L_f}$ is necessarily the n 'th power of a generator of
 318 $m_{L_f,0}$. It follows that $T_{\mathcal{R}_h}^1 f$ is generated by the first n non-negative powers of any linear form whose zero locus
 319 is transverse to the line L_f .

320 (iii)(c) Since f is \mathcal{R}_D finite, L_f is a line transverse to $\{f = 0\}$. The critical points of $f|_{D_t}$ are those points
 321 $p \in D_t$ where $T_p D_t = \{f = 0\}$; thus $L_f \cap D_t$ at each critical point. In \mathcal{O}_{D_t} , the ideals $df(\text{Der}(-\log h))$ and
 322 $J_{f|_{D_t}}$ coincide. Thus the intersection number of L_f with D_t at p , which we already know is equal to 1, is also
 323 equal to the Milnor number of $f|_{D_t}$ at p . The fact that there are n critical points, counting multiplicity, is just
 324 the fundamental theorem of algebra, applied to the single-variable polynomial $(h - t)|_{L_f}$. The fact that these n
 325 points form an orbit under the diagonal action of the group \mathbb{G}_n of n -th roots of unity is a consequence simply
 326 of the fact that h is \mathbb{G}_n -invariant and L_f is preserved by the action. \square

327 For homogeneous functions f of higher degree, the implications of 3.5(iii) hold only with the additional assump-
 328 tion that f has an isolated singularity at $0 \in \mathbb{C}^n$.

329 If D is a linear free divisor, there may be no \mathcal{R}_h -finite linear forms, or even no \mathcal{R}_D finite linear forms, as the
 330 following examples shows.

331 *Example 3.6.* Let D be the free divisor in the space V of 2×5 complex matrices defined by the vanishing of the
 332 product of the 2×2 minors $m_{12}, m_{13}, m_{23}, m_{34}$ and m_{35} . Then D is a linear free divisor ([GMNS06, Example
 333 5.7(2)]), but ρ^\vee has no open orbit in V^\vee : it is easily checked that $h^\vee = 0$. It follows by 3.5 (i) that no linear
 334 function $f \in V^\vee$ is \mathcal{R}_D -finite, and so by 3.5(iii) that none is \mathcal{R}_h -finite.

335 In Example 3.6, the group G_D^0 is not reductive. Results of Sato and Kimura in [SK77, §4] show that if G_D^0 is
 336 reductive then $(G_D^0, \rho^\vee, V^\vee)$ is prehomogeneous, so that almost all $f \in V^\vee$ are \mathcal{R}_D -finite, and moreover imply
 337 that all f in the open orbit in V^\vee are \mathcal{R}_h -finite. We briefly review their results. First, the complement of the
 338 open orbit in V^\vee is a divisor whose equation, in suitable coordinates x on V , and dual coordinates y on V^\vee ,
 339 is of the form $h^*(y) = \overline{h(\bar{y})}$. The coordinates in question are chosen as follows: as G_D^0 is reductive, it has a
 340 Zariski dense compact subgroup K . In suitable coordinates on $V = \mathbb{C}^n$ the representation ρ places K inside
 341 $U(n)$. Call such a coordinate system *unitary*. From this it follows that if f is any rational semi-invariant on V
 342 with associated character χ then the function $f^* : V^\vee \rightarrow \mathbb{C}$ defined by $f^*(y) = \overline{f(\bar{y})}$ is also a semi-invariant *for*
 343 *the representation of K* with associated character $\bar{\chi}$, which is equal to χ^{-1} since $\chi(K) \subset S^1$ by compactness.
 344 Note that f^* cannot be the zero polynomial. As K is Zariski-dense in G_D^0 , the rational equality

$$f^*(\rho^\vee(g)y) = \frac{1}{\chi(g)} f^*(y)$$

345 holds for all $g \in G_D^0$.

346 **Proposition 3.7.** *Let $D \subset \mathbb{C}^n$ be a linear free divisor with equation h . If G_D^0 is reductive then*

- 347 (i) $(G_D^0, \rho^\vee, V^\vee)$ is a prehomogeneous vector space.
- 348 (ii) D^\vee , the complement of the open orbit in V^\vee , has equation h^* , with respect to dual unitary coordinates on
 349 V^\vee .
- 350 (iii) D^\vee is a linear free divisor.

351 *Proof.* As \mathbb{C} -basis of the Lie algebra \mathfrak{g}_D of G_D^0 we can take a real basis of the Lie algebra of K . With respect
 352 to unitary coordinates, ρ represents K in $U(n)$, so $d\rho(\mathfrak{g}_D) \subset \mathfrak{gl}_n(\mathbb{C})$ has \mathbb{C} -basis A_1, \dots, A_n such that $A_i \in \mathfrak{u}_n$,
 353 i.e. ${}^t A_i = -\bar{A}_i$, for $i = 1, \dots, n$. It follows that the determinant of the matrix of coefficients of the matrix (3.4)
 354 above is equal to h^* , and in particular is not zero. This proves (i) and (ii).

355 That D^\vee is free follows from Saito's criterion ([Sai80]): the n vector fields (3.4) are logarithmic with respect
 356 to D^\vee , and h^* , the determinant of their matrix of coefficients, is not identically zero, and indeed is a reduced
 357 equation for D^\vee because h is reduced. \square

358 We now prove the main result of this section. In order to make the argument clear, we postpone some steps in
 359 the proof to a sequence of lemmas, 3.9, 3.10 and 3.11, which we prove immediately afterwards.

360 **Theorem 3.8.** *If G_D^0 is reductive then $f \in V^\vee$ is \mathcal{R}_h -finite if and only if it is \mathcal{R}_D -finite. In particular, f is
 361 \mathcal{R}_h -finite if and only if $f \in V^\vee \setminus D^*$.*

362 *Proof.* Let $p \in D_t$ (for $t \neq 0$) and suppose that $T_p D_t$ has equation f , i.e. that $\nabla h(p)$ is a non-zero multiple
 363 of f . We claim that f is \mathcal{R}_h -finite. For by Lemma (3.10) below, $H(p) \neq 0$, and it follows by 3.5(ii) that the
 364 restriction of f to D_t has a non-degenerate critical point at p . The critical locus of $f|_{D_t}$ is precisely $L_f \cap D_t$;
 365 so L_f must be a line (recall that it is a linear subspace of V), and must meet D_t transversely at p . By the
 366 homogeneity of D , it follows that $L_f \cap D = \{0\}$, so f is \mathcal{R}_h -finite. Thus

$$f \text{ } \mathcal{R}_D\text{-finite} \xrightarrow{3.5} f \in V^\vee \setminus D^* \xrightarrow{3.11} f = \nabla h(p) \text{ for some } p \notin D \implies f \text{ } \mathcal{R}_h\text{-finite.}$$

367 We have already proved the opposite implication, in 3.5. □

368 **Lemma 3.9.** *Let $D \subset \mathbb{C}^n$ be a linear free divisor with homogeneous equation h , let h^\vee be the determinant of
 369 the matrix of coefficients of (3.4), and let H be the Hessian determinant of h . Then*

$$h^\vee \left(\frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n} \right) = (n-1)Hh.$$

370 *Proof.* Choose the basis $A_1 = I_n, \dots, A_n$ for \mathfrak{gl}_D so that the associated vector fields ξ_2, \dots, ξ_n are in $\text{Der}(-\log h)$.
 371 The matrix I gives rise to the Euler vector field E . Write $A_i = [a_{ij}^k]$, with the upper index k referring to
 372 columns and the lower index j referring to rows. Let $\alpha_{ji} = \sum_k a_{ij}^k x_k$ denote the coefficient of $\partial/\partial x_j$ in ξ_i for
 373 $i = 2, \dots, n-1$. Then

$$0 = dh(\xi_i) = \sum_j \alpha_{ji} \frac{\partial h}{\partial x_j},$$

374 so differentiating with respect to x_k ,

$$0 = \sum_j \frac{\partial \alpha_{ji}}{\partial x_k} \frac{\partial h}{\partial x_j} + \sum_j \alpha_{ji} \frac{\partial^2 h}{\partial x_k \partial x_j} = \sum_j a_{ij}^k \frac{\partial h}{\partial x_j} + \sum_j \alpha_{ji} \frac{\partial^2 h}{\partial x_k \partial x_j}. \quad (3.10)$$

375 For the Euler field ξ_1 we have

$$nh = dh(E) = \sum_j \alpha_{j1} \frac{\partial h}{\partial x_j}$$

376 so

$$n \frac{\partial h}{\partial x_k} = \sum_j a_{1j}^k \frac{\partial h}{\partial x_j} + \sum_j \alpha_{j1} \frac{\partial^2 h}{\partial x_k \partial x_j} = \frac{\partial h}{\partial x_k} + \sum_j \alpha_{j1} \frac{\partial^2 h}{\partial x_k \partial x_j}. \quad (3.11)$$

377 Putting the n equations (3.10) and (3.11) together in matrix form we get

$$\begin{bmatrix} {}^t E \\ {}^t \xi_1 \\ \cdot \\ {}^t \xi_{n-1} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 h}{\partial x_k \partial x_j} \end{bmatrix} = - \begin{bmatrix} (n-1)\nabla h \cdot E \\ \nabla h \cdot A_1 \\ \cdot \\ \nabla h \cdot A_{n-1} \end{bmatrix}.$$

378 Now take determinants of both sides. The determinant on the right hand side is

$$(n-1)h^\vee \left(\frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n} \right).$$

379 The determinants of the two matrices on the left are, respectively, h and H . □

Lemma 3.10. ([SK77]) *If D is a reductive linear free divisor, then for all $p \in \mathbb{C}^n$*

$$h(p) \neq 0 \implies H(p) \neq 0.$$

380 *Proof.* In [SK77, Page 72], Sato and Kimura show that if g is a homogeneous rational semi-invariant of degree
 381 r with associated character χ_g then there is a polynomial $b(m)$ of degree r (the b -function of g) such that, with
 382 respect to unitary coordinates on \mathbb{C}^n ,

$$g^* \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \cdot g^m = b(m)g^{m-1} \quad (3.12)$$

383 This is proved by showing that the left hand side is a semi-invariant with associated character χ_g^{m-1} , and noting
 384 that the semi-invariant corresponding to a given character is unique up to scalar multiple, since the quotient of
 385 two semi-invariants with the same character is an absolute invariant, and therefore must be constant (since G_D^0
 386 has a dense orbit). From this it follows ([SK77, page 72]) that

$$g^* \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right) = b_0 g^{r-1}, \quad (3.13)$$

387 where b_0 is the (non-zero) leading coefficient of the polynomial $b(m)$, and hence, by 3.9, that

$$(n-1)H = b_0 h^{n-2}. \quad (3.14)$$

388 □

389 **Proposition 3.11.** *If D is a linear free divisor with reductive group G_D^0 and homogeneous equation h with*
 390 *respect to unitary coordinates, then*

- 391 (i) *the gradient map ∇h maps the fibres $D_t, t \neq 0$ of h diffeomorphically to the fibres of h^* .*
 392 (ii) *the gradient map ∇h^* maps Milnor fibres of h^* diffeomorphically to Milnor fibres of h .*

393 *Proof.* The formula (3.13) shows that ∇h maps fibres of h into fibres of h^* . Each fibre of h is a single orbit
 394 of the kernel of $\chi_h : G_D^0 \rightarrow \mathbb{C}^*$, and each fibre of h^* is a single orbit of the kernel of χ_{h^*} . These two subgroups
 395 coincide because $\chi_{h^*} = (\chi_h)^{-1}$. The map is equivariant: $\nabla h(\rho(g)x) = \rho^*(g)^{-1} \nabla h(x)$. It follows that ∇h maps
 396 D_t surjectively onto a fibre of h^* . By (3.10), this mapping is a local diffeomorphism. It is easy to check that it
 397 is 1-1. Since $(h^*)^* = h$ and dual unitary coordinates are themselves unitary, the same argument, interchanging
 398 the roles of h and h^* , gives (ii). □

399 *Question 3.12.* If we drop the condition that D be a linear free divisor, what condition could replace reductivity
 400 to guarantee that for (linear) functions $f \in \mathcal{O}_{\mathbb{C}^n}$, \mathcal{R}_D -finiteness implies \mathcal{R}_h finiteness?

401 *Remark 3.13.* The following will be used in the proof of lemma 3.19. Let $AT_x D_t := x + T_x D_t$ denote the affine
 402 tangent space at x . Lemma 3.11 implies that the affine part $D_t^\vee = \{AT_x D_t : x \in D_t\}$ of the projective dual of
 403 D_t is a Milnor fibre of h^* . For $AT_x D_t$ is the set

$$(y_1, \dots, y_n) \in \mathbb{C}^n : d_x h(y_1, \dots, y_n) = d_x h(x_1, \dots, x_n);$$

404 by homogeneity of h the right hand side is just nt , and thus in dual projective coordinates $AT_x D_t$ is the point
 405 $(-nt : \partial h / \partial x_1(x) : \dots : \partial h / \partial x_n(x))$. In affine coordinates on U_0 , this is the point

$$\left(\frac{-1}{nt} \frac{\partial h}{\partial x_1}(x), \dots, \frac{-1}{nt} \frac{\partial h}{\partial x_n}(x) \right).$$

406 By (3.13), the function h^* takes the value $b_0 t^{n-1} / (nt)^n = b_0 / n^{n-1} t$ at this point, independent of $x \in D_t$, and so
 407 $D_t^\vee \subset (h^*)^{-1}(b_0 / n^{n-1} t)$. The opposite inclusion holds by openness of the map ∇h , which, in turn, follows from
 408 Proposition 3.9.

409 3.3 Tameness

410 In this subsection, we study a property of the polynomial functions $f|_{D_t}$ known as tameness. It describes the
 411 topological behavior of f at infinity, and is needed in order to use the general results from [Sab06] and [DS03]
 412 on the Gauß-Manin system and the construction of Frobenius structures. In fact we discuss two versions,
 413 cohomological tameness and M -tameness. Whereas the first will be seen to hold for all \mathcal{R}_h -finite linear functions
 414 on a linear free divisor D , we show M -tameness only if D is reductive. Cohomological tameness is all that is
 415 needed in our later construction of Frobenius manifolds, but we feel that the more evidently geometrical condition
 416 of M -tameness is of independent interest.

417 **Definition 3.14** ([Sab06]). Let X be an affine algebraic variety and $f : X \rightarrow \mathbb{C}$ a regular function. Then f is
418 called cohomologically tame if there is a partial compactification $X \xrightarrow{j} Y$ with Y quasiprojective, and a proper
419 regular function $F : Y \rightarrow \mathbb{C}$ extending f , such that for any $c \in \mathbb{C}$, the complex $\varphi_{F-c}(\mathbb{R}j_*\mathbb{Q}_X)$ is supported in
420 a finite number of points, which are contained in X . Here φ is the functor of vanishing cycles of Deligne, see,
421 e.g., [Dim04].

422 It follows in particular that a cohomologically tame function f has at most isolated critical points.

423 **Proposition 3.15.** *Let $D \subset V$ be linear free and $f \in \mathbb{C}[V]_1$ be an \mathcal{R}_h -finite linear section. Then the restriction*
424 *of f to $D_t := h^{-1}(t)$ is cohomologically tame.*

425 *Proof.* A similar statement is actually given without proof in [NS99] as an example of a so-called weakly tame
426 function. We consider the standard graph compactification of f : $\overline{\Gamma}(f)$ is the closure of the graph $\Gamma(f) \subset D_t \times \mathbb{C}$
427 of f in $\overline{D}_t \times \mathbb{C}$ (where \overline{D}_t is the projective closure of D_t in \mathbb{P}^n), we identify f with the projection $\Gamma(f) \rightarrow \mathbb{C}$,
428 and extend f to the projection $F : \overline{\Gamma}(f) \rightarrow \mathbb{C}$. Refine the canonical Whitney stratification of \overline{D}_t by dividing
429 the open stratum, which consists of $D_t \cup (B_h)_{\text{reg}}$, into the two strata D_t and $(B_h)_{\text{reg}}$. Evidently this new
430 stratification \mathcal{S} is still Whitney regular. From \mathcal{S} we obtain a Whitney stratification \mathcal{S}' of $\overline{\Gamma}(f)$, since $\overline{\Gamma}(f)$ is
431 just the transversal intersection of a hyperplane with $\overline{D}_t \times \mathbb{C}$. The isosingular locus of \overline{D}_t through any point
432 $(0 : x_1 : \dots : x_n) \in B_h$ contains the projectivised isosingular locus of D through (x_1, \dots, x_n) , and so by the
433 \mathcal{R}_h -finiteness of f , $\{f = 0\}$ is transverse to the strata of \mathcal{S} . This translates into the fact that the restriction
434 of F (i.e., the second projection) to the strata of the stratification \mathcal{S}' (except the stratum over D_t) is regular.
435 It then follows from [Dim04, proposition 4.2.8] that the cohomology sheaves of $\varphi_{F-c}(\mathbb{R}j_*\mathbb{Q}_{D_t})$ are supported in
436 D_t in a finite number of points, namely the critical points of $f|_{D_t}$. Therefore f is cohomologically tame. \square

437 **Definition 3.16** ([NS99]). Let $X \subset \mathbb{C}^n$ be an affine algebraic variety and $f : X \rightarrow \mathbb{C}$ a regular function. Set

$$M_f := \{x \in X : f^{-1}(f(x)) \not\cong S_{\|x\|}\},$$

438 where $S_{\|x\|}$ is the sphere in \mathbb{C}^n centred at 0 with radius $\|x\|$. We say that $f : X \rightarrow \mathbb{C}$ is M -tame if there is no
439 sequence $(x^{(k)})$ in M_f such that

440
441 (i) $\|x^{(k)}\| \rightarrow \infty$ as $k \rightarrow \infty$ and (ii) $f(x^{(k)})$ tends to a limit $\ell \in \mathbb{C}$ as $k \rightarrow \infty$.

442 Suppose $x^{(k)}$ is a sequence in M_f satisfying (i) and (ii). After passing to a subsequence, we may suppose also
443 that as $k \rightarrow \infty$,

444 (iii) $(x^{(k)}) \rightarrow x^{(0)} \in H_\infty$, where H_∞ is the hyperplane at infinity in \mathbb{P}^n ,

445 (iv) $T^{(k)} \rightarrow T^{(0)} \in G_{d-1}(\mathbb{P}^n)$ where $T^{(k)}$ denotes the affine tangent space $AT_{x^{(k)}}f^{-1}(f(x^{(k)}))$, $d = \dim X$ and
446 $G_{d-1}(\mathbb{P}^n)$ is the Grassmannian of $(d-1)$ -planes in \mathbb{P}^n .

