Not all codimension 1 germs have good real pictures

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1 Introduction

Some time ago the first author conjectured (in [9]) that every $\mathcal{A}_e$-codimension 1 equivalence class of map-germs $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n \geq p-1$ and (n,p) nice dimensions, should have a real form with a "good real perturbation" — that is, the $\mathcal{A}$-equivalence class should contain a real germ (one whose power-series expansion has purely real coefficients) which moreover should have a real perturbation whose real discriminant (if $n \geq p$) or real image (if $n = p-1$) carries the vanishing homology of its complexification. The purpose of this note is to give an example for which this does not hold — thus proving the conjecture false.

The example is not hard to describe: it is the simplest $\mathcal{A}_e$-codimension 1 class of map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ of corank 2 (i.e. such that $\ker(d_0 f)$ has dimension 2). Since the codimension in jet space $J^1(n, n+1)$ of the set Σ^2 of matrices of kernel rank 2 is equal to 6, such a singularity cannot occur if $n < 5$.

In fact one can easily construct map-germs $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ of $\mathcal{A}_e$-codimension 1 and corank 2. From the point of view of this paper, the most straightforward procedure is by transverse pull-back from the stable map-germ $F : (\mathbb{C}^6, 0) \to (\mathbb{C}^7, 0)$ defined by

$$F(A, B, C, D, x, y) = (A, B, C, D, x^2 + Ay, xy + Bx + Cy, y^2 + Dx).$$

That is, if $i : \mathbb{C}^6 \to \mathbb{C}^7$ is transverse to F, then the fibre square

$$\begin{array}{ccc} (\mathbb{C}^6, 0) & \xrightarrow{F} & (\mathbb{C}^7, 0) \\ \uparrow & & \uparrow i \\ (\mathbb{C}^6 \times_{\mathbb{C}^7} \mathbb{C}^6, 0) & \xrightarrow{i^*(F)} & (\mathbb{C}^6, 0) \end{array}$$

gives us a map-germ $i^*(F)$ from the smooth 5-dimensional complex manifold germ $(\mathbb{C}^6 \times_{\mathbb{C}^7} \mathbb{C}^6, 0)$ to $(\mathbb{C}^6, 0)$. If i is suitably generic (in a sense we will make precise below) then $i^*(F)$ has $\mathcal{A}_e$-codimension 1. Every map-germ $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ with local algebra isomorphic to that of F (which is the simplest local algebra of type Σ^2, isomorphic to $\mathcal{O}_{\mathbb{C}^2,0}/(m_{\mathbb{C}^2,0})^2$), can be obtained by transverse pull-back from F.

To give a concrete example, one can compute that the germ

$$f(u, v, w, x, y) = (u, v, w, x^2 + uy, xy + vx + wy, y^2 - ux)$$

has $\mathcal{A}_e$-codimension 1. This germ is obtained as $i^*(F)$ by taking

$$i(u, v, w, x, y, z) = (u, v, w, -u, x, y, z).$$

Of course there is no canonical choice of coordinates on the fibre product $(\mathbb{C}^6 \times_{\mathbb{C}^7} \mathbb{C}^6, 0)$, and we could equally well have written $i^*(F)$ in a different form. In fact the $\mathcal{A}$-equivalence class of $i^*(F)$ is determined by the image L of i, and so a more economical way of studying the codimension 1 map-germs obtained in this way is to consider the space of smooth hypersurface-germs transverse to F. For the same reason we also adopt the notation $F_{|L}$ in place of $i^*(F)$, although the reader will of course appreciate that since L is contained in the *target* of F, this denotes base-change rather than restriction.

In Section 2 we show first that to obtain every $\mathcal{A}_e$-equivalence class we need consider only hyperplanes, and among hyperplanes only those of the form $kA + lB + mC + nD = 0$. We then show that within the space $\mathcal{L}$ of these hyperplanes there is a discriminant Δ, itself a hypersurface, such that if $L \notin \Delta$ then $F_{|L}$ has $\mathcal{A}_e$-codimension 1.

In Section 3 we study the topology of the complement in $\mathcal{L}_{\mathbb{R}}$ of $\Delta_{\mathbb{R}}$; we do not determine its homology, but we do locate at least one point in each connected component. If L_1 and L_2 are in the same connected component of $\mathcal{L}_{\mathbb{R}} \setminus \Delta_{\mathbb{R}}$, the two germs $F_{|L_1}$ and $F_{|L_2}$ are $\mathcal{A}$-equivalent (over $\mathbb{R}$) (Remark 2.5 below) and thus we do not need to examine both of them.

In Section 4, we compute the homotopy type of the image of a stable perturbation of map-germs $F_{|L}$, for each hyperplane L in the list that we determine in Section 3. Up to homeomorphism each real codimension 1 map-germ has two real stable perturbations, corresponding to the sign of the deformation parameter; however here it turns out that the two are isomorphic. This implies a rather strong version of triviality of the monodromy of the family of complex stable perturbations, over $\mathbb{C}^*$. Using Morse theory, we show that each of these complex images has the homotopy type of a 5-sphere; in consequence, the algebraic closure of the real image of each stable perturbation has the homotopy type of an n-sphere for some n between 0 and 5. For none of the examples in our list does n equal 5.

It turns out that $\Delta \subset \mathcal{L}$ is a free divisor, with some curious properties; in Section 5 we record some of these.

We are grateful to Jan Stevens for a geometrical observation concerning Δ, and to the organisers of the VIth Workshop for running such an enjoyable meeting.

2 $\mathcal{A}_e$-codimension 1 germs $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ of type Σ^2

Lemma 2.1 *Let $i : (\mathbb{C}^6, 0) \to (\mathbb{C}^7, 0)$ be a map germ such that $i^*(F)$ has $\mathcal{A}_e$-codimension 1. Then*

1. *i is an immersion, and*
2. *$i^*(F)$ is $\mathcal{A}$-equivalent to $i_\ell^*(F)$, where i_ℓ is the linear part of i.*

Proof Write $f = i^*(F)$. (1) By a theorem of J. Damon ([6], but see also [10], Section 8),

$$\frac{\theta(f)}{T\mathcal{A}_e f} \simeq N\mathcal{K}_{E,e} i := \frac{\theta(i)}{ti(\theta_{\mathbb{C}^6}) + i^*(\mathrm{Der}(\log E))}$$

where $E = F(\mathbb{C}^6)$. As F is not a trivial unfolding of any lower-dimensional germ, $\mathrm{Der}(\log E) \subset$

$m_{\mathbb{C}^7,0}\theta_{\mathbb{C}^7}$; hence

$$1 = \mathcal{A}_e\text{-codimension } f \geq \dim_{\mathbb{C}} \frac{\theta(i)}{ti(\theta_{\mathbb{C}^6}) + i^*(\mathrm{Der}(\log E)) + m_{\mathbb{C}^7,0}\theta(i)} = \dim_{\mathbb{C}} \frac{\mathbb{C}^7}{i_\ell(\mathbb{C}^6)}.$$

It follows that the inequality in the centre is an equality, and i is an immersion.
(2) is essentially Proposition 2.6 of [1], and can also be found, in this formulation, as of [13]. For completeness we outline the proof.

Since $i^*(F)$ has $\mathcal{A}_e$-codimension 1, we must have

$$ti(m_{\mathbb{C}^6,0}\theta_{\mathbb{C}^6}) + i^*(\mathrm{Der}(\log E)) = m_{\mathbb{C}^7,0}\theta(i);$$

by Nakayama's lemma this holds if and only if

$$ti(m_{\mathbb{C}^6,0}\theta_{\mathbb{C}^6}) + i^*(\mathrm{Der}(\log E)) + m^2_{\mathbb{C}^7,0}\theta(i) = m_{\mathbb{C}^7,0}\theta(i);$$

hence it depends only on the linear part of i. Let $i_\lambda = i_\ell + \lambda(i - i_\ell)$ be the linear interpolation between i_ℓ and i. For each value of λ, $T\mathcal{K}_E i_\lambda = m_{\mathbb{C}^7,0}\theta(i_\lambda)$, and a standard argument using Mather's Lemma (of [7]) and finite $\mathcal{K}_E$-determinacy now shows that the family i_λ is $\mathcal{K}_E$-trivial (over $\mathbb{R}$). It follows that the family $i^*_\lambda(F)$ is $\mathcal{A}$- trivial, and $i^*(F)$ is $\mathcal{A}$-equivalent to $i^*_\ell(F)$. $\square$