Let f and h be homogeneous polynomials on \mathbb{C}^n and $X = D_t = h^{-1}(t)$ for some $t \neq 0$. Let

$$B_f = \{(0, x_1, \dots, x_n) \in \mathbb{P}^n : f(x_1, \dots, x_n) = 0\} \quad B_h = \{(0, x_1, \dots, x_n) \in \mathbb{P}^n : h(x_1, \dots, x_n) = 0\}.$$

447 Note that B_f and B_h are contained in the projective closure of every affine fibre of f and h respectively. We
448 continue to denote the restriction of f to D_t by f . Let $x^{(k)}$ be a sequence satisfying 3.16(i)–(iv).

449 **Lemma 3.17.** $x^{(0)} \in B_f \cap B_h$.

450 *Proof.* Evidently $x^{(0)} \in \overline{D}_t \cap H_\infty = B_h$. Let $U_1 = \{(x_0, x_1, \dots, x_n) \in \mathbb{P}^n : x_1 \neq 0\}$. After permuting the
451 coordinates x_1, \dots, x_n and passing to a subsequence we may assume that $|x_1^{(k)}| \geq |x_j^{(k)}|$ for $j \geq 1$. It follows that
452 $x^{(0)} \in U_1$. In local coordinates $y_0 = x_0/x_1, y_2 = x_2/x_1, \dots, y_n = x_n/x_1$ on U_1 , B_f is defined by the two equations
453 $y_0 = 0, f(1, y_2, \dots, y_n) = 0$. Since $f(x_1^{(k)}, \dots, x_n^{(k)}) \rightarrow \ell$ and $x_1^{(k)} \rightarrow \infty$, we have $f(1, x_2^{(k)}/x_1^{(k)}, \dots, x_n^{(k)}/x_1^{(k)}) \rightarrow 0$.
454 It follows that $f(x_1^{(0)}, \dots, x_n^{(0)}) = 0$, and $x^{(0)} \in B_f$. \square

455 **Lemma 3.18.** *If f is a linear function then $T^{(0)} = B_f$.*

456 *Proof.* For all k we have $T^{(k)} \subset AT_{x^{(k)}}S_{\|x^{(k)}\|}$. Let x^\perp denote the Hermitian orthogonal complement of the
457 vector x in \mathbb{C}^n . Then $T^{(k)}$ is contained in $(x^{(k)} + x^{(k)\perp}) \cap AT_{x^{(k)}}D_t$, since this is the maximal complex subspace
458 of $AT_{x^{(k)}}D_t \cap AT_{x^{(k)}}S_{\|x^{(k)}\|}$. With respect to dual homogeneous coordinates on $(\mathbb{P}^n)^\vee$, $x^{(k)} + x^{(k)\perp} = (-\|x^{(k)}\|^2 : x_1^{(k)} : \dots : x_n^{(k)})$. So

$$\lim_{k \rightarrow \infty} x^{(k)} + x^{(k)\perp} = \lim_{k \rightarrow \infty} (1 : x_1^{(k)} / \|x^{(k)}\|^2 : \dots : x_n^{(k)} / \|x^{(k)}\|^2) = (1 : 0 : \dots : 0) = H_\infty.$$

460 It follows that $T^{(0)} \subset H_\infty$. To see that $T^{(0)} \subset B_f$, note that $T^{(k)} \subset f^{-1}(f(x^{(k)}))$. Since $f(x^{(k)}) \rightarrow \ell$,
461 $f^{-1}(f(x^{(k)})) \rightarrow f^{-1}(\ell)$ and so in the limit $T^{(0)} \subseteq \overline{f^{-1}(\ell)}$. Since $T^{(0)} \subset H_\infty$, we conclude that $T^{(0)} \subset$
462 $\overline{f^{-1}(\ell)} \cap H_\infty = B_f$. As $\dim B_f = \dim T^{(0)}$, the two spaces must be equal. \square

463 By passing to a subsequence, we may suppose that $AT_{x^{(k)}}D_t$ tends to a limit L as $k \rightarrow \infty$.

464 **Lemma 3.19.** *If $D = h^{-1}(0)$ is a reductive linear free divisor then $L \neq H_\infty$.*

465 *Proof.* It is only necessary to show that H_∞ does not lie in the projective closure of the dual D_t^\vee of D_t . By
466 Remark 3.13, $D_t^\vee = (h^*)^{-1}(c)$ for some $c \neq 0$. Its projective closure is thus $\{(y_0 : y_1 : \dots : y_n) \in (\mathbb{P}^n)^\vee : h^*(y_1, \dots, y_n) = cy_0^n\}$, which does not contain $H_\infty = (1 : 0 : \dots : 0)$. \square

468 Let $\{X_\alpha\}_{\alpha \in A}$ be a Whitney stratification of \overline{D}_t , with regular stratum D_t , and suppose $x^{(0)} \in X_\alpha$. By Whitney
469 regularity, $L \supset AT_{x^{(0)}}X_\alpha$. Clearly $T^{(0)} \subset L$. As $L \neq H_\infty$ then since $T^{(0)} \subset H_\infty$, for dimensional reasons we
470 must have $T^{(0)} = L \cap H_\infty$. It follows that $T^{(0)} \supset AT_{x^{(0)}}X_\alpha$, and thus, by Lemma 3.18,

$$B_f \supset AT_{x^{(0)}}X_\alpha.$$

471 We have proved

472 **Proposition 3.20.** *If $D = \{h = 0\}$ is a reductive linear free divisor, $D_t = h^{-1}(t)$ for $t \neq 0$, and $f : D_t \rightarrow \mathbb{C}$ is
473 not M -tame, then B_f is not transverse to the Whitney stratification $\{X_\alpha\}_{\alpha \in A}$ of \overline{D}_t . \square*

474 Now we can prove the main result of this section.

475 **Theorem 3.21.** *If D is a reductive linear free divisor with homogeneous equation h , and if the linear function
476 f is \mathcal{R}_h finite, then the restriction of f to D_t is M -tame.*

477 *Proof.* \mathcal{R}_h -finiteness of f implies that for all $x \in D \cap \{f = 0\} \setminus \{0\}$,

$$T_x\{f = 0\} + \text{Der}(-\log h)(x) = T_x\mathbb{C}^n. \quad (3.15)$$

478 The strata of the canonical Whitney stratification \mathcal{S} ([Tei82] and [TT83, Corollary 1.3.3]) of D are unions of
479 isosingular loci. So for any $x \in X_\alpha \in \mathcal{S}$, $T_x X_\alpha \supset \text{Der}(-\log D)(x)$. It follows from (3.15) that $\{f = 0\} \pitchfork \mathcal{S}$.
480 Because D is homogeneous, the strata of \mathcal{S} are homogeneous too, and so we may form the projective quotient
481 stratification $\mathbb{P}\mathcal{S}$ of B_h . Transversality of $\{f = 0\}$ to D outside 0 implies that B_f is transverse to $\mathbb{P}\mathcal{S}$. The
482 conclusion follows by Proposition 3.20. \square

483 *Remark 3.22.* Reductivity is needed in Lemma 3.19 to conclude that $L \neq H_\infty$. Indeed, the example given
484 by Broughton in [Bro88, Example 3.2] of a non-tame function on \mathbb{C}^2 , $g(x_1, x_2) = x_1(x_1x_2 - 1)$, becomes the
485 defining equation of a non-reductive linear free divisor, $h(x_1, x_2, x_3) = x_1(x_1x_2 - x_3^2)$, on homogenisation (in
486 fact this is the case $n = 2$ of the non-special linear free divisors described after Definition 2.1). The sequence
487 $x^{(k)} = (1/k, k^2, \sqrt{2k})$ lies in D_{-1} and tends to $x^{(0)} = (0 : 0 : 1 : 0)$ in \mathbb{P}^3 , and $AT_{x^{(k)}}D_{-1}$ has dual projective
488 coordinates $(3 : 0 : 1/n^2 : -2n^{-1/2})$ and thus tends to H_∞ as $n \rightarrow \infty$.

489 4 Gauß-Manin systems and Brieskorn lattices

490 In this section we introduce the family of Gauß-Manin systems and Brieskorn lattices attached to an \mathcal{R}_h -finite
491 linear section of the fibration defined by the equation h . All along this section, we suppose that h defines a
492 linear free divisor.

493 Under this hypothesis, we show the freeness of the Brieskorn lattice, and prove that a particular basis can
494 be found yielding a solution of the so called Birkhoff problem. The proof of the freeness relies on two facts,

495 first, we need that the deformation algebra $T_{\mathcal{R}_h/\mathbb{C}}^1 f$ is generated by the powers of f (this would follow only
496 from the hypotheses of lemma 3.1) and second on a division theorem, whose essential ingredient is lemma 4.3
497 below, which in turn uses the relative logarithmic de Rham complex associated to a LFD which was studied
498 in subsection 2.2. The particular form of the connection that we obtain on the Brieskorn lattice allows us to
499 prove that a solution to the Birkhoff problem always exists. This solution defines an extension to infinity (i.e.,
500 a family of trivial algebraic bundles on \mathbb{P}^1) of the Brieskorn lattice. However, these solutions miss a crucial
501 property needed in the next section: The extension is not compatible with the canonical V -filtration at infinity,
502 in other words, it is not a V^+ -solution in the sense of [DS03, Appendix B]. We provide a very explicit algorithm
503 to compute these V^+ -solutions. In particular, this gives the spectral numbers at infinity of the functions $f|_{D_t}$.
504 Using the tameness of the functions $f|_{D_t}$ it is shown in [Sab06] that the Gauß-Manin systems are equipped with
505 a non-degenerate pairing with a specific pole order property on the Brieskorn lattices. A solution to the Birkhoff
506 problem compatible with this pairing is called S -solution in [DS03, Appendix B]. One needs such a solution in
507 order to construct Frobenius structures, see the next section. We prove that our solution is a (V^+, S) -solution
508 under an additional hypothesis, which is nevertheless satisfied for many examples.
509 Let us start by defining the two basic objects we are interested in this section. We recall that we work in the
510 algebraic category.

511 **Definition 4.1.** Let D be a linear free divisor with defining equation $h \in \mathbb{C}[V]_n$ and $f \in \mathbb{C}[V]_1$ linear and
512 \mathcal{R}_h -finite. Let

$$\mathbf{G} := \frac{\Omega^{n-1}(\log h)[\tau, \tau^{-1}]}{(d - \tau df \wedge)(\Omega^{n-2}(\log h)[\tau, \tau^{-1}])}$$

513 be the family of *algebraic* Gauß-Manin systems of (f, h) and

$$G := \text{Image of } \Omega^{n-1}(\log h)[\tau^{-1}] \text{ in } \mathbf{G}$$

514 be the family of *algebraic* Brieskorn lattices of (f, h) .

515 **Lemma 4.2.** \mathbf{G} is a free $\mathbb{C}[t, \tau, \tau^{-1}]$ -module of rank n , and G is free over $\mathbb{C}[t, \tau^{-1}]$ and is a lattice inside \mathbf{G} , i.e.,
516 $\mathbf{G} = G \otimes_{\mathbb{C}[t, \tau^{-1}]} \mathbb{C}[t, \tau, \tau^{-1}]$. A $\mathbb{C}[t, \tau, \tau^{-1}]$ -basis of \mathbf{G} (resp. a $\mathbb{C}[t, \tau^{-1}]$ -basis of G) is given by $(f^i \alpha)_{i \in \{0, \dots, n-1\}}$,
517 where $\alpha := n \cdot \text{vol}/dh = \iota_E(\text{vol}/h)$.

518 *Proof.* As it is clear that $\mathbf{G} = G \otimes \mathbb{C}[t, \tau, \tau^{-1}]$, we only have to show that the family $(f^i \alpha)_{i \in \{0, \dots, n-1\}}$ freely
519 generates G . This is done along the line of [dG07, proposition 8]. Remember from the discussion in subsection
520 2.2 that $\Omega^{n-1}(\log h)$ is $\mathbb{C}[V]$ -free of rank one, generated by the form α . The next fact needed is that

$$G/\tau^{-1}G \cong \frac{\Omega^{n-1}(\log h)}{df \wedge \Omega^{n-2}(\log h)} \cong \left(h_* T_{\mathcal{R}_h/\mathbb{C}}^1 f \right) \alpha = \left(\frac{\mathbb{C}[V]}{\xi_1(f), \dots, \xi_{n-1}(f)} \right) \alpha$$

521 which is a graded free $\mathbb{C}[t]$ -module of rank $n = \deg(h)$ by proposition 3.4 and proposition 3.5. Let $1, f, f^2, \dots, f^{n-1}$
522 be the homogeneous $\mathbb{C}[t]$ -basis of $h_* T_{\mathcal{R}_h/\mathbb{C}}^1 f$ constructed in proposition 3.5, and $\omega = g\alpha$ be a representative for
523 a section $[\omega]$ of G , where $g \in \mathbb{C}[V]_l$ is a homogeneous polynomial of degree l . Then g can be written as
524 $g(x) = \tilde{g}(h) \cdot f^i + \eta(f)$ where $\tilde{g} \in \mathbb{C}[t]_{\lfloor l/n \rfloor}$, $i = l \bmod n$ and $\eta \in \text{Der}(-\log h)$. Using the linear basis ξ_1, \dots, ξ_{n-1}
525 of $\text{Der}(-\log h)$, we find homogeneous functions $k_j \in \mathbb{C}[V]_{l-1}$, $j = 1, \dots, n-1$ such that

$$\omega = \tilde{g}(h) f^i \alpha + \sum_{j=1}^{n-1} k_j \xi_j(f) \alpha$$

526 It follows from the next lemma that in \mathbf{G} we have

$$[\omega] = \tilde{g}(h) f^i \alpha + \tau^{-1} \sum_{j=1}^{n-1} (\xi_j(k_j) + \text{trace}(\xi_j) \cdot k_j) \alpha$$

527 As $\deg(\xi_j(k_j) + \text{trace}(\xi_j) \cdot k_j) = \deg(g) - 1$, we see by iterating the argument (i.e., applying it to all the classes
528 $[(\xi_j(k_j) + \text{trace}(\xi_j) \cdot k_j) \alpha] \in G$) that $(f^i \alpha)_{i=0, \dots, n-1}$ gives a system of generators for G over $\mathbb{C}[t, \tau^{-1}]$. To show
529 that they freely generate, let us consider a relation

$$\sum_{i=0}^{n-1} a_i(t, \tau^{-1}) f^i \alpha = (d - \tau df \wedge) \sum_{i=-l}^L \tau^i \omega_i, \quad \omega_i \in \Omega^{n-2}(\log h)$$

530 Rewriting the left-hand side as a polynomial in τ^{-1} with coefficients in $\Omega^{n-1}(\log h)$, the above equation becomes
531

$$\sum_{i=-m}^M \tau^i \eta_i = (d - \tau df \wedge) \sum_{i=-l}^L \tau^i \omega_i \quad (4.1)$$

532 where we have written $\eta_i = \sum_{j=0}^{n-1} b_{ij}(t)(f^j \alpha)$ and set $\eta_i = 0, i > 0$ for convenience. The above implies
533 $\eta_{L+1} = df \wedge \omega_L$, so that if $L > -1$, we must have $df \wedge \omega_L = 0$. This in turn implies $\omega_L = df \wedge \omega'_L$
534 for some $\omega'_L \in \Omega^{n-2}(\log h)$ and hence $\eta_L = d\omega_L + df \wedge \omega_{L-1} = df \wedge (d\omega'_L - \omega_{L-1})$. By induction we obtain
535 $\eta_0 \in df \wedge \Omega^{n-2}(\log h)$ and since $f^i \alpha, i = 0, \dots, n-1$ form a $\mathbb{C}[t]$ -basis of $\Omega^{n-1}(\log h)/df \wedge \Omega^{n-2}(\log h)$, it follows
536 that $b_{0j} = 0, j = 0, \dots, n-1$. Notice that the same conclusion is obtained if $L \leq -1$. In both cases, successive
537 multiplication by τ in both sides of (4.1) yields $\eta_{-i} \in df \wedge \Omega^{n-2}(\log h)$.
538 □

539 **Lemma 4.3.** *For any ξ in $\text{Der}(-\log h)_0$ and $g \in \mathbb{C}[V]$, the following relation holds in \mathbf{G}*

$$\tau g \xi(f) \alpha = (\xi(g) + g \cdot \text{trace}(\xi)) \alpha$$

Proof. We have that

$$\begin{aligned} \tau g \xi(f) \alpha &= \tau g i_\xi(df) \alpha = \tau g (i_\xi(df \wedge \alpha) + df \wedge i_\xi \alpha) = \tau g df \wedge i_\xi \alpha \\ &= d(g i_\xi \alpha) = dg \wedge i_\xi \alpha + g di_\xi \alpha = i_\xi(dg \wedge \alpha) + dg \wedge i_\xi \alpha + g di_\xi \alpha \\ &= i_\xi(dg) \alpha + g di_\xi \alpha = \xi(g) \alpha + g di_\xi \alpha \\ &= (\xi(g) + g \cdot \text{trace}(\xi)) \alpha. \end{aligned}$$

540 In this computation, we have twice used the fact that for any function $r \in \mathbb{C}[V]$, the class $i_\xi(dr \wedge \alpha)$ is zero
541 in $\Omega^{n-1}(\log h)$. This holds because for $\xi \in \text{Der}(-\log h)$ and for $r \in \mathbb{C}[V]$ the operations i_ξ and $dr \wedge$ are well
542 defined on $\Omega^\bullet(\log h)$ and moreover, $\Omega^n(\log h) = 0$, so that already $dr \wedge \alpha = 0 \in \Omega^\bullet(\log h)$. □

543 We denote by $T := \text{Spec } \mathbb{C}[t]$ the base of the family defined by h . Then \mathbf{G} corresponds to a rational vector
544 bundle of rank n over $\mathbb{P}^1 \times T$, with poles along $\{0, \infty\} \times T$. Here we consider the two standard charts of \mathbb{P}^1
545 where τ is a coordinate centered at infinity. G defines an extension over $\{0\} \times T$, i.e., an algebraic bundle over
546 $\mathbb{C} \times T$ of the same rank as \mathbf{G} .

547 We define a (relative) connection operator on \mathbf{G} by

$$\nabla_{\partial_\tau} \left(\sum_{i=i_0}^{i_1} \omega_i \tau^i \right) := \sum_{i=i_0-1}^{i_1} ((i+1)\omega_{i+1} - f \cdot \omega_i) \tau^i$$

548 where $\omega_{i_1+1} := 0, \omega_{i_0-1} := 0$. Then it is easy to check that this gives a well defined operator on the quotient
549 \mathbf{G} , and that it satisfies the Leibniz rule, so that we obtain a relative connection

$$\nabla : \mathbf{G} \longrightarrow \mathbf{G} \otimes \Omega_{\mathbb{C} \times T/T}^1(*\{0\} \times T).$$

550 As multiplication with f leaves invariant the module $\Omega^{n-1}(\log h)$, we see that the operator ∇_{∂_τ} sends G to
551 itself, in other words, G is stable under $\nabla_{\partial_\tau} = -\tau^{-2} \nabla_{\partial_{\tau^{-1}}} = -\theta^2 \nabla_{\partial_\theta}$, where we write $\theta := \tau^{-1}$. This shows
552 that the relative connection ∇ has a pole of order at most two on G along $\{0\} \times T$ (i.e., along $\tau = \infty$).

553 Consider, for any $t \in T$, the restrictions $\mathbf{G}_t := \mathbf{G}/\mathfrak{m}_t \mathbf{G}$ and $G_t := G/\mathfrak{m}_t G$. Then \mathbf{G}_t is a free $\mathbb{C}[\tau, \tau^{-1}]$ -module
554 and G_t is a $\mathbb{C}[\tau^{-1}]$ -lattice in it. It follows from the definition that this is exactly the (localized partial Fourier-
555 Laplace transformation of the) Gauß-Manin system resp. the Brieskorn lattice of the function $f : D_t \rightarrow \mathbb{C}$, as
556 studied in [Sab06]. We will make use of the results of loc.cit. applied to $f|_{D_t}$ in the sequel. Let us remark
557 that the freeness of the individual Brieskorn lattices G_t (and consequently also of the Gauß-Manin systems \mathbf{G}_t)
558 follows from the fact that $f|_{D_t}$ is cohomological tame ([Sab06, theorem 10.1]). In our situation we have the
559 stronger statement of lemma 4.2, which gives the $\mathbb{C}[\tau^{-1}, t]$ -freeness of the whole module G .

560 Our next aim is to consider the so-called Birkhoff problem, that is, to find a basis $\underline{\omega}^{(1)}$ of G such that the
561 connection take the particularly simple form

$$\partial_\tau(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot (\Omega_0 + \tau^{-1} A_\infty),$$

562 (from now on, we write ∂_τ instead of ∇_{∂_τ} for short) where we require additionally that A_∞ is diagonal. We
 563 start with the basis $\underline{\omega}^{(0)}$, defined by

$$\omega_i^{(0)} := (-f)^{i-1} \cdot \alpha \quad \forall i \in \{1, \dots, n\}. \quad (4.2)$$

564 Then we have $\partial_\tau(\omega_i^{(0)}) = \omega_{i+1}^{(0)}$ for all $i \in \{1, \dots, n-1\}$ and

$$\partial_\tau(\omega_n^{(0)}) = (-f)^n \alpha.$$

565 As $\deg((-f)^n) = n$, $(-f)^n$ is a non-zero multiple of h in the Jacobian algebra $\mathbb{C}[V]/(df(Der(-\log h)))$, so that
 566 we have an expression $(-f)^n = c_0 \cdot h + \sum_{j=1}^{n-1} d_j^{(1)} \xi_j(f)$, where $c_0 \in \mathbb{C}^*$, $d_j^{(1)} \in \mathbb{C}[V]_{n-1}$. This gives by using
 567 lemma 4.3 again that

$$\partial_\tau(\omega_n^{(0)}) = (-f)^n \alpha = \left(c_0 t + \sum_{j=1}^{n-1} d_j^{(1)} \xi_j(f) \right) \alpha = \left(c_0 t + \tau^{-1} \sum_{j=1}^{n-1} \left(\xi_j(d_j^{(1)}) + \text{trace}(\xi_j) \right) \right) \alpha.$$

As $\deg(\xi_j(d_j^{(1)}) + \text{trace}(\xi_j)) = n-1$, there exist $c_1 \in \mathbb{C}$ and $d_r^{(2)} \in \mathbb{C}[V]_{n-2}$ such that

$$\begin{aligned} \left(\sum_{j=1}^{n-1} \left(\xi_j(d_j^{(1)}) + \text{trace}(\xi_j) \right) \right) \alpha &= \left(c_1 (-f)^{n-1} + \sum_{r=1}^{n-1} d_r^{(2)} \xi_r(f) \right) \alpha \\ &= \left(c_1 (-f)^{n-1} + \tau^{-1} \sum_{r=1}^{n-1} \left(\xi_r(d_r^{(2)}) + \text{trace}(\xi_r) d_r^{(2)} \right) \right) \alpha, \end{aligned}$$

568 and $\deg(\xi_r(d_r^{(2)}) + \text{trace}(\xi_r) d_r^{(2)}) = n-2$. We see by iteration that the connection operator ∂_τ takes the
 569 following form with respect to $\underline{\omega}^{(0)}$:

$$\partial_\tau(\underline{\omega}^{(0)}) = \underline{\omega}^{(0)} \cdot \begin{pmatrix} 0 & 0 & \dots & 0 & c_0 t + c_n \tau^{-n} \\ 1 & 0 & \dots & 0 & c_{n-1} \tau^{-n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & c_2 \tau^{-2} \\ 0 & 0 & \dots & 1 & c_1 \tau^{-1} \end{pmatrix} =: \underline{\omega}^{(0)} \cdot \Omega =: \underline{\omega}^{(0)} \cdot \left(\sum_{k=0}^n \Omega_k \tau^{-k} \right). \quad (4.3)$$

570 Notice that if D is special then $c_n = 0$, i.e., $\Omega_n = 0$.

571 The matrix Ω_0 has a very particular form, due to the fact that the jacobian algebra $h_* \mathcal{T}_{\mathcal{R}_n/\mathbb{C}}^1 f$ is generated
 572 by the powers of f . Notice also that the restriction $(\Omega_0)|_{t=0}$ is nilpotent, with a single Jordan block with
 573 eigenvalue zero. This reflects the fact that (G_0, ∇) is *regular singular* at $\tau = \infty$, which is not the case for any
 574 $t \neq 0$. Remember that although D is singular itself, so that it is not quite true that there is only one critical
 575 value of f on D , we have that f is regular on $D \setminus \{0\}$ in the stratified sense (see the proof of proposition 3.15).
 576 The particular form of the matrix Ω_0 is the key ingredient to solve the Birkhoff problem, which can actually be
 577 done by a triangular change of basis.

578 **Lemma 4.4.** *There exists a base change*

$$\omega_j^{(1)} := \omega_j^{(0)} + \sum_{i=1}^{j-1} b_i^j \tau^{-i} \omega_{j-i}^{(0)}, \quad (4.4)$$

579 such that the matrix of the connection with respect to $\underline{\omega}^{(1)}$ is given by

$$\Omega_0 + \tau^{-1} A_\infty,$$

580 where $A_\infty^{(1)}$ is diagonal. Moreover, if D is special, then b_i^j can be chosen such that $b_i^{i+1} = 0$ for $i = 1, \dots, n-1$.

581 *Proof.* Let us regard b_i^j as unknown constants to be determined and then let

$$B := (b_{j-k}^j \tau^{k-j})_{kj} =: \sum_{i=0}^{n-1} B_i \tau^{-i} = \text{Id} + \sum_{i=1}^{n-1} B_i \tau^{-i}.$$

582 Here $b_j^i = 0$ for $j < 0$. Notice that B_i is a matrix whose only non-zero entries are in the position $(j, j + i)$ for
 583 $j = 1, \dots, n - i$.

584 The matrix of the action of ∂_τ changes according to the formula:

$$X := B^{-1} \cdot \frac{dB}{d\tau} + B^{-1}\Omega B =: \sum_{i=0}^n X_i \tau^{-i}. \quad (4.5)$$

585 Multiplying by B both sides of the above equation we find

$$BX = \sum_{i=0}^n \left(\sum_{j=0}^i B_j X_{i-j} \right) \tau^{-i} = \sum_{i=0}^n (-(i-1)B_{i-1} + \Omega_0 B_i + \Omega_i) \tau^{-i}. \quad (4.6)$$

586 where $B_{-1} := 0$. Let $N = (n_{ij})$ be the matrix with $n_{ij} = 1$ if $j = i - 1$ or 0 otherwise. Hence $\Omega_0 = N + C_0$
 587 where C_0 is the matrix whose only non-zero entry is $c_0 t$ in the right top corner. It follows that $X_0 = \Omega_0$ and
 588 that

$$X_i = - \sum_{j=1}^{i-1} B_j X_{i-j} - [B_i, N] - (i-1)B_{i-1} + \Omega_i. \quad (4.7)$$

We are looking for a solution to the system $X_1 = A_\infty^{(1)}$, $X_i = 0$, $i = 2, \dots, n$, where $A_\infty^{(1)}$ is diagonal with entries yet to be determined. In view of the above, this system is equivalent to:

$$\begin{aligned} X_1 &= -[B_1, N] + \Omega_1 = A_\infty^{(1)}, \\ [B_{i+1}, N] &= -B_i X_1 - iB_i + \Omega_{i+1}, \quad i = 1, \dots, n-1. \end{aligned} \quad (4.8)$$

589 We are going to show that this system of polynomial equations in the variables b_i^j can always be reduced to a
 590 triangular system in b_1^j , so that there exists a solution. In particular, this determines the entries of the diagonal
 591 matrix $-[B_1, N] + \Omega_1$, i.e., the matrix $A_\infty^{(1)}$ we are looking for.

A direct calculation shows that if we substitute the first equation of (4.8) into the right hand side of the second one, we obtain $[B_{i+1}, N] = B_i([B_1, N] - \Omega_1 + i\text{Id}) + \Omega_{i+1} =: P^i$, where the only non-zero coefficients of the matrix P^i are $P_{j,i+j}^i$, namely:

$$\begin{aligned} P_{j,i+j}^i &= b_i^{i+j} (b_1^{i+j+1} - b_1^{i+j} + i), \quad j = 1, \dots, n-i-1, \\ P_{n-i,n}^i &= b_i^n (-b_1^n - c_1 + i) + c_{i+1}. \end{aligned} \quad (4.9)$$

592 A matrix B_{i+1} satisfying $[B_{i+1}, N] = P^i$ exists if and only if $Q^i := \sum_{j=1}^{n-i} P_{j,i+j}^i = 0$, and if this is case, the
 593 solution is given by setting:

$$b_{i+1}^{i+k+1} = - \sum_{j=k+1}^{n-i} P_{j,j+i}^i, \quad k = 1, \dots, n-i-1. \quad (4.10)$$

594 For $i = 1$ and $j = 1, \dots, n-1$, we have from (4.9)

$$P_{j,j+1}^1 = -(b_1^{j+1})^2 + \text{lower degree terms in } b_1^{j+1} \text{ with coefficients in } b_1^k, \quad k > j+1$$

595 so that substituting (4.10) in (4.9) for $i = 2$

$$P_{j,2+j}^2 = -(b_1^{j+2})^3 + \text{lower degree terms in } b_1^{j+2} \text{ with coefficients in } b_1^k, \quad k > j+2$$

596 By induction we see that after substitution we have

$$P_{j,i+j}^i = -(b_1^{i+j})^{i+1} + \text{lower degree terms in } b_1^{i+j} \text{ with coefficients in } b_1^k, \quad k > i+j$$

597 from which it follows that

$$Q^i = -(b_1^{i+1})^{i+1} + \text{lower degree terms in } b_1^{i+1} \text{ with coefficients in } b_1^k, \quad k > i+1$$

598 The system $Q^i = 0$, $i = 1, \dots, n-1$ is triangular (e.g., $Q^{n-1} \in \mathbb{C}[b_1^n]$) and thus has a solution.

599 In the case where D is special, the vanishing of Ω_n can be used to set $b_1^{i+1} = 0$ from the start. The above proof
 600 then works verbatim. \square

601 Notice that we can assume by a change of coordinates on T that the non-zero constant c_0 is actually normalized
602 to 1. We will make this assumption from now on.

603 In the next section, we are interested in constructing Frobenius structures associated to the tame functions $f|_{D_t}$
604 and to study their limit behaviour when t goes to zero. For that purpose, it is desirable to complete the relative
605 connection ∇ from above to an absolute one, which will acquire an additional pole at $t = 0$. This can be done in
606 general, however, in order to give an explicit expression for this connection, we will need the special form it takes
607 in the basis $\underline{\omega}^{(1)}$ as well as theorem 2.7. For this reason, we restrict to the reductive case here. It is however
608 true that the definition given below defines an integrable connection on \mathbf{G} in all cases, more precisely, it defines
609 the (partial Fourier-Laplace transformation of the) Gauß-Manin connection for the complete intersection given
610 by the two functions (f, h) . We will not discuss this in detail here.

611 The completion of the relative connection ∇ on \mathbf{G} referred to above is given by the formula

$$\nabla_{\partial_t}(\omega) := \frac{1}{n \cdot t} (L_E(\omega) - \tau L_E(f) \cdot \omega), \quad (4.11)$$

for any $[\omega] \in \Omega^{n-1}(\log h)$ and extending τ -linearly. One checks that

$$(t\nabla_{\partial_t})((d - \tau df \wedge)(\Omega^{n-2}(\log h)[\tau, \tau^{-1}])) \subset (d - \tau df \wedge)(\Omega^{n-2}(\log h)[\tau, \tau^{-1}]),$$

612 so that we obtain operator

$$\nabla : \mathbf{G} \longrightarrow \mathbf{G} \otimes \tau \Omega_{\mathbb{C} \times T}^1(\log \mathcal{D}), \quad (4.12)$$

613 where \mathcal{D} is the divisor $(\{0\} \times T) \cup (\mathbb{C} \times \{0\}) \subset \mathbb{C} \times T$.

614 **Proposition 4.5.** *Let D be reductive. Then:*

615 (i) *The elements of the basis $\underline{\omega}^{(1)}$ constructed above can be represented by differential forms $\omega_i^{(1)} = [g_i \alpha]$ with
616 g_i homogeneous of degree $i = 0, \dots, n-1$, i.e., by elements outside of $\tau^{-1} \Omega^{n-1}(\log h)[\tau^{-1}]$.*

617 (ii) *The connection operator defined above is flat outside $\theta = 0, t = 0$. We denote by \mathbf{G}^∇ the corresponding
618 local system and by \mathbf{G}^∞ its space of multivalued flat sections.*

(iii) *Consider the Gauß-Manin system, localized at $t = 0$, i.e.*

$$\mathbf{G}[t^{-1}] := \mathbf{G} \otimes_{\mathbb{C}[\tau, \tau^{-1}, t]} \mathbb{C}[\tau, \tau^{-1}, t, t^{-1}] \cong \frac{\Omega_{V/T}^{n-1}(*D)[\tau, \tau^{-1}]}{(d - \tau df \wedge) \Omega_{V/T}^{n-2}(*D)[\tau, \tau^{-1}]}$$

and similarly, the localized Brieskorn lattice

$$G[t^{-1}] := G \otimes_{\mathbb{C}[\tau^{-1}, t]} \mathbb{C}[\tau^{-1}, t, t^{-1}] \cong \frac{\Omega_{V/T}^{n-1}(*D)[\tau^{-1}]}{(\tau^{-1}d - df \wedge) \Omega_{V/T}^{n-2}(*D)[\tau^{-1}]} \subset \mathbf{G}[t^{-1}].$$

619 Then $\underline{\omega}^{(1)}$ provides a solution to the Birkhoff problem for $(G[t^{-1}], \nabla)$ “in a family”, i.e., an extension to
620 a trivial algebraic bundle $\widehat{G}[t^{-1}] \subset \widetilde{i}_* G[t^{-1}]$ (here $\widetilde{i} : \mathbb{C} \times (T \setminus \{0\}) \hookrightarrow \mathbb{P}^1 \times (T \setminus \{0\})$) on $\mathbb{P}^1 \times (T \setminus \{0\})$, on
621 which the connection has a logarithmic pole along $\{\infty\} \times (T \setminus \{0\})$ (and, as before, a pole of type one along
622 $\{0\} \times (T \setminus \{0\})$).

623 (iv) *Let γ resp. γ' be a small counterclockwise loop around the divisor $\{0\} \times T$ resp. $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times T$. Let
624 M resp. M' denote the monodromy endomorphisms on \mathbf{G}^∞ corresponding to γ resp. γ' . Then*

$$M^{-1} = (M')^n.$$

625 (v) *Let $u : \mathbb{C}^2 \rightarrow \mathbb{C} \times T$, $(\theta, s) \mapsto (\theta, s^n)$. Consider the pullback $u^*(G, \nabla)$ and denote by (\widetilde{G}, ∇) the restriction
626 to $\mathbb{C} \times \mathbb{C}^*$ of the analytic bundle corresponding to $u^*(G, \nabla)$. Then \widetilde{G} underlies a Sabbah-orbit of TERP-
627 structures, as defined in [HS07, definition 4.1].*

628 *Proof.* (i) It follows from theorem 2.7 that for $g \in \mathbb{C}[V]_i$ with $1 < i < n$, the $n-1$ -form $g\alpha$ is exact in the
629 complex $\Omega^\bullet(\log h)$. Therefore in \mathbf{G} we have $\tau^{-1}g\alpha = \tau^{-1}d\omega' = df \wedge \omega' = g'\alpha$ for some $\omega' \in \Omega^{n-2}(\log h)$
630 and $g' \in \mathbb{C}[V]$. Note that necessarily $g' \in \mathbb{C}[V]_{i+1}$. Moreover, in the above constructed base change matrix
631 we had $B_{1l} = \delta_{1l}$ (as D is reductive hence special), which implies that for all $i > 0$, $\omega_i^{(1)}$ is represented
632 by an element in $f\mathbb{C}[V]\alpha[\tau^{-1}]$, i.e., by a sum of terms of the form $\tau^{-k}g\alpha$ with $g \in \mathbb{C}[V]_{\geq 1}$. This proves
633 that we can successively erase all negative powers of τ , i.e., represent all $\omega_i^{(1)}$, $i > 0$ by pure forms (i.e.,
634 without τ^{-1} 's), and $\omega_0^{(1)} = \omega_0^{(0)} = \alpha$ is pure anyhow.

635 (ii) From (i) and the definition of ∇_{∂_t} in (4.11) we obtain

$$\nabla(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left[(\Omega_0 + \tau^{-1} A_\infty^{(1)}) d\tau + (D + \tau\Omega_0 + A_\infty^{(1)}) \frac{dt}{nt} \right]$$

where $D = \text{diag}(0, \dots, n-1)$. The flatness conditions of an arbitrary connection of the form

$$\nabla(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left[(\tau A + B) \frac{d\tau}{\tau} + (\tau A' + B') \frac{dt}{t} \right]$$

with $A, A' \in M(n \times n, \mathbb{C}[t])$ and $B, B' \in M(n \times n, \mathbb{C})$ is given by the following system of equations:

$$[A, A'] = 0 \quad ; \quad [B, B'] = 0 \quad ; \quad (t\partial_t)A - A' = [A, B'] - [A', B]$$

636 One checks that for $A = \Omega_0$, $A' = \frac{1}{n}\Omega_0$, $B = A_\infty^{(1)}$ and $B' = \frac{1}{n}(A_\infty^{(1)} + D)$, these equations are satisfied.

637 (iii) The extension defined by $\underline{\omega}^{(1)}$, i.e., $\widehat{G}[t^{-1}] := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1 \times T}[t^{-1}] \omega_i^{(1)}$ provides the solution in a family to the
638 Birkhoff problem, i.e., we have $\nabla_X \widehat{G}[t^{-1}] \subset \widehat{G}[t^{-1}]$ for any $X \in \text{Der}(-\log(\{\infty\} \times (T \setminus \{0\})))$.

639 (iv) If we restrict (\mathbf{G}, ∇) to the curve $C := \{(\tau, t) \in (\mathbb{C}^*)^2 \mid \tau^n t = 1\}$ we obtain

$$\nabla|_C = -D \frac{d\tau}{\tau}.$$

640 As the diagonal entries of D are integers, the monodromy of $(\mathbf{G}, \nabla)|_C$ is trivial which implies the result
641 (notice that the composition of γ_1 and γ_2 is homotopic to a loop around the origin in C).

642 (v) That the restriction to $\mathbb{C} \times (T \setminus \{0\})$ of the (analytic bundle corresponding to) G underlies a variation of
643 pure polarized TERP-structures is a general fact, due to the tameness of the functions $f|_{D_t}$ (see [Sab06]
644 and [Sab08], [HS07, theorem 11.1]). Using the connection matrix from (ii), it is an easy calculation that
645 $\nabla_{s\partial_s - \tau\partial_\tau}(\widetilde{\omega}^{(1)}) = 0$, where $\widetilde{\omega}^{(1)} := u^* \underline{\omega}^{(1)} \cdot s^{-D}$ so that $(\widetilde{G}, \widetilde{\nabla})$ satisfies condition 2.(a) in [HS07, definition
646 4.1].

647 □

648 For the purpose of the next section, we need to find a much more special solution to the Birkhoff problem,
649 which is called V^+ -solution in [DS03]. It takes into account the Kashiwara-Malgrange filtration of \mathbf{G} at infinity
650 (i.e., at $\tau = 0$). We briefly recall the notations and explain how to construct the V^+ -solution starting from our
651 basis $\underline{\omega}^{(1)}$.

652 Fix $t \in T$ and consider, as before, the restrictions \mathbf{G}_t resp. G_t of the family of Gauß-Manin-systems resp.
653 Brieskorn lattices \mathbf{G} resp. G . As already pointed out, for $t \neq 0$, these are the Gauß-Manin-system resp. the
654 Brieskorn lattice of the tame of the function $f|_{D_t}$. \mathbf{G}_t is known to be a holonomic left $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ -module, with
655 singularities at $\tau = 0$ and $\tau = \infty$ only. The one at infinity, i.e. $\tau = 0$ is regular singular, but not necessarily the
656 one at zero (at $\tau = \infty$). Similarly to the notation used above, we have the the local system \mathbf{G}_t^∇ and its space
657 of multivalued global flat sections \mathbf{G}_t^∞ . Recall that for any $t \neq 0$, the monodromy of \mathbf{G}_t^∇ is quasi-unipotent,
658 so any logarithm of any of its eigenvalues is a rational number. As we will see in section 6, the same is true
659 in all examples for $t = 0$, but this is not proved for the moment. Let \mathbb{K} be either \mathbb{C} or \mathbb{Q} , depending on
660 whether $t = 0$ or $t \neq 0$. In the former case, we chose the lexicographic ordering on \mathbb{C} which extends the usual
661 ordering of \mathbb{R} . Recall that there is a unique increasing exhaustive filtration $V_\bullet \mathbf{G}_t$ indexed by \mathbb{K} , called the
662 Kashiwara-Malgrange or canonical V-filtration on \mathbf{G}_t with the properties

663 (i) It is a good filtration with respect to the V-filtration $V_\bullet \mathbb{C}[\tau]\langle\partial_\tau\rangle$ of the Weyl-algebra, i.e. it satisfies
664 $V_k \mathbb{C}[\tau]\langle\partial_\tau\rangle V_l \mathbf{G}_t \subset V_{k+l} \mathbf{G}_t$ and this is an equality for any $k \leq 0, l \leq -l_0, k \geq 0, l \geq l_0$ for some sufficiently
665 large positive integer l_0 .

666 (ii) For any $\alpha \in \mathbb{K}$, the operator $\tau\partial_\tau + \alpha$ is nilpotent on the quotient $gr_\alpha^V \mathbf{G}_t$

667 We have an induced V-filtration on the Brieskorn lattice G_t , and we denote by

$$\text{Sp}(G_t, \nabla) := \sum_{\alpha \in \mathbb{K}} \dim_{\mathbb{C}} \left(\frac{V_\alpha \cap G_t}{V_{<\alpha} \cap G_t + \tau^{-1} G_t \cap V_\alpha} \right) \alpha \in \mathbb{Z}[\mathbb{K}]$$

668 the spectrum of G_t at infinity (for $t \neq 0$ it is also called the spectrum at infinity associated to $f|_{D_t}$). We also
669 write it as an ordered tuple of (possibly repeated) rational numbers $\alpha_1 \leq \dots \leq \alpha_n$. We recall the following
670 notions from [DS03, appendix B].

671 **Lemma and Definition 4.6.** (i) *The following conditions are equivalent.*

672 (a) *There is a solution to the Birkhoff problem, i.e, a basis $\underline{\omega}$ of G_t with $\partial_\tau(\underline{\omega}) = \underline{\omega}(\Omega_0 + \tau^{-1}A_\infty)$ (where*
673 *A_∞ is not necessarily semi-simple).*

674 (b) *There is a $\mathbb{C}[\tau]$ -lattice G'_t of \mathbf{G}_t which is stable under $\tau\partial_\tau$, and such that $G_t = (G_t \cap G'_t) \oplus \tau^{-1}G_t$.*

675 (c) *There is an extension to a free $\mathcal{O}_{\mathbb{P}^1}$ -module $\widehat{G}_t \subset \widetilde{i}_*G_t$ (where $\widetilde{i} : \mathbb{C} \hookrightarrow \mathbb{P}^1$) with the property that*
676 *$(\tau\nabla_\tau)\widehat{G}_t \subset \widehat{G}_t$.*

677 (ii) *A solution to the Birkhoff problem G'_t is called a V -solution iff*

$$G_t \cap V_\alpha \mathbf{G}_t = (G_t \cap G'_t \cap V_\alpha \mathbf{G}_t) \oplus (\tau^{-1}G_t \cap V_\alpha \mathbf{G}_t).$$

678 (iii) *G'_t is called a V^+ -solution if moreover we have*

$$(\tau\partial_\tau + \alpha)(G_t \cap G'_t \cap V_\alpha \mathbf{G}_t) \subset (G_t \cap G'_t \cap V_{<\alpha} \mathbf{G}_t) \oplus \tau(G_t \cap G'_t \cap V_{\alpha+1} \mathbf{G}_t).$$

679 *In this case, a basis as in (i) (a) can be chosen such that the matrix A_∞ is diagonal, and the diagonal*
680 *entries are the spectral numbers of (G_t, ∇) at infinity.*

681 (iv) *Suppose that we are moreover given a non-degenerate flat Hermitian pairing on \mathbf{G}_t which has weight $n-1$*
682 *on G_t , more precisely (see [DS03, section 1.f.] or [DS04, section 4]) a morphism $S : \mathbf{G}_t \otimes_{\mathbb{C}[\tau, \tau^{-1}]} \overline{\mathbf{G}}_t \rightarrow$*
683 *$\mathbb{C}[\tau, \tau^{-1}]$ (where $\overline{\mathbf{G}}_t$ denotes the module \mathbf{G}_t on which τ acts as $-\tau$) with the following properties*

684 (a) $\tau\partial_\tau S(a, \bar{b}) = S(\tau\partial_\tau a, \bar{b}) + S(a, \tau\partial_\tau \bar{b})$,

685 (b) $S : V_0 \otimes \overline{V}_{<1} \rightarrow \mathbb{C}[\tau]$,

686 (c) $S(G_t, \overline{G}_t) \subset \tau^{-n+1}\mathbb{C}[\tau^{-1}]$, and the induced symmetric pairing $G_t/\tau^{-1}G_t \otimes G_t/\tau^{-1}G_t \rightarrow \tau^{-n+1}\mathbb{C}$ is
687 non-degenerate.

688 *In particular, the spectral numbers then obey the symmetry $\alpha_1 + \alpha_{n+1-i} = n-1$. A V^+ -solution G'_t is*
689 *called (V^+, S) -solution if $S(G_t \cap G'_t, \overline{G}_t \cap \overline{G}'_t) \subset \mathbb{C}\tau^{-n+1}$.*

690 We will see in the sequel (theorem 4.13) that under a technical hypothesis (which is however satisfied in many
691 examples) we are able to construct directly a (V^+, S) -solution. Without this hypothesis, we can for the moment
692 only construct a V^+ -solution. In order to obtain Frobenius structures in all cases, we need the following general
693 result, which we quote from [Sab06] and [DS03].

694 **Theorem 4.7.** *Let Y be a smooth affine complex algebraic variety and $f : Y \rightarrow \mathbb{C}$ be a cohomologically tame*
695 *function. Then the Gauß-Manin system of f is equipped a pairing S as above, and there is a canonical (V^+, S) -*
696 *solution to the Birkhoff problem for the Brieskorn lattice of f , defined by a (canonical choice of an) opposite*
697 *filtration to the Hodge filtration of the mixed Hodge structure associated to f .*

698 The key tool to compute the spectrum and to obtain such a V^+ -solution to the Birkhoff problem is the following
699 result.

700 **Proposition 4.8.** *Let $t \in T$ be arbitrary, $G_t \subset \mathbf{G}_t$ as before and consider any solution to the Birkhoff problem*
701 *for (G_t, ∇) , given by a basis $\underline{\omega}$ of G_t such that $\partial_\tau(\underline{\omega}) = \underline{\omega}(\Omega_0 + \tau^{-1}A_\infty)$ with Ω_0 as above and such that*
702 *$A_\infty = \text{diag}(-\nu_1, \dots, -\nu_n)$ is diagonal. Suppose moreover that $\nu_i - \nu_{i-1} \leq 1$ for all $i \in \{1, \dots, n\}$ and additionally*
703 *that $\nu_1 - \nu_n \leq 1$ if $t \neq 0$.*

704 *Then $\underline{\omega}$ is a V^+ -solution to the Birkhoff problem and the numbers $(-\nu_i)_{i=1, \dots, n}$ give the spectrum $\text{Sp}(G_t, \nabla)$ of*
705 *G_t at infinity.*

Proof. The basic idea is similar to [DS04] and [dG07], namely, that the spectrum of A_∞ can be used to define
a filtration which turns out to coincide with the V -filtration using that the latter is unique with the above
properties. More precisely, we define a \mathbb{K} -grading on \mathbf{G}_t by $\deg(\tau^k \omega_i) := \nu_i - k$ and consider the associated
increasing filtration $\widetilde{V}_\bullet \mathbf{G}_t$ given by

$$\begin{aligned} \widetilde{V}_\alpha \mathbf{G}_t &:= \left\{ \sum_{i=1}^n c_i \tau^{k_i} \omega_i \in \mathbf{G}_t \mid \max_i (\nu_i - k_i) \leq \alpha \right\} \\ \widetilde{V}_{<\alpha} \mathbf{G}_t &:= \left\{ \sum_{i=1}^n c_i \tau^{k_i} \omega_i \in \mathbf{G}_t \mid \max_i (\nu_i - k_i) < \alpha \right\}. \end{aligned}$$

706 By definition $\partial_\tau \widetilde{V}_\bullet \mathbf{G}_t \subset \widetilde{V}_{\bullet+1} \mathbf{G}_t$ and $\tau \widetilde{V}_\bullet \mathbf{G}_t \subset \widetilde{V}_{\bullet-1} \mathbf{G}_t$ and moreover, τ is obviously bijective on \mathbf{G} . Thus to
707 verify that $V_\bullet \mathbf{G}_t = \widetilde{V}_\bullet \mathbf{G}_t$, we only have to show that $\tau \partial_\tau + \alpha$ is nilpotent on $gr_\alpha^{\widetilde{V}} \mathbf{G}_t$. This will prove both
708 statements of the proposition: the conditions in definition 4.6 for $\underline{\omega}$ to be a V^+ -solution are trivially satisfied if
709 we replace V by \widetilde{V} . The nilpotency of $\tau \partial_\tau + \alpha \in \text{End}_{\mathbb{C}}(gr_\alpha^{\widetilde{V}} \mathbf{G}_t)$ follows from the assumption $\nu_i - \nu_{i-1} \leq 1$:
710 First define a block decomposition of the ordered tuple $(1, \dots, n)$ by putting $(1, \dots, n) = (I_1, \dots, I_s)$, where
711 $I_r = (i_r, i_r + 1, \dots, i_r + l_r = i_{r+1} - 1)$ such that $\nu_{i_r+l_r+1} - \nu_{i_r+l_r} = 1$ for all $l \in \{0, \dots, l_r - 1\}$ and $\nu_{i_r} - \nu_{i_r-1} < 1$,
712 $\nu_{i_{r+1}} - \nu_{i_r+l_r} < 1$. Then in \mathbf{G}_t we have $(\tau \partial_\tau + (\nu_i - k_i))(\tau^{k_i} \omega_i) = \tau^{k_i+1} \omega_{i+1}$ for $i \in \{1, \dots, n-1\}$ and
713 $(\tau \partial_\tau + (\nu_n - k_n))(\tau^{k_n} \omega_n) = t \tau^{k_n+1} \omega_1$, so that $(\tau \partial_\tau + (\nu_i - k_i))^{i_{r+1}-i}(\tau^{k_i} \omega_i) = 0$ in $gr_{\nu_i - k_i}^{\widetilde{V}} \mathbf{G}_t$ for all $i \in I_r$ (here
714 we put $i_{s+1} := n+1$, note also that if $t \neq 0$ we suppose that $\nu_1 - \nu_n \leq 1$). \square

715 As a by-product, a solution with the above properties also makes it possible to compute the monodromy of \mathbf{G}_t .
716 Consider the local system \mathbf{G}_t^∇ and the space \mathbf{G}_t^∞ of its multivalued flat sections. There is a natural isomorphism
717 $\oplus_{\alpha \in (0,1]} gr_\alpha^V \mathbf{G}_t \xrightarrow{\psi} \mathbf{G}_t^\infty$. The monodromy $M \in \text{Aut}(\mathbf{G}_t^\infty)$, which corresponds to a counter-clockwise loop around
718 $\tau = \infty$, decomposes as $M = M_s \cdot M_u$ into semi-simple and unipotent part, and we write $N := \log(M_u)$ for
719 the nilpotent part of M . N corresponds under the isomorphism ψ , up to a constant factor, to the operator
720 $\oplus_{\alpha \in (0,1]} (\tau \partial_\tau + \alpha) \in \oplus_{\alpha \in (0,1]} \text{End}_{\mathbb{C}}(gr_\alpha^V \mathbf{G})$. This gives the following result, notice that a similar statement and
721 proof are given in [DS04, end of section 3].

722 **Corollary 4.9.** *Consider the basis of \mathbf{G}_t^∞ induced from a basis $\underline{\omega}$ as above, i.e.,*

$$\mathbf{G}_t^\infty = \oplus_{i=1}^n \mathbb{C} \psi^{-1}([\tau^{l_i} \omega_i]),$$

723 where $l_i = \lfloor \nu_i \rfloor + 1$. Then $M_s \psi^{-1}[\tau^{l_i} \omega_i] = e^{-2\pi i \nu_i} \cdot \psi^{-1}[\tau^{l_i} \omega_i]$ and

$$N(\psi^{-1}[\tau^{l_i} \omega_i]) = \begin{cases} 2\pi i \psi^{-1}[\tau^{l_i} \omega_{i+1}] & \text{if } \nu_{i+1} - \nu_i = 1 \\ 0 & \text{else,} \end{cases}$$

724 where $\omega_{n+1} = \omega_1$ if $t \neq 0$ and $\omega_{n+1} = 0$ if $t = 0$. Thus the Jordan blocks of N are exactly the blocks appearing
725 above in the decomposition of the tuple $(1, \dots, n)$.

726 We can now use proposition 4.8 to compute a V^+ -solution and the spectrum of G_t . We give an explicit algorithm,
727 which we split into two parts for the sake of clarity. Once again it should be emphasized that the special form
728 of the matrix Ω_0 is the main ingredient for this algorithm.

729 *Algorithm 1.* Given $\underline{\omega}^{(1)}$ from lemma 4.4, i.e., $\partial_\tau(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)}(\Omega_0 + \tau^{-1} A_\infty)$ and $A_\infty^{(1)} = \text{diag}(-\nu_1^{(1)}, \dots, -\nu_n^{(1)})$,
730 whenever there is $i \in \{2, \dots, n\}$ with $\nu_i^{(1)} - \nu_{i-1}^{(1)} > 1$, put

$$\begin{aligned} \widetilde{\omega}_i^{(1)} &:= \omega_i^{(1)} + \tau^{-1}(\nu_i^{(1)} - \nu_{i-1}^{(1)} - 1)\omega_{i-1}^{(1)} \\ \widetilde{\omega}_j^{(1)} &:= \omega_j^{(1)} \quad \forall j \neq i \end{aligned} \tag{4.13}$$

731 so that $\partial_\tau(\widetilde{\omega}^{(1)}) = \widetilde{\omega}^{(1)}(\Omega_0 + \tau^{-1} \widetilde{A}_\infty^{(1)})$ and $\widetilde{A}_\infty^{(1)} = \text{diag}(-\widetilde{\nu}_1^{(1)}, \dots, -\widetilde{\nu}_n^{(1)})$, where $\widetilde{\nu}_i^{(1)} = \nu_{i-1}^{(1)} + 1$, $\widetilde{\nu}_{i-1}^{(1)} = \nu_i^{(1)} - 1$
732 and $\widetilde{\nu}_j^{(1)} = \nu_j$ for any $j \notin \{i, i-1\}$. Restart algorithm 1 with input $\widetilde{\omega}^{(1)}$.

733 Now we have

734 **Lemma 4.10.** *Given any basis $\underline{\omega}^{(1)}$ of G_t as above, algorithm 1 terminates. Its output $\underline{\omega}^{(2)}$ is a V^+ -solution
735 for G_t if $t = 0$.*

736 *Proof.* The first statement is a simple analysis on the action of the algorithm on the array $(\nu_1^{(1)}, \dots, \nu_n^{(1)})$,
737 namely, if $(\nu_1^{(1)}, \dots, \nu_k^{(1)})$ is ordered (i.e., $\nu_i^{(1)} - \nu_{i-1}^{(1)} \leq 1$ for all $i \in \{2, \dots, k\}$), then after a finite number of steps
738 the array $(\widetilde{\nu}_1^{(1)}, \dots, \widetilde{\nu}_{k+1}^{(1)})$ will be ordered. This shows that the algorithm will eventually terminate. Its output
739 is then a V^+ -solution for G_t if $t = 0$ by proposition 4.8. \square

740 If we want to compute the spectrum and a V^+ -solution of G_t for $t \neq 0$, we also have to make sure that
741 $\nu_1 - \nu_n \leq 1$. This is done by the following procedure.

742 *Algorithm 2.* Run algorithm 1 on the input $\underline{\omega}^{(1)}$ with output $\underline{\omega}^{(2)}$ where $A_\infty^{(2)} = (-\nu_1^{(2)}, \dots, -\nu_n^{(2)})$. As long as
743 $\nu_1^{(2)} - \nu_n^{(2)} > 1$, put

$$\begin{aligned} \widetilde{\omega}_1^{(2)} &:= t \omega_1^{(2)} + \tau^{-1}(\nu_1^{(2)} - \nu_n^{(2)} - 1)\omega_n^{(2)} \\ \widetilde{\omega}_i^{(2)} &:= t \omega_i^{(2)} \quad \forall i \neq 1. \end{aligned} \tag{4.14}$$

744 so that $\partial_\tau(\tilde{\omega}^{(2)}) = \tilde{\omega}^{(2)}(\Omega_0 + \tau^{-1}\tilde{A}_\infty^{(2)})$ where $\tilde{A}_\infty^{(2)} = \text{diag}(-\tilde{\nu}_1^{(2)}, \dots, -\tilde{\nu}_n^{(2)})$, where $\tilde{\nu}_1^{(2)} = \nu_n^{(2)} + 1$, $\tilde{\nu}_n^{(2)} = \nu_1^{(2)} - 1$
745 and $\tilde{\nu}_i^{(2)} = \nu_i^{(2)}$ for any $i \notin \{1, n\}$. Run algorithm 2 again on input $\tilde{\omega}^{(2)}$.

746 **Lemma 4.11.** *Let $t \neq 0$, given any solution $\omega^{(1)}$ to the Birkhoff problem for G_t , such that $\partial_\tau(\omega^{(1)}) = \omega^{(1)}(\Omega_0 +$
747 $\tau^{-1}A_\infty^{(1)})$ with Ω_0 as above and $A_\infty^{(1)}$ diagonal, then algorithm 2 with input $\underline{\omega}^{(1)}$ terminates and yields a basis
748 $\underline{\omega}^{(3)}$ with $\partial_\tau(\underline{\omega}^{(3)}) = \underline{\omega}^{(3)}(\Omega_0 + \tau^{-1}A_\infty^{(3)})$, where $A_\infty^{(3)} = (-\nu_1^{(3)}, \dots, -\nu_n^{(3)})$ with $\nu_{i+1}^{(3)} - \nu_i^{(3)} \leq 1$ for $i \in \{1, \dots, n\}$
749 (here $\nu_{n+1}^{(3)} := \nu_1^{(3)})$.*

750 *Proof.* We only have to prove that algorithm 2 terminates. This is easily be done by showing that in each
751 step, the number $\tilde{\nu}_1^{(2)} - \tilde{\nu}_n^{(2)}$ does not increase, that it strictly decreases after a finite number of steps, and that
752 the possible values for this number are contained in the set $\{a - b \mid a, b \in \{\nu_1, \dots, \nu_n\}\} + \mathbb{Z}$ (which have no
753 accumulation points), so that after a finite number of steps we necessarily have $\tilde{\nu}_1^{(2)} - \tilde{\nu}_n^{(2)} \leq 1$. \square

754 Note that for any fixed $t \neq 0$, algorithm 2 produces a base change of G_t , but this does not lift to a base change
755 of G itself, i.e., $G^{(3)} := \bigoplus_{i=1}^n \mathbb{C}[\tau^{-1}, t]\omega_i^{(3)}$ is a proper free submodule of G which coincides with G only after
756 localization off $t = 0$. In other words, it is a $\mathbb{C}[t]$ -lattice of $G[t^{-1}]$ which is in general different from G .

757 Summarizing the above calculations, we have shown the following.

758 **Corollary 4.12.** (i) *Let $D \subset V$ be a linear free divisor with defining equation $h \in \mathbb{C}[V]_n$, seen as a morphism
759 $h : V \rightarrow T$. Let $f \in \mathbb{C}[V]_1$ be linear and \mathcal{R}_h -finite. Then for any $t \in T$, there is a V^+ -solution of the
760 Birkhoff problem for (G_t, ∇) , defined by bases $\underline{\omega}^{(2)}$ if $t = 0$ resp. $\underline{\omega}^{(3)}$ if $t \neq 0$ as constructed above. If
761 $\nu_1^{(2)} - \nu_n^{(2)} \leq 1$ then $\underline{\omega}^{(3)} = \underline{\omega}^{(2)}$. Moreover, we have that $\underline{\omega}_i^{(2)} - (-f)^{i-1}\alpha$ and $\omega_i^{(3)} - (-f)^{i-1}\alpha$ lie in
762 $\tau^{-1}G_t$ for all $i \in \{1, \dots, n\}$.*

(ii) *Let D be reductive. Then the integrable connection ∇ on $\mathbf{G}[t^{-1}]$ defined by formula (4.11) takes the
following form in the basis $\underline{\omega}^{(3)}$:*

$$\nabla(\underline{\omega}^{(3)}) = \underline{\omega}^{(3)} \cdot \left[(\Omega_0 + \tau^{-1}A_\infty^{(3)})d\tau + (\tilde{D} + \tau\Omega_0 + A_\infty^{(3)})\frac{dt}{nt} \right]$$

763 where $\tilde{D} := D + k \cdot n \cdot \text{Id}$, here k is the number of times the (meromorphic) base change (4.14) in algorithm
764 2 is performed.

765 Hence, in the reductive case, $\underline{\omega}^{(3)}$ gives a V^+ -solution to the Birkhoff problem for $(G[t^{-1}], \nabla)$.

766 *Proof.* Starting with the basis $\omega_i^{(0)} = (-f)^{i-1}\alpha$ of G_t , we construct $\underline{\omega}^{(2)}$ resp. $\underline{\omega}^{(3)}$ using lemma 4.4, proposition
767 4.8 and lemma 4.10 resp. lemma 4.11. In both cases, the base change matrix $P \in \text{Gl}(n, \mathbb{C}[\tau^{-1}])$ defined by
768 $\underline{\omega}^{(2)} = \underline{\omega}^{(0)} \cdot P$ resp. $\underline{\omega}^{(3)} = \underline{\omega}^{(0)} \cdot P$ has the property that $P - \text{Id} \in \tau^{-1}\text{Gl}(n, \mathbb{C}[\tau^{-1}])$ which shows the second
769 statement of the first part. As to the second part, one checks that the base change steps (4.13) performed in
770 algorithm 1 have the effect that $n \cdot t\partial_t(\tilde{\omega}^{(1)}) = \tilde{\omega}^{(1)}(\tau\Omega_0 + D + \tilde{A}_\infty^{(1)})$, whereas step (4.14) in algorithm 2 gives
771 $n \cdot t\partial_t(\tilde{\omega}^{(2)}) = \tilde{\omega}^{(2)}(\tau\Omega_0 + D + n \cdot \text{Id} + \tilde{A}_\infty^{(2)})$. \square

772 As already indicated above, we can show that the solution obtained behaves well with respect to the pairing S ,
773 provided that a technical hypothesis holds true. More precisely, we have the following statement.

774 **Theorem 4.13.** *Let $t \neq 0$. Suppose that the minimal spectral number of the tame function $f|_{D_t}$ is of multiplicity
775 one, i.e., there is a unique $i \in \{1, \dots, n\}$ such that $-\nu_i^{(3)} = \min_{j \in \{1, \dots, n\}}(-\nu_j^{(3)})$. Then $\underline{\omega}^{(3)}$ is a (V^+, S) -solution
776 of the Birkhoff problem for (G_t, ∇) , i.e., $S(G_t \cap G'_t, \bar{G}_t \cap \bar{G}'_t) \subset \mathbb{C}\tau^{-w}$, where $G'_t := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1 \setminus \{0\} \times \{t\}} \omega_i^{(3)}$.*

777 *Proof.* The proof is essentially a refined version of the proof of the similar statement [DS04, lemma 4.1].
778 Denote by $\alpha_1, \dots, \alpha_n$ a non-decreasing sequence of rational numbers such that we have an equality of sets
779 $\{-\nu_1^{(3)}, \dots, -\nu_n^{(3)}\} = \{\alpha_1, \dots, \alpha_n\}$. Then, as was stated in lemma 4.6 (iv), we have $\alpha_i + \alpha_{n+1-i} = n - 1$ for all
780 $k \in \{1, \dots, n\}$.

Let i be the index of the smallest spectral number $-\nu_i^{(3)}$. The symmetry $\alpha_k + \alpha_{n+1-k} = n - 1$ implies that there
is a unique $j \in \{1, \dots, n\}$ such that $\nu_i^{(3)} + \nu_j^{(3)} = n - 1$, or, equivalently, that $-\nu_j^{(3)} = \max_{l \in \{1, \dots, n\}}(-\nu_l^{(3)})$.
Then, as in the proof of loc.cit., we have that for all $l \in \{1, \dots, n\}$

$$S(\omega_i^{(3)}, \bar{\omega}_l^{(3)}) = \begin{cases} 0 & \text{if } l \neq j \\ c \cdot \tau^{-n+1}, c \in \mathbb{C} & \text{if } l = j \end{cases}$$

781 This follows from the compatibility of S with the V -filtration and the pole order property of S on the Brieskorn
782 lattice G_t (i.e., properties (iv) (b) and (c) in definition 4.6). Suppose without loss of generality that $i < j$, if
783 $i = j$, i.e., if there is only one spectral number, then the result is clear. Now the proof of the theorem follows
784 from the next lemma. \square

785 **Lemma 4.14.** *Let i and j as above. Then*

(i) *For any $k \in \{i, \dots, j\}$, we have*

$$S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) = \begin{cases} 0 & \text{for all } l \neq i + j - k \\ S(\omega_i^{(3)}, \bar{\omega}_j^{(3)}) \text{ and } \nu_k^{(3)} + \nu_l^{(3)} = -n + 1 & \text{for } l = i + j - k \end{cases}$$

(ii) *For any $k \in \{1, \dots, n\} \setminus \{i, \dots, j\}$, we have that*

$$S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) = \begin{cases} 0 & \text{for all } l \neq i + j - k \\ c_{kl} \cdot S(\omega_i^{(3)}, \bar{\omega}_j^{(3)}) \text{ and } \nu_k^{(3)} + \nu_l^{(3)} = -n + 1 & \text{for } l = i + j - k \end{cases}$$

786 where $c_{kl} \in \mathbb{C}$.

Proof. (i) We will prove the statement by induction over k . It is obviously true for $k = i$ by the hypothesis above. Hence we suppose that there is $r \in \{i, \dots, j\}$ such that statement in (i) is true for all k with $i \leq k < r \leq j$. The following identity is a direct consequence of property (iv) (a) in definition (4.6).

$$\begin{aligned} (\tau \partial_\tau + (n-1))S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) &= S(\tau \partial_\tau \omega_k^{(3)}, \bar{\omega}_l^{(3)}) + S(\omega_k^{(3)}, \tau \partial_\tau \bar{\omega}_l^{(3)}) + (n-1)S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) \\ &= S(\tau \partial_\tau \omega_k^{(3)}, \bar{\omega}_l^{(3)}) + S(\omega_k^{(3)}, \overline{\tau \partial_\tau \omega_l^{(3)}}) + (n-1)S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) = \\ &= S(\tau \omega_{k+1}^{(3)} - \nu_k^{(3)} \omega_k^{(3)}, \bar{\omega}_l^{(3)}) + S(\omega_k^{(3)}, \overline{\tau \omega_{l+1}^{(3)} - \nu_l^{(3)} \omega_l^{(3)}}) + (n-1)S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) = \\ &= (n-1 - \nu_k^{(3)} - \nu_l^{(3)})S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) + \tau \left(S(\omega_{k+1}^{(3)}, \bar{\omega}_l^{(3)}) - S(\omega_k^{(3)}, \bar{\omega}_{l+1}^{(3)}) \right). \end{aligned}$$

787 By induction hypothesis, we have that $(\tau \partial_\tau + (n-1))S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) = 0$ for all $l \in \{1, \dots, n\}$.

788 Now we distinguish several cases depending on the value of l : If $l \notin \{i + j - k, i + j - k - 1\}$, then by the
789 induction hypothesis, both $S(\omega_k^{(3)}, \bar{\omega}_l^{(3)})$ and $S(\omega_k^{(3)}, \bar{\omega}_{l+1}^{(3)})$ are zero. Hence it follows that $S(\omega_{k+1}^{(3)}, \bar{\omega}_l^{(3)}) = 0$
790 in this case.

791 If $l = i + j - k$ then again by the induction hypothesis we know that $(n-1) - \nu_k^{(3)} - \nu_l^{(3)} = 0$ and that
792 moreover $S(\omega_k^{(3)}, \bar{\omega}_{l+1}^{(3)}) = 0$. Thus we have $S(\omega_{k+1}^{(3)}, \bar{\omega}_{i+j-k}^{(3)}) = 0$.

Finally, if $l = i + j - k - 1$, then $S(\omega_k^{(3)}, \bar{\omega}_l^{(3)}) = 0$, and so $S(\omega_{k+1}^{(3)}, \bar{\omega}_l^{(3)}) = S(\omega_k^{(3)}, \bar{\omega}_{l+1}^{(3)})$ in other words:
 $S(\omega_{k+1}^{(3)}, \bar{\omega}_{i+j-(k+1)}^{(3)}) = S(\omega_k^{(3)}, \bar{\omega}_{i+j-k}^{(3)})$. In conclusion, we obtain that

$$S(\omega_{k+1}^{(3)}, \bar{\omega}_l^{(3)}) = \begin{cases} 0 & \text{if } l \neq i + j - (k+1) \\ S(\omega_k^{(3)}, \bar{\omega}_{i+j-k}^{(3)}) & \text{if } l = i + j - (k+1). \end{cases}$$

793 In order to make the induction work, it remains to show that $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} = n-1$. It is obvious
794 that $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} \geq n-1$ for otherwise we would have $S(\omega_{k+1}^{(3)}, \bar{\omega}_{i+j-(k+1)}^{(3)}) = 0$. (Remember that it
795 follows from the flatness of S , i.e. from condition (iv) (a) in 4.6, that $S : V_\alpha \otimes \bar{V} <_{1-\alpha+m} \rightarrow \tau^{-m} \mathbb{C}[\tau]$ for any
796 $\alpha \in \mathbb{Q}, m \in \mathbb{Z}$, so that $S(\omega_{k+1}^{(3)}, \bar{\omega}_{i+j-(k+1)}^{(3)}) \in \tau^{-n+2} \mathbb{C}[\tau]$ if $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} < n-1$, which is impossible
797 since $S : G_t \otimes_{\mathbb{C}[\tau^{-1}]} \bar{G}_t \rightarrow \tau^{-n+1} \mathbb{C}[\tau^{-1}]$). Thus the only case to exclude is $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} > n-1$.

First notice that it follows from property (iv) (c) of definition 4.6 that S induces an isomorphism

$$\overline{\tau^{n-1} G_t} \cong G_t^* := \text{Hom}_{\mathbb{C}[\tau^{-1}]}(G_t, \mathbb{C}[\tau^{-1}]).$$

On the other hand, we deduce from [Sab06, remark 3.6] that for any $\alpha \in \{-\nu_1^{(3)}, \dots, -\nu_n^{(3)}\}$,

$$\text{gr}_\alpha^{V^*}(G_t^*/\tau^{-1}G_t^*) \cong \text{gr}_{-\alpha}^V(G_t/\tau^{-1}G_t),$$

where V^* denotes the canonical V -filtration on the dual module $(G_t, \nabla)^*$. In conclusion, S induces a non-degenerate pairing

$$S : gr_\alpha^V(G_t/\tau^{-1}G_t) \otimes gr_{n-1-\alpha}^V(G_t/\tau^{-1}G_t) \rightarrow \tau^{-n+1}\mathbb{C}$$

which yields a non-degenerate pairing on the sum $gr_\bullet^V(G_t/\tau^{-1}G_t) := \bigoplus_{\alpha \in \mathbb{Q}} gr_\alpha^V(G_t/\tau^{-1}G_t)$. However, we know that $\underline{\omega}^{(3)}$ induces a basis of $gr_\bullet^V(G_t/\tau^{-1}G_t)$, compatible with the above decomposition. This, together with the fact that $S(\omega_{k+1}^{(3)}, \bar{\omega}_l^{(3)}) \in \tau^{-n+1}\mathbb{C}\delta_{i+j, k+1+l}$, yields that $-\nu_{k+1}^{(3)} - \nu_{i+j-(k+1)}^{(3)} = n-1$, as required.

- (ii) For this second statement, we consider the constant (in τ^{-1}) base change given by $\omega'_{k+1}{}^{(3)} := \omega_{j+k}^{(3)}$ for all $k \in \{0, \dots, n-j\}$ and $\omega'_{k+1+n-j}{}^{(3)} := t\omega_k^{(3)}$ for all $k \in \{1, \dots, j-1\}$. Then we have

$$\partial_\tau(\underline{\omega}'^{(3)}) = \underline{\omega}'^{(3)} \cdot (\Omega_0 + \tau^{-1}(A_\infty^{(3)})'),$$

where $(A_\infty^{(3)})' = \text{diag}(-\nu_j^{(3)}, -\nu_{j+1}^{(3)}, \dots, -\nu_n^{(3)}, -\nu_1^{(3)}, \dots, -\nu_{j-1}^{(3)})$. Now the proof of (i) works verbatim for the basis $\underline{\omega}'^{(3)}$, with the index i from above replaced by 1 and the index j from above replaced by $n-j+i+2$. Notice that then the spectral number corresponding to 1 is the biggest one and the one corresponding to $n-j+i+2$ is the smallest one, but this does not affect the proof. Depending on the value of the indices k and l , we have that $c_{kl}(t)$ is either t^{-1} , 1 or t . □

5 Frobenius structures

5.1 Frobenius structures for linear functions on Milnor fibres

In this subsection, we derive one of the main results of this paper: the existence of a Frobenius structure on the unfolding space of the function $f|_{D_t}$, $t \neq 0$. Depending on whether we restrict to the class of examples satisfying the hypotheses of theorem 4.13, the Frobenius structure can be derived directly from the (V^+, S) -solution $\underline{\omega}^{(3)}$ of the Birkhoff problem constructed in the last section, or otherwise is obtained by appealing to theorem 4.7. We refer to [Her02] or [Sab02] for the definition of a Frobenius manifold. It is well known that a Frobenius structure on a complex manifold M is equivalent to the following set of data.

- (i) a holomorphic vector bundle E on $\mathbb{P}^1 \times M$ such that $\text{rank}(E) = \dim(M)$, which is fibrewise trivial, i.e. $\mathcal{E} = p^*p_*\mathcal{E}$, (where $p : \mathbb{P}^1 \times M \rightarrow M$ is the projection) equipped with an integrable connection with a logarithmic pole along $\{\infty\} \times M$ and a pole of type one along $\{0\} \times M$,
- (ii) an integer w ,
- (iii) A non-degenerate, $(-1)^w$ -symmetric pairing $S : \mathcal{E} \otimes j^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1 \times M}(-w, w)$ (here $j(\tau, u) = (-\tau, u)$, with, as before, τ a coordinate on \mathbb{P}^1 centered at infinity and u a coordinate on M and we write $\mathcal{O}_{\mathbb{P}^1 \times M}(a, b)$ for the sheaf of meromorphic functions on $\mathbb{P}^1 \times M$ with a pole of order a along $\{0\} \times M$ and order b along $\{\infty\} \times M$) the restriction of which to $\mathbb{C}^* \times M$ is flat,
- (iv) A global section $\xi \in H^0(\mathbb{P}^1 \times M, \mathcal{E})$, which is flat with respect to the residue connection $\nabla^{res} : \mathcal{E}/\tau\mathcal{E} \rightarrow \mathcal{E}/\tau\mathcal{E} \otimes \Omega_M^1$ with the following two properties

- (a) The morphism

$$\begin{aligned} \Phi_\xi : \mathcal{T}_M &\longrightarrow \mathcal{E}/\tau^{-1}\mathcal{E} \cong p_*\mathcal{E} \\ X &\longmapsto -[\tau^{-1}\nabla_X](\xi) \end{aligned}$$

is an isomorphism of vector bundles (a section ξ with this property is called primitive),

- (b) ξ is an eigenvector of the residue endomorphism $[\tau\nabla_\tau] \in \text{End}_{\mathcal{O}_M}(p_*\mathcal{E}) \cong \text{End}_{\mathcal{O}_M}(\mathcal{E}/\tau\mathcal{E})$ (a section with this property is called homogeneous).

829 In many application one is only interested in constructing a Frobenius structure on a germ at a given point, in
 830 that case M is a sufficiently small representative of such a germ.

831 We now come back to our situation of a \mathcal{R}_h -finite linear section f on the Milnor fibration $h : V \rightarrow T$. In this
 832 subsection, we are interested to construct Frobenius structures on the (germ of a) semi-universal unfolding of
 833 the function $f|_{D_t}$, $t \neq 0$. It is well known that in contrast to the local case, such an unfolding does not have
 834 obvious universality properties. One defines, according to [DS03, 2.a.], a deformation

$$F = f + \sum_{i=1}^n u_i g_i : B_t \times M \rightarrow D$$

835 of the restriction $f|_{B_t}$ to some intersection $D_t \cap B_\epsilon$ such that the critical locus C of F is finite over M via
 836 the projection $q : B_t \times M \rightarrow M$ to be a semi-universal unfolding if the Kodaira-Spencer map $\mathcal{T}_M \rightarrow q_* \mathcal{O}_C$,
 837 $X \mapsto [X(F)]$ is an isomorphism.

838 From proposition 3.4 we know that any basis g_1, \dots, g_n of $T_{\mathcal{R}_h}^1 f$ gives a representative

$$F = f + \sum_{i=1}^n u_i g_i : B_t \times M \rightarrow D$$

839 of this unfolding, where M is a sufficiently small neighborhood of the origin in \mathbb{C}^n , with coordinates u_1, \dots, u_n .
 840 In order to exhibit Frobenius structures via the approach sketched in the beginning of this section, one has
 841 to find a (V^+, S) -solution to the Birkhoff problem for G_t . If the minimal spectral number of (G_t, ∇) has
 842 multiplicity one, then, according to corollary 4.12 and theorem 4.13, the basis $\underline{\omega}^{(3)}$ yields such a solution, which
 843 we denote by \widehat{G}_t (which is, if D is reductive, the restriction of \widehat{G} from corollary 4.12 (ii) to $\mathbb{P}^1 \times \{t\}$). Otherwise,
 844 we consider the canonical solution from theorem 4.7, which is denoted by \widehat{G}_t^{can} . The bundle called \mathcal{E} in the
 845 beginning of this subsection is then obtained by *unfolding* the solution \widehat{G}_t resp. \widehat{G}_t^{can} . We will not describe \mathcal{E}
 846 explicitly, but use a standard result due to Dubrovin which gives directly the corresponding Frobenius structure
 847 provided that one can construct a homogenous and primitive form for \widehat{G}_t resp. \widehat{G}_t^{can} , i.e., a section called ξ
 848 above at the point t .

849 We can now state and prove the main result of this section.

850 **Theorem 5.1.** *Let $f \in \mathbb{C}[V]_1$ be an \mathcal{R}_h -finite linear function. Write M_t for the parameter space of a semi-*
 851 *universal unfolding $F : B_t \times M_t \rightarrow D$ of $f|_{B_t}$, $t \neq 0$ as described above. Let $\alpha_{\min} = \alpha_1$ be the minimal spectral*
 852 *number of (G_t, ∇) .*

853 (i) *Suppose that α_{\min} has multiplicity one, i.e., $\alpha_2 > \alpha_1$. Then any of the sections $t^{-k} \omega_i^{(3)} \in H^0(\mathbb{P}^1, \widehat{G}_t)$ is*
 854 *primitive and homogenous. Any choice of such a section yields a Frobenius structure on M_t , which we*
 855 *denote by $(M_t, \circ, g, e, E)^{(i)}$.*

856 (ii) *Without any assumption on the multiplicity of α_{\min} , we have that any of the sections $\omega_i^{(3)}$ which lies in*
 857 *$H^0(\mathbb{P}^1, \widehat{G}_t^{can}) \cap V_\alpha$, where $\alpha \in [\alpha_{\min}, \alpha_{\min} + 1)$ is primitive and homogenous and hence yields a Frobenius*
 858 *structure on M_t , denoted by $(M_t, \circ, g, e, E)^{(i), can}$.*

859 **Remark:** In (i), we can also use the sections $\omega_i^{(3)}$ or any other non-zero multiple of them to construct Frobenius
 860 structures. However, for technical reasons, it is preferable to work with the rescaled sections $t^{-k} \omega_i^{(3)}$ (see
 861 proposition 5.4).

862 *Proof.* In both cases, we use the universal semi-simple Frobenius structure defined by a finite set of given initial
 863 data as constructed by Dubrovin ([Dub96], see also [Sab02, théorème VII.4.2]). The initial set of data we need
 864 to construct is

- 865 (i) an n -dimensional complex vector space W ,
- 866 (ii) a symmetric, bilinear, non-degenerate pairing $g : W \otimes_{\mathbb{C}} W \rightarrow \mathbb{C}$,
- 867 (iii) two endomorphisms $B_0, B_\infty \in \text{End}_{\mathbb{C}}(W)$ such that B_0 is semi-simple with distinct eigenvalues and g -
 868 selfadjoint and such that $B_\infty + B_\infty^* = (n-1)\text{Id}$, where B_∞^* is the g -adjoint of B_∞ .
- 869 (iv) an eigenvector $\xi \in W$ for B_∞ , which is a cyclic generator of W with respect to B_0 .

870 In both cases of the theorem, the vector space W will be identified with $G_t/\tau^{-1}G_t$. Dubrovin's theorem yields
871 a germ of a universal Frobenius structure on a certain n -dimensional manifold such that its first structure
872 connection restricts to the data $(W, B_0, B_\infty, g, \xi)$ over the origin. The universality property then induces a
873 Frobenius structure on the germ $(M_t, 0)$, as the tangent space of the latter at the origin is canonically identified
874 with $T_{\mathcal{A}_h}^1 f \cong T_{\mathcal{A}_h}^1 f \cdot \alpha \cong G_t/\tau^{-1}G_t$.
875 Let us show how to construct the initial data needed in case (i) and (ii) of the theorem:

876 (i) We put $W := H^0(\mathbb{P}^1, \widehat{G}_t)$, $g := \tau^{-n+1}S$ (notice that this is possible due to theorem 4.13), $B_0 := [\nabla_\tau] \in$
877 $\text{End}_{\mathbb{C}}(G_t/\tau^{-1}G_t) \cong \text{End}_{\mathbb{C}}(W)$ and $B_\infty := [\tau\nabla_\tau] \in \text{End}_{\mathbb{C}}(\widehat{G}_t/\tau\widehat{G}_t) \cong \text{End}_{\mathbb{C}}(W)$. In order to verify the
878 conditions from above on these initial data, consider the basis $t^{-k}\omega_i^{(3)}$ of W . Then B_0 is given by the
879 matrix Ω_0 , which is obviously semi-simple with distinct eigenvalues (these are the critical values of $f|_{D_t}$).
880 Its self-adjointness follows from the flatness of S . The endomorphism B_∞ corresponds to the matrix \widetilde{A}_∞ ,
881 so that the symmetry of the spectrum as well as the proof of lemma 4.14 show that $B_\infty + B_\infty^* = (n-1)\text{Id}$.
882 Finally, it follows from corollary 4.12 that for all $i \in \{1, \dots, n\}$, the class of $t^{-k}\omega_i^{(3)}$ in $G_t/\tau^{-1}G_t$ is equal
883 to the class of $f^{i-1}\alpha$. By definition, $B_0 = [\nabla_\tau]$ is the multiplication by $-f$ on $W \cong G_t/\tau^{-1}G_t$, hence, any
884 of the classes of the sections $t^{-k}\omega_i^{(3)}$ is a cyclic generator of W with respect to $[\nabla_\tau]$. It is homogenous,
885 i.e., an eigenvector of B_∞ by construction. This proves the theorem in case (i).

886 (ii) We use the canonical solution to the Birkhoff problem described in theorem 4.7. Namely, we put $W :=$
887 $H^0(\mathbb{P}^1, \widehat{G}_t^{can})$ and again $g := \tau^{-n+1}S$, $B_0 := [\nabla_\tau] \in \text{End}_{\mathbb{C}}(G_t/\tau^{-1}G_t) \cong \text{End}_{\mathbb{C}}(W)$ and $B_\infty := [\tau\nabla_\tau] \in$
888 $\text{End}_{\mathbb{C}}(\widehat{G}_t^{can}/\tau\widehat{G}_t^{can}) \cong \text{End}_{\mathbb{C}}(W)$. The eigenvalues of the endomorphism B_∞ are always the critical values
889 of $f|_{D_t}$ so as in (i) it follows that B_∞ is semi-simple with distinct eigenvalues. It is g -self-adjoint by the same
890 argument as in (i). B_∞ is also semi-simple, as \widehat{G}_t^{can} is a V^+ -solution. The property $B_\infty + B_\infty^* = (n-1)\text{Id}$
891 follows then as in (i) by the fact that it is also a (V^+, S) -solution (more precisely, by choosing a basis \underline{w}
892 of W such that B_∞ is again given by the matrix \widetilde{A}_∞ and such that $g(w_i, w_j) = 1$ if $-\tilde{\nu}_i - \tilde{\nu}_j = n-1$ and
893 $g(w_i, w_j) = 0$ otherwise). In order to obtain a primitive form, we use the fact (see [DS03, appendix B.b.])
894 that the space $H^0(\mathbb{P}^1, G_t) \cap V_\alpha$ for $\alpha \in [\alpha_{min}, \alpha_{min} + 1)$ is independent of the choice of the V^+ solution
895 \widehat{G}_t of the Birkhoff problem for (G_t, ∇) . Hence the same argument as in (i) shows that any section $\omega_i^{(3)}$
896 which is an element in $H^0(\mathbb{P}^1, \widehat{G}_t^{can}) \cap V_\alpha$ is primitive and homogenous.

897 □

898 The previous theorem yields for fixed i Frobenius structures $M_t^{(i)}$ resp. $M_t^{(i),can}$ for any t . One might ask
899 whether they are related in some way. It turns out that for a specific choice of the index i they are (at least
900 in the reductive case), namely, one of them can be seen as analytic continuation of the other. The proof relies
901 on the fact that it is possible to construct a Frobenius structure from the bundle G simultaneously for all values
902 of t at least on a small disc outside of $t = 0$. This is done using a generalization of Dubrovins theorem, due
903 to Hertling and Manin [HM04, theorem 4.5]. In loc.cit., Frobenius manifolds are constructed from so-called
904 “trTLEP-structures”. The following result shows how they arise in our situation.

905 **Lemma 5.2.** *Suppose that D is reductive. Fix $t \in T$ and suppose that the minimal spectral number α_{min}*
906 *of (G_t, ∇) has multiplicity one, so that theorem 4.13 applies. Let Δ_t be a sufficiently small disc centered*
907 *at t . Denote by $\widehat{H}^{(t)}$ the restriction to Δ_t of the analytic bundle corresponding to \widehat{G} . Then $\widehat{H}^{(t)}$ underlies*
908 *a trTLEP-structure on Δ_t , and any of the sections $t^{-k}\omega_i^{(3)}$ satisfy the conditions (IC), (GC) and (EC) of*
909 *[HM04, theorem 4.5]. Hence, the construction in loc.cit. yields a universal Frobenius structure on a germ*
910 *$(\widetilde{M}^{(i)}, t) := (\Delta_t \times \mathbb{C}^{n-1}, (t, 0))$.*

911 *Proof.* That $\widehat{H}^{(t)}$ underlies a trTLEP-structure is a consequence of corollary 4.12 (i) and theorem 4.13. We have
912 already seen that the sections $t^{-k}\omega_i^{(3)}$ are homogenous and primitive, i.e., satisfy conditions (EC) and (GC)
913 of loc.cit. It follows from the connection form computed in corollary 4.12 (ii), that they also satisfy condition
914 (IC). Thus the theorem of Hertling and Manin gives a universal Frobenius structure on $\widetilde{M}^{(i)}$ such that its first
915 structure connection restricts to $\widehat{H}^{(t)}$ on Δ_t . □

916 In order to apply this lemma we need to find a homogenous and primitive section of $\widehat{H}^{(t)}$ which is also ∇_t -flat.
917 This is done in the following lemma.

918 **Lemma 5.3.** *Let D be reductive. Consider the V^+ -solution to the Birkhoff-problem for (G_0, ∇) resp. (G_t, ∇)*
919 *given by $\underline{\omega}^{(2)}$ resp. $\underline{\omega}^{(3)}$. Then there is an index $j \in \{1, \dots, n\}$ such that $\deg(\omega_j^{(2)}) = -\nu_i^{(2)}$ and an index*
920 *$i \in \{1, \dots, n\}$ such that $\deg(\omega_i^{(3)}) = -\nu_i^{(3)} + k \cdot n$. In particular, $\nu_j^{(2)}, \nu_i^{(3)} \in \mathbb{N}$. Moreover, $\nabla_t^{res}(t^{-k}\omega_i^{(3)}) = 0$,*
921 *where $\nabla_t^{res} : \widehat{G}/\tau\widehat{G} \rightarrow \widehat{G}/\tau\widehat{G}$ is the residue connection.*

922 *Proof.* By construction we have $\omega_1^{(1)} = \omega_1^{(0)} = \alpha$, so in particular $\deg(\omega_1^{(1)}) = 0$. We also have $\nu^{(1)} = 0$.
923 Now it suffices to remark that in algorithm 1 (formula (4.13)), whenever we have an index $j \in \{1, \dots, n\}$ with
924 $\deg(\omega_j^{(1)}) = -\nu_j^{(1)}$, then either $\deg(\tilde{\omega}_j^{(1)}) = -\tilde{\nu}_j^{(1)}$ (this happens if the index i in formula (4.13) is different
925 from $j + 1$) or $\deg(\tilde{\omega}_{j+1}^{(1)}) = -\tilde{\nu}_{j+1}^{(1)}$ (if $i = j + 1$). It follows that we always conserve some index k with
926 $\deg(\tilde{\omega}_k^{(1)}) = -\tilde{\nu}_k^{(1)}$. A similar argument works for algorithm 2, which gives the second statement of the first
927 part. The residue connection is given by the matrix $\frac{1}{nt} (\tilde{D} + A_\infty^{(3)})$ in the basis $\underline{\omega}^{(3)}$ of $\widehat{G}/\tau\widehat{G}$ (see corollary 4.12
928 (ii)). This yields the ∇^{res} -flatness of $t^{-k}\omega_i^{(3)}$. \square

929 Finally, the comparison result can be stated as follows.

930 **Proposition 5.4.** *Let i be the index from the previous lemma such that $\nabla^{res}(t^{-k}\omega_i^{(3)}) = 0$. Then for any*
931 *$t' \in \Delta_t$, the germs of Frobenius structures $(\widetilde{M}^{(i)}, t')$ (from lemma 5.2) and $(M^{(i)}, t')$ (from theorem 5.1) are*
932 *isomorphic.*

933 *Proof.* We argue as in [Dou08, proposition 5.5.2]: The trTLEP-structure $\widehat{H}^{(t)}$ is a deformation (in the sense of
934 [HM04, definition 2.3]) of the fibre $\widehat{G}/t'\widehat{G}$, hence contained in the universal deformation of the latter. Thus
935 the (germs at t' of the) universal deformations of $\widehat{H}^{(t)}$ and $\widehat{G}/t'\widehat{G}$ are isomorphic. This gives the result as the
936 homogenous and primitive section $t^{-k}\omega_i^{(3)}$ of $\widehat{H}^{(t)}$ that we choose in order to apply lemma 5.2 is ∇^{res} -flat. \square

937 5.2 Limit Frobenius structures

938 In the last subsection, we constructed Frobenius structures on the unfolding spaces M_t . It is a natural question
939 to know whether one can attach a “limit” Frobenius structure to the restriction of f on D . In order to carry
940 this out, one is faced with the difficulty that the pairing S from theorem 4.7 is not, a priori, defined on \mathbf{G}_0 .
941 Hence a more precise control on this pairing on $\mathbf{G}[t^{-1}]$ is needed in order to make a statement on the limit.
942 The following conjecture provides exactly this additional information.

943 **Conjecture 5.5.** The pairing S from theorem 4.7 is defined on $\mathbf{G}[t^{-1}]$ and meromorphic at $t = 0$, i.e., induces
944 a pairing $S : \mathbf{G}[t^{-1}] \otimes \overline{\mathbf{G}}[t^{-1}] \rightarrow \mathbb{C}[\tau, \tau^{-1}, t, t^{-1}]$. Moreover, consider the natural grading of \mathbf{G} resp. on $\mathbf{G}[t^{-1}]$
945 induced from the grading of $\Omega^{n-1}(\log h)$ by putting $\deg(\tau) = -1$ and $\deg(t) = n$. Then

946 (i) S is homogenous, i.e., it sends $(\mathbf{G}[t^{-1}]_k) \otimes (\overline{\mathbf{G}}[t^{-1}]_l)$ into $\mathbb{C}[\tau, \tau^{-1}, t, t^{-1}]_{k+l}$ (remember that the ring
947 $\mathbb{C}[\tau, \tau^{-1}, t, t^{-1}]$ is graded by $\deg(\tau) = -1$, $\deg(t) = n$).

948 (ii) S sends G into $\tau^{-n+1}\mathbb{C}[\tau^{-1}, t]$.

949 Some evidence supporting the first part of this conjecture comes from the computation of the examples in
950 section 6. Namely, it appears that in all cases, there is an extra symmetry satisfied by the spectral numbers,
951 i.e., we have $\nu_k^{(3)} + \nu_{n+1-k}^{(3)} = -n + 1$, and not only $\alpha_k + \alpha_{n+1-k} = n - 1$ for all $k \in \{1, \dots, n\}$ (Remember that
952 $\alpha_1, \dots, \alpha_n$ was the ordered sequence of spectral numbers). Moreover, the eigenvalues of the residue of $t\partial_t$ on
953 $(G/tG)|_{\tau \neq 0}$ are constant in τ and symmetric around zero, which indicates that S extends without poles and as
954 a non-degenerate pairing to G . In particular, one obtains a pairing on G_0 , which would explain the symmetry
955 $\nu_k^{(2)} + \nu_{n+1-k}^{(2)} = -n + 1$ observed in the examples (notice that even the symmetry of the spectral numbers at
956 $t = 0$, written as an order sequence, is not a priori clear).

957 The following corollary draws some consequences of the above conjecture.

958 **Corollary 5.6.** *Suppose that conjecture 5.5 holds true and that the minimal spectral number α_{min} of (G_t, ∇) ,*
959 *$t \neq 0$ has multiplicity one so that theorem 4.13 holds. Then*

(i) *The pairing S is expressed in the basis $\underline{\omega}^{(3)}$ as*

$$S(\omega_i^{(3)}, \bar{\omega}_j^{(3)}) = \begin{cases} c \cdot t^{2k} \cdot \tau^{-n+1} & \text{if } i + j = n + 1 \\ 0 & \text{else} \end{cases}$$

960 for some constant $c \in \mathbb{C}$, where, as before, $k \in \mathbb{N}$ counts the number of meromorphic base changes in
 961 algorithm 2. Moreover, we have $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n + 1$ for all $i \in \{1, \dots, n\}$.

(ii) The pairing S is expressed in the basis $\underline{\omega}^{(2)}$ as

$$S(\omega_i^{(2)}, \bar{\omega}_j^{(2)}) = \begin{cases} c \cdot \tau^{-n+1} & \text{if } i + j = n + 1 \\ 0 & \text{else} \end{cases}$$

962 for the same constant $c \in \mathbb{C}$ as in (i).

963 (iii) S extends to a non-degenerate pairing on G , i.e., it induces a pairing $S : G_0 \otimes_{\mathbb{C}[\tau^{-1}]} \bar{G}_0 \rightarrow \tau^{-n+1} \mathbb{C}[\tau^{-1}]$
 964 with all the properties of definition 4.6 (iv). Moreover, $\underline{\omega}^{(2)}$ defines a (V^+, S) -solution for the Birkhoff
 965 problem for (G_0, ∇) with respect to S .

966 *Proof.* (i) Following the construction of the bases $\underline{\omega}^{(1)}$, $\underline{\omega}^{(2)}$ and $\underline{\omega}^{(3)}$, starting from the basis $\underline{\omega}^{(0)}$ (via lemma
 967 4.4 and algorithms 1 and 2), it is easily seen that $\deg(\omega_i^{(1)}) = \deg(\omega_i^{(2)}) = i - 1$ and that $\deg(\omega_i^{(3)}) =$
 968 $k \cdot n + i - 1$. The (conjectured) homogeneity of S yields that $S(\omega_i^{(2)}, \bar{\omega}_j^{(2)}) = i + j - 2$ and $\deg(S(\omega_i^{(3)}, \bar{\omega}_j^{(3)})) =$
 969 $2kn + i + j - 2$.

970 The proof of lemma 4.14 shows that $\tau^{n-1} S(\omega_i^{(3)}, \bar{\omega}_j^{(3)})$ is either zero or constant in τ , hence, by part (ii)
 971 of conjecture 5.5, $S(\omega_i^{(3)}, \bar{\omega}_j^{(3)}) = c(t) \cdot \tau^{-n+1}$, with $c(t) \in \mathbb{C}[t]$, which is actually homogenous by part (i)
 972 of conjecture 5.5. Now since $i + j - 2 < 2(n - 1)$, $\deg(c(t) \cdot \tau^{-n+1}) = 2kn + (i + j - 2)$ is only possible if
 973 $i + j = n + 1$, and then $c(t) = c \cdot t^{2k}$, in particular, the numbers c_{kl} in lemma 4.14 (ii) are always equal to
 974 one, and we have $\nu_i^{(3)} + \nu_j^{(3)} = -n + 1$.

975 (ii) Using (i), one has to analyse the behaviour of S under the base changes inverse to 4.13 (algorithm 1) and
 976 4.14 (algorithm 2). Suppose that $\underline{\omega}$ is a basis of $\mathbf{G}[t^{-1}]$ with $\deg(\omega_i) = l \cdot n + i - 1$, $l \in \{0, \dots, k\}$ and
 977 such that $S(\omega_i, \bar{\omega}_j) = c \cdot t^{2l} \cdot \tau^{-n+1} \cdot \delta_{i+j, n+1}$, then if we define for any $i \in \{1, \dots, n\}$ a new basis $\underline{\omega}'$ by

$$\begin{aligned} \omega'_i &:= \omega_i - \tau^{-1} \cdot \nu \cdot \omega_{i-1}, \\ \omega'_j &:= \omega_j \quad \forall j \neq i, \end{aligned} \tag{5.1}$$

978 where $\nu \in \mathbb{C}$ is any constant, we see that we still have $S(\omega'_i, \bar{\omega}'_j) = c \cdot t^{2l} \cdot \tau^{-n+1} \cdot \delta_{i+j, n+1}$. Notice that if
 979 $j = i + 1$ and $i + j = n + 1$, then in order to show $S(\omega'_i, \bar{\omega}'_i) = 0$, one uses that if $i + (i - 1) = n + 1$, then
 980 $S(\omega_i, \bar{\omega}_{i-1}) = (-1)^{n-1} \overline{S(\omega_{i-1}, \bar{\omega}_i)} = S(\omega_{i-1}, \bar{\omega}_i)$ since $S(\omega_{i-1}, \bar{\omega}_i)$ is homogenous in τ^{-1} of degree $-n + 1$.
 981 Similarly, if we put, for any constant $\nu \in \mathbb{C}$,

$$\begin{aligned} \omega''_1 &:= t^{-2} \omega_1 - t^{-1} \tau^{-1} \cdot \nu \cdot \omega_n, \\ \omega''_i &:= t^{-1} \omega_i \quad \forall i \neq 1, \end{aligned} \tag{5.2}$$

982 then we have $S(\omega''_i, \bar{\omega}''_j) = c \cdot t^{2(l-1)} \cdot \tau^{-n+1} \cdot \delta_{i+j, n+1}$.

983 (iii) This follows from (ii) and the fact that $\underline{\omega}^{(2)}$ is a V^+ -solution for (G_0, ∇) . □

985 As a consequence, we show that under the hypothesis of conjecture 5.5, we obtain indeed a limit Frobenius
 986 structure.

987 **Theorem 5.7.** *Suppose that conjecture 5.5 holds true and that the minimal spectral number α_{min} of (G_t, ∇) for*
 988 *$t \neq 0$ has multiplicity one, so that theorem 4.13 applies. Then the (germ at the origin of the) \mathcal{R}_h -deformation*
 989 *space of f , which we call M_0 , carries a Frobenius structure, which is constant, i.e., given by a potential of degree*
 990 *at most three (or, expressed otherwise, such that the structure constants c_{ij}^k defined by $\partial_{t_i} \circ \partial_{t_j} = \sum_k c_{ij}^k \partial_{t_k}$*
 991 *are constant in the flat coordinates t_1, \dots, t_n). It is the limit of the Frobenius structures $M_t^{(1)}$ for $t \rightarrow 0$ if*
 992 *$\underline{\omega}^{(3)} = \underline{\omega}^{(2)}$.*

Proof. Remember that $(M_0, 0)$ is a smooth germ of dimension n , with tangent space given by $T_{\mathcal{R}_h}^1 f \cong G_0 / \tau^{-1} G_0$
 (notice that the deformation functor in question is evidently unobstructed). As usual, a \mathcal{R}_h -semi-universal
 unfolding of f is given as

$$F = f + \sum_{i=1}^n u_i g_i : V \times M_0 \longrightarrow \mathbb{C},$$

993 where u_1, \dots, u_n are coordinates on M_0 and g_1, \dots, g_n is a basis of $T_{\mathcal{R}_h}^1 f$.
994 In order to endow M_0 with a Frobenius structure, we will use a similar strategy as in subsection 5.1, namely,
995 we construct a germ of an n -dimensional Frobenius manifold which induces a Frobenius structure on M_0 by a
996 universality property. However, we have to use, as in lemma 5.2, the generalization due to Hertling and Manin
997 instead of Dubrovins classical result as the Frobenius manifold we obtain will not be semi-simple. We will
998 use a particular case of this result, namely, [HM04, remark 4.6]. Thus we have to construct a *Frobenius type*
999 *structure* on a point, and a section satisfying the conditions called (GC) and (EC) in loc.cit. This is nothing
1000 but a tuple $(W, g, B_0, B_\infty, \xi)$ as in the proof of theorem 5.1, except that we do not require the endomorphism
1001 B_0 to be semi-simple, but to be *regular*, i.e., its characteristic and minimal polynomial must coincide. Consider
1002 the (V^+, S) -solution defined by $\widehat{G}_0 := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1 \times \{0\}} \omega_i^{(2)}$, and put, as before, $W := H^0(\mathbb{P}^1, \widehat{G}_0)$, $g := \tau^{-n+1} S$,
1003 $B_0 := [\nabla_\tau]$ and $B_\infty := [\tau \nabla_\tau]$. Considering the matrices $(\Omega_0)|_{t=0}$ resp. $A_\infty^{(2)}$ of B_0 resp. B_∞ with respect to
1004 the basis $\underline{\omega}^{(2)}$ of W , we see immediately that $g(B_0 -, -) = g(-, B_0 -)$, $g(B_\infty -, -) = g(-, (n-1)\text{Id} - B_\infty -)$
1005 and that B_0 is regular since $(\Omega_0)|_{t=0}$ is nilpotent with a single Jordan block. Notice that the assumption that
1006 conjecture 5.5 holds is used through corollary 5.6 (ii), (iii). The section $\xi := \omega_1^{(2)}$ is obviously homogenous
1007 and primitive, i.e., satisfies (EC) and (GC). Notice that it is, up to constant multiplication, the only primitive
1008 section, contrary to the case $t \neq 0$, where we could chose any of the sections $\omega_i^{(3)}$, $i \in \{1, \dots, n\}$. We have
1009 thus verified all conditions of the theorem of Hertling and Manin, and obtain, as indicated above, a Frobenius
1010 structure on M_0 , which might be considered as a limit $\lim_{t \rightarrow 0} M_t^{(1)}$ if the two modules G and $G^{(3)}$ coincide, i.e.,
1011 if $\omega^{(2)} = \omega^{(3)}$ (equivalently, if $k = 0$).
1012 It remains to show that it is given by potential of degree at most three. The argument is exactly the same as in
1013 [Dou08, lemma 6.4.1 and corollary 6.4.2.] so that we omit the details here. The main point is that if one uses
1014 the unfolding theorem [HM04, theorem 4.5], then it is easily seen that the matrices of the Higgs field in the
1015 natural basis given by the flat extension of $\underline{\omega}^{(2)}$ are polynomials in the matrix $(\Omega_0)|_{t=0}$, but their first columns
1016 are fixed, by the fact that $\omega_1^{(2)}$ is the primitive form, i.e., corresponds to the unit field on M_0 . In particular,
1017 they are constant. \square

1018 5.3 Logarithmic Frobenius structures

1019 The pole order property of the connection ∇ on G (see formula 4.12) suggests that the family of germs of
1020 Frobenius manifolds M_t studied above can be put together in a single Frobenius manifold with a *logarithmic*
1021 degeneration behavior at the divisor $t = 0$. We show that this is actually the case for the normal crossing
1022 divisor, the same result has been obtained from a slightly different viewpoint in [Dou08]. In the general case, we
1023 observe a phenomenon which also occurs in loc.cit.: one obtains a Frobenius manifold where the multiplication
1024 is defined on the logarithmic tangent bundle, but the metric might be degenerate on it (see cit.loc., section 7.1.).
1025 We recall the following definition from [Rei08], which we extend to the more general situation studied here.

1026 **Definition 5.8.** (i) Let M be a complex manifold and $\Sigma \subset M$ be a normal crossing divisor. Suppose
1027 that $(M \setminus \Sigma, \circ, g, E, e)$ is a Frobenius manifold. One says that it has a logarithmic pole along Σ if $\circ \in$
1028 $\Omega^1(\log \Sigma)^{\otimes 2} \otimes \text{Der}(-\log \Sigma)$, $g \in \Omega^1(\log \Sigma)^{\otimes 2}$ and g is non-degenerate as a pairing on $\text{Der}(-\log \Sigma)$.

1029 (ii) If, in the previous definition, we relax the condition of g being non-degenerate on $\text{Der}(-\log \Sigma)$, then we
1030 say that (M, Σ) is a weak logarithmic Frobenius manifold.

1031 In [Rei08], logarithmic Frobenius manifolds are constructed using a generalisation of the main theorem of
1032 [HM04]. More precisely, universal deformations of so called “log Σ -trTLEP-structures” (see [Rei08, definition
1033 1.8.]) are constructed. In our situation, the base of such an object is the space T , and the divisor $\Sigma := \{0\} \subset T$.
1034 In order to adapt the construction to the more general situation that we discuss here, we define a weak log Σ -
1035 trTLEP-structures to be such a vector bundle on $\mathbb{P}^1 \times T$ with connection and pairing, where the latter is
1036 supposed to be non-degenerate only on $\mathbb{P}^1 \times T \setminus \Sigma$. The result can then be stated as follows.

Theorem 5.9. *Let D be reductive, $i \in \{1, \dots, n\}$ be the index from lemma 5.3 such that $\deg(t^{-k} \omega_i) = -\nu_i^{(3)}$
and suppose that the minimal spectral number α_{\min} of (G_t, ∇) has multiplicity one (so that theorem 4.7 applies).*

Then the (analytic bundle corresponding to the) module

$$\begin{aligned}\widehat{G}' &:= \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}^1 \times T} \omega_j^{(4)} \quad \text{where} \\ \omega_j^{(4)} &:= t^{-k} \omega_j^{(3)} \quad \forall j \in \{i, \dots, n\} \\ \omega_j^{(4)} &:= t^{-k+1} \omega_j^{(3)} \quad \forall j \in \{1, \dots, i-1\}\end{aligned}$$

1037 underlies a weak log Σ -trTLEP-structure, and a log Σ -trTLEP-structure if conjecture 5.5 holds true and if $i = 1$.
 1038 The form $t^{-k} \omega^{(3)}$ is homogenous and primitive and yields a weak logarithmic Frobenius manifold. It yields a
 1039 logarithmic Frobenius manifold if conjecture 5.5 holds true and if $i = 1$, e.g., in the case of a linear section f of
 1040 the normal crossing divisor.

Proof. It is clear by definition that $(\widehat{G}', \nabla, S)$ is a weak log Σ -trTLEP-structure (of weight $n - 1$). It is easy to see that the connection takes the form

$$\nabla(\underline{\omega}^{(4)}) = \underline{\omega}^{(4)} \cdot \left[(\Omega_0 \tau + A_\infty^{(4)}) \frac{d\tau}{\tau} + (\Omega_0 \tau + \widetilde{A}_\infty^{(4)}) \frac{dt}{nt} \right],$$

where

$$\begin{aligned}A_\infty^{(4)} &= \text{diag}(-\nu_i^{(3)}, \dots, -\nu_n^{(3)}, -\nu_1^{(3)}, \dots, -\nu_{i-1}^{(3)}) \\ \widetilde{A}_\infty^{(4)} &= A_\infty^{(4)} + \text{diag}(\deg(\omega_i^{(4)}), \dots, \deg(\omega_n^{(4)}), \deg(\omega_1^{(4)}), \dots, \deg(\omega_{i-1}^{(4)})).\end{aligned}$$

1041 In particular, $\omega_1^{(4)}$ is ∇^{res} -flat, $[\nabla_\tau]$ -homogenous and a cyclic generator of $H^0(\mathbb{P}^1 \times \{0\}, \widehat{G}'/t\widehat{G}')$ with respect
 1042 to $[\nabla_\tau]$ and $[\tau^{-1} \nabla_{t\partial_t}]$ (even with respect to $[\nabla_\tau]$ alone). Moreover, $[\tau^{-1} \nabla_{t\partial_t}(\omega_1^{(4)})]$ is non-zero in $H^0(\mathbb{P}^1 \times$
 1043 $\{0\}, \widehat{G}'/t\widehat{G}')$, so that $\omega_1^{(4)}$ satisfies the conditions (EC), (GC) and (IC) of [Rei08, theorem 1.12], except that the
 1044 form S might be degenerate on $\widehat{G}'_{|t=0}$ (correspondingly, the metric g on $K := \widehat{G}'/\tau\widehat{G}'$ from loc.cit. might be
 1045 degenerate on $K_{|t=0}$). Hence theorem 1.12 of loc.cit yields a weak logarithmic Frobenius structure.

1046 Now assume conjecture 5.5 and suppose that $i = 1$. Then $\underline{\omega}^{(4)} = t^{-k} \underline{\omega}^{(3)}$, and we get that S is non-degenerate
 1047 on \widehat{G}' by corollary 5.6. In particular, $(\widehat{G}', \nabla, S)$ underlies a log Σ -trTLEP-structure in this case. This yields
 1048 a logarithmic Frobenius structure by applying [Rei08, theorem 1.12]. That the pairing is non-degenerate and
 1049 that $i = 1$ holds for the normal crossing divisor case follows, e.g., from the computations in [DS04] (which, as
 1050 already pointed out above, have been taken up in [Dou08] to give the same result as here). \square

1051 Let us remark that one might consider the result for the normal crossing divisor as being “well-known” by
 1052 the mirror principle: as already stated in the introduction, the Frobenius structure for fixed t is known to
 1053 be isomorphic to the quantum cohomology ring of the ordinary projective space. But in fact we have more:
 1054 the parameter t corresponds exactly to the parameter in the small quantum cohomology ring (note that the
 1055 convention for the name of the coordinate on the parameter space differs from the usual one in quantum
 1056 cohomology, our t is usually called q and defined as $q = e^t$, where this t corresponds to a basis vector in the
 1057 second cohomology of the underlying variety, e.g., \mathbb{P}^{n-1}). Using this interpretation, the logarithmic structure
 1058 as defined above is the same as the one obtained in [Rei08, subsection 2.1.2].

1059 6 Examples

1060 We have computed the spectrum and monodromy for some of the discriminants in quiver representation spaces
 1061 described in [BM06]. In some cases, we have implemented the methods explained in the previous sections in
 1062 SINGULAR ([GPS05]). For the infinite families given in the Table 1 below, we have solved the Birkhoff problem by
 1063 essentially building the seminvariants h_i , where $h = h_1 \cdots h_k$ is the equation of D , by successive multiplication
 1064 by $(-f)$.

1065 We will present two types of examples. On the one hand, we will explain in detail some specific examples,
 1066 namely, the normal crossing divisor, the star quiver with three exterior vertices (denoted by \star_n in example
 1067 2.3 (i)), and the non-reductive example discussed after definition 2.1 for $k = 2$. We also give the spectral
 1068 numbers for the linear free divisor associated to the E_6 quiver (see example 2.3 (ii)), but we do not write down
 1069 the corresponding good basis, which is quite complicated (remember that already the equation 2.4 was not
 1070 completely given).

1071 On the other hand, we are able to determine the spectrum for (G_t, ∇) ($t \neq 0$) and (G_0, ∇) for the whole D_n -
1072 and \star_n -series by a combinatorial procedure. The details are rather involved, therefore, we present the results,
1073 but refer to the forthcoming paper [dGS09] for full details and proofs. It should be noticed that except in the
1074 case of the normal crossing divisor and in very small dimensions for other examples, it is very hard to write
1075 down explicitly elements for the good bases $\underline{\omega}^{(2)}$ and $\underline{\omega}^{(3)}$ as already the equation for the divisor becomes quickly
1076 quite involved.

1077 Let us start with the three explicit examples mentioned above. Notice that in all examples discussed below the
1078 linear function is the sum of the coordinates, which is, in each case, easily seen to lie in the open orbit of the
1079 dual action.

1080 **The case of the normal crossing divisor:** As noticed in the first section, this is the discriminant in the
1081 representation space of any quiver with a tree as underlying (oriented) graph. In particular, it is the discriminant
1082 of the Dynkin A_{n+1} -quiver. Choosing coordinates x_0, \dots, x_n on V , we have $h = x_0 \cdot \dots \cdot x_n$. A direct calculation
1083 (i.e., without using lemma 4.4 and algorithm 1) shows that $\underline{\omega}^{(1)} = \underline{\omega}^{(2)} = \underline{\omega}^{(3)} = \left(\prod_{j=0}^{i-1} x_j \cdot \alpha \right)_{i=1, \dots, n}$. This is
1084 consistent with the basis found in [DS04, proposition 3.2]. In particular, we have $A^{(2)} = A^{(3)} = -\text{diag}(0, \dots, n-1)$,
1085 so the spectral numbers of (G_t, ∇) for $t \neq 0$ and (G_0, ∇) are $(0, \dots, n-1)$. We also see that $(nt\partial_t)\underline{\omega}^{(2)} =$
1086 $\underline{\omega}^{(2)} \cdot \tau\Omega_0$, which is a well known result from the calculation of the quantum cohomology of \mathbb{P}^{n-1} (see the last
1087 remark in subsection 5.3).

1088 **The case \star_3 (see example 2.3 (i)):** Here the bases of the Brieskorn lattice G produced by lemma 4.4 is

$$\begin{aligned} \omega_1^{(1)} &= \alpha & ; & \quad \omega_2^{(1)} = ??? \cdot \alpha & ; & \quad \omega_3^{(1)} = ??? \cdot \alpha \\ \omega_4^{(1)} &= ??? \cdot \alpha & ; & \quad \omega_5^{(1)} = ??? \cdot \alpha & ; & \quad \omega_6^{(1)} = ??? \cdot \alpha \end{aligned} \tag{6.1}$$

1089 and we have $A_\infty^{(1)} = \text{diag}(-0, -3, -2, -3, -4, -3)$. Algorithm 1 yields that $\omega_2^{(2)} = \omega_2^{(1)} + 2\tau^{-1}\omega_1^{(1)}$ and $\omega_i^{(2)} = \omega_i^{(1)}$
1090 for all $i \neq 2$, and we obtain $A_\infty^{(2)} = \text{diag}(-2, -1, -2, -3, -4, -3)$. As $\nu_1^{(2)} - \nu_6^{(2)} = -1 \leq 1$, we have $\underline{\omega}^{(2)} = \underline{\omega}^{(3)}$,
1091 hence $G^{(3)} = G$ and $(2, 1, 2, 3, 4, 3)$ is the spectrum for (G_t, ∇) , $t \neq 0$ as well as for (G_0, ∇) . We see that
1092 the minimal spectral number is unique, therefore, $\underline{\omega}^{(2)}$ yields a (V^+, S) -solution for any t . Moreover, we have
1093 $(nt\partial_t)\underline{\omega}^{(2)} = \underline{\omega}^{(2)} \cdot [\tau\Omega_0 + \text{diag}(-2, 0, 0, 0, 0, 2)]$, so that in this case the ∇^{res} -flat section $\omega_i^{(3)}$ from lemma 5.3 is
1094 $\omega_2^{(2)}$, which is an eigenvector of $A_\infty^{(2)}$ with respect to the minimal spectral number.

The case E_6 (see example 2.3 (ii)): The spectrum of both (G_t, ∇) , $t \neq 0$ and (G_0, ∇) is

$$\left(\frac{44}{5}, \frac{25}{3}, \frac{28}{3}, \frac{31}{3}, \frac{34}{3}, \frac{47}{5}, 6, \dots, 15, \frac{58}{5}, \frac{29}{3}, \frac{32}{3}, \frac{35}{3}, \frac{38}{3}, \frac{61}{5} \right)$$

Again we have a unique minimal spectral, hence, theorem 4.13 applies. The symmetry $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n + 1$
holds. Moreover, we obtain the following eigenvalues for the residue of $t\partial_t$ on $G|_{\mathbb{C}^* \times T}$ at $t = 0$:

$$\frac{1}{12} \cdot \left(-\frac{44}{5}, \left(-\frac{22}{3} \right)^4, -\frac{22}{5}, 0^{10}, \frac{22}{5}, \left(\frac{22}{3} \right)^4, \frac{44}{5} \right)$$

1095 which are (again) symmetric around zero (hence supporting conjecture 5.5 (ii)).

1096 **A non-reductive example in dimension 3 (see (2.2)):** The linear free divisor in \mathbb{C}^3 with equation $h =$
1097 $x(xz - y^2)$ is not special and therefore not reductive. The basis $\underline{\omega}^{(1)}$ is given as

$$\omega_1^{(1)} = \alpha & ; & \quad \omega_2^{(1)} = ??? \cdot \alpha & ; & \quad \omega_3^{(1)} = ??? \cdot \alpha \tag{6.2}$$

1098 and we have $A_\infty^{(1)} = \text{diag}(0, ?, ?)$. Algorithm 1 yields

$$\omega_1^{(2)} = ??? \cdot \alpha & ; & \quad \omega_2^{(2)} = ??? \cdot \alpha & ; & \quad \omega_3^{(2)} = ??? \cdot \alpha \tag{6.3}$$

and $A_\infty^{(1)} = \text{diag}(-\frac{3}{4}, -1, -\frac{5}{4})$. Again, as $\nu_1^2 - \nu_3^2 = -\frac{1}{2} \leq 1$, we obtain $\underline{\omega}^{(2)} = \underline{\omega}^{(3)}$, $G = G^{(3)}$, and $(\frac{3}{4}, 1, \frac{5}{4})$ is the
spectrum of both (G_t, ∇) , $t \neq 0$ and (G_0, ∇) . We can also compute the spectral numbers for the case $k = 3, 4$

and 5 (these are again the same for (G_t, ∇) , $t \neq 0$ and (G_0, ∇) , namely:

size of matrices	$\dim(V)$	Spectrum of (G_t, ∇)
$k = 3$	$n = 6$	$(2, \frac{5}{2}, 2, 3, \frac{5}{2}, 3)$
$k = 4$	$n = 10$	$(\frac{15}{4}, \frac{13}{3}, \frac{9}{2}, \frac{17}{4}, 4, 5, \frac{19}{4}, \frac{9}{2}, \frac{14}{3}, \frac{21}{4})$
$k = 5$	$n = 15$	$(6, \frac{53}{8}, 7, \frac{27}{4}, 7, \frac{55}{8}, 6, 7, 8, \frac{57}{8}, 7, \frac{29}{4}, 7, \frac{59}{8}, 8)$

1099 The case $k = 5$ is an example where the minimal spectral number is not unique, hence, theorem 4.13 does not
1100 apply. However, we observe that the “extra symmetry” $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n-1$ from corollary 5.6 still holds, which
1101 supports conjecture 5.5. One might speculate that although the eigenspace of the smallest spectral number is
1102 two-dimensional (generated by $\omega_1^{(3)}$ and $\omega_7^{(3)}$), we still have $\tau^{n-1}S(\omega_1^{(3)}, \omega_j^{(3)}) \in \mathbb{C}\delta_{j,15}$ (resp. $\tau^{n-1}S(\omega_7^{(3)}, \omega_j^{(3)}) \in$
1103 $\mathbb{C}\delta_{j,9}$) which would imply that the conclusions of theorem 4.13 still hold.

1104 Now we turn to the series D_m resp. \star_m . The results are given in table 1 below. We write $(p_1, q_1), \dots, (p_k, q_k)$
1105 to indicate that the output of algorithm 1 resp. algorithm 2 is a basis $\underline{\omega}^{(2)}$ resp. $\underline{\omega}^{(3)}$ which decomposes into k
1106 blocks as in the proof of proposition 4.8, where in each block (p_i, q_i) the eigenvalues of the residue endomorphism
1107 $\tau\partial_\tau$ along $\tau = 0$ are $-p_i, -p_i - 1, \dots, -p_i - q_i + 1$. In particular, this gives the monodromy of (G_t, ∇) according
1108 to corollary 4.9. We observe that in all cases the symmetries $\nu_i^{(2)} + \nu_{n+1-i}^{(2)} = -n+1$ and $\nu_i^{(3)} + \nu_{n+1-i}^{(3)} = -n+1$
1109 hold, and that the eigenvalues of the residue of $t\partial_t$ on G_0 are symmetric around zero.

	D_m	$\star_{m:=2k+1}$	$\star_{m:=2k}$
$\dim(D)=n-1$	$4m-11$	m^2-m-1	m^2-m-1
$Sp(G_0, \nabla)$	$(\frac{4m-10}{3}, m-3),$ $(m-3, 2m-4),$ $(\frac{5m-11}{3}, m-3)$???	???
$Sp(G_t, \nabla)$ $t \neq 0$	$(\frac{4m-10}{3}, m-3),$ $(m-3, 2m-4),$ $(\frac{5m-11}{3}, m-3)$	$\overbrace{(2k^2, m-2), (2k^2-1, m-2), \dots, (2k^2-k+1, m-2)}^k,$ $(2k^2-k, 2m-2),$ $\overbrace{(2k^2+k, m-2), (2k^2+k-1, m-2), \dots, (2k^2+1, m-2)}^k$	$\overbrace{(mk-k, k-1), (mk-m, m-2), (mk-m-1, m-2), \dots, (mk-3k+2, m-2)}^{k-1},$ $(2k^2-3k+1, 2m-2),$ $\overbrace{(mk-k, m-2), (mk-k-1, m-2), \dots, (mk-m+2, m-2), (mk-m+1, k-1)}^{k-1}$
$\text{Res}[t\partial_t]$ on $(G_0/tG_0) _{\tau \neq \infty}$	$(-\frac{4m-10}{3}, m-3),$ $(0, 2m-4),$ $(\frac{4m-10}{3}, m-3)$???	???

Table 1: Spectra of f on the Milnor resp. zero fibre for fibrations defined by some quiver discriminants.

1110 *Remark 6.1.* (i) We see that the jumping phenomenon (i.e., the fact that the spectrum of (G_t, ∇) , $t \neq 0$
1111 and (G_0, ∇) are different) occurs in our examples only for the star quiver for $n \geq 5$. However, there are
1112 probably many more examples where this happens, if the divisor D has sufficiently high degree.

1113 (ii) Each Dynkin diagram supports many different quivers, distinguished by their edge orientations. Neverthe-
1114 less, each of these quivers has the same set of roots. For quivers of type A_n and D_n , the discriminants in
1115 the corresponding representation spaces are also the same, up to isomorphism. However, for the quivers of
1116 type E_6 , there are three non-isomorphic linear free divisors associated to the highest root (the dimension
1117 vector shown). Their generic hyperplane sections all have the same spectrum and monodromy.

1118 (iii) For the case of the star quiver with $n = 2k$, the last and first blocks actually form a single block. We
1119 have split them into two to respect the order given by the weight of the corresponding elements in the
1120 Gauß-Manin system.

1121 (iv) In all examples presented above the ∇^{res} -flat basis element $t^{-k}\omega_i^{(3)}$ from lemma 5.3 was an eigenvector
 1122 of $A_\infty^{(3)}$ for the smallest spectral number. An example where the latter does not hold is provided by
 1123 the *bracelet*, the discriminant in the space of binary cubics (the last example in 4.4 of [GMNS06]). The
 1124 spectrum of the generic hyperplane section is $(\frac{2}{3}, 1, 2, \frac{7}{3})$, and hence the minimal spectral number is not
 1125 an integer. However, the ∇^{res} -flat section is an eigenvector for a spectral number within the intervall
 1126 $[\alpha_{min}, \alpha_{min} + 1)$, which is thus still a primitive and homogenous section for the canonical solution \widehat{G}_t^{can}
 1127 used in theorem 5.1 (ii).

1128 Let us finish the paper by a few remarks on open questions and problems related to the results obtained.
 1129 In [DS04], where similar questions for certain Laurent polynomials are studied, it is shown that the (V^+, S) -
 1130 solution constructed coincides in fact with the canonical solution as described in theorem 4.7 (see of loc.cit.).
 1131 A natural question is to ask whether the same holds true in our situation.

1132 A second problem is to come to a more conceptual understanding of the degeneration behavior of the various
 1133 Frobenius structures M_t as discussed in theorem 5.9 in those cases (i.e., all examples except the normal crossing
 1134 case), where only a weak logarithmic Frobenius manifold is found. As already pointed out, a rather similar
 1135 phenomenon occurs in [Dou08].

1136 Another very interesting point is to better understand the relation of the Frobenius structure constructed to the
 1137 called tt^* -geometry (also known as variation of TERP- resp. integrable twistor structures, see, e.g., [Her03]).
 1138 We know from corollary 4.5 (v) that the families studied here are examples of *Sabbah orbits*. The degeneration
 1139 behavior of such variations of integrable twistor structures has been studied in [HS07] using methods from
 1140 [Moc07]. However, the extensions over the boundary point $0 \in T$ used in loc.cit. is in general different from the
 1141 lattices G resp. $G^{(3)}$ considered here, as the eigenvalues of the residue $[t\partial_t]$ computed above does not always lie
 1142 in a half-open interval of length one (i.e., $G|_{\mathbb{C}^* \times T}$ is not always a Deligne extension of $G|_{\mathbb{C}^* \times (T \setminus \{0\})}$). One might
 1143 want to better understand what kind of information is exactly contained in the extension G . Again, a similar
 1144 problem is studied to some extend for Laurent polynomials in [Dou08].

1145 Finally, as we already remarked, the connection ∂_τ is regular singular at $\tau = \infty$ on \mathbf{G}_0 but irregular for $t \neq 0$.
 1146 Irregular connections are characterized by a subtle set of topological data, the so-called Stokes matrices. It
 1147 might be interesting to calculate these matrices for the examples we studied, extending the calculations done
 1148 in [Guz99] for the A_n -case.

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