Lemma 2.2 *If the $\mathcal{A}_e$-codimension of $i^*_\ell(F)$ is equal to 1 then i_ℓ is $\mathcal{K}_E$- equivalent to a linear immersion whose image has equation of the form $kA + lB + mC + nD = 0$.*

Proof Let p define the image of i_ℓ. By the preceding lemma we may assume that p is linear. We can express the $\mathcal{K}_{E,e}$- normal space of i_ℓ (and hence the $\mathcal{A}_e$-normal space of $i^*_\ell(F)$) in terms of p:

$$N\mathcal{K}_{E,e}(i_\ell) \simeq \frac{\theta(p)}{tp(\mathrm{Der}(\log E)) + (p)\theta(p)}.$$

For if we write $\mathbb{C}^7 = L \times \mathbb{C}$ and identify p with projection to $\mathbb{C}$, then

$$\frac{\theta(p)}{tp(\mathrm{Der}(\log E)) + (p)\theta(p)} \simeq \frac{\theta_{L\times\mathbb{C}}}{\theta_{L\times\mathbb{C}/\mathbb{C}} + \mathrm{Der}(\log E) + m_{\mathbb{C},0}\theta_{L\times\mathbb{C}}}.$$

Now

$$\theta(i_\ell) = (\theta_{L\times\mathbb{C}})_{|\ L} = \theta_{L\times\mathbb{C}}/m_{\mathbb{C},0}\theta_{L\times\mathbb{C}}$$

and within $\theta_{L\times\mathbb{C}}/m_{\mathbb{C},0}\theta_{L\times\mathbb{C}}$,

$$\theta_{L\times\mathbb{C}/\mathbb{C}} = ti_\ell(\theta_L),$$

so

$$\frac{\theta_{L\times\mathbb{C}}}{\theta_{L\times\mathbb{C}/\mathbb{C}} + \mathrm{Der}(\log E) + m_{\mathbb{C},0}\theta_{L\times\mathbb{C}}} \simeq \frac{\theta(i_\ell)}{ti_\ell(\theta_L) + i^*(\mathrm{Der}(\log E))}.$$

If $i^*_\ell(F)$ has $\mathcal{A}_e$-codimension 1, then $tp(\mathrm{Der}(\log E)) + (p)\theta(p)$ must be equal to $m_{\mathbb{C}^7,0}$. Now the module $tp(\mathrm{Der}(\log E)) + (p)\theta(p)$ is the tangent space of p under the action of the group $\mathcal{K}(E)$, the semi-direct product of the group $\mathcal{R}_E$ of diffeomorphisms of $(\mathbb{C}^7, 0)$ preserving E, with the group $\mathcal{C}$. This is a geometric group in the sense of Damon [5]; it follows that p is finitely $\mathcal{K}(E)$-determined.

There is a 7-dimensional family of linear maps p,

$$\mathcal{L}_1 := \{p = kA + lB + mC + nD + pU + qV + rW : k, l, m, n, p, q, r \in \mathbb{C}\},$$

and within $\mathcal{L}_1$ a discriminant, Δ_1, in principle possibly equal to all of $\mathcal{L}_1$, or reduced to the zero map, consisting of linear maps for which $tp(\mathrm{Der}(\log E)) + (p)\theta(p)$ fails to be equal to $m_{\mathbb{C}^7,0}$. A calculation with *Macaulay* shows that the annihilator in $\mathbb{C}[A, \ldots, W, k, \ldots, r]$ of

$$\frac{(A, \ldots, W, k, \ldots, r)}{T\mathcal{K}(E)p}$$

is generated by

$$f(k, l, m, n) := 8kl^3 + 3l^2m^2 + 24klmn + 8m^3n - 16k^2n^2$$

together with polynomials in the ideal $(A, \ldots, W)$. Thus, the equation of the discriminant is $f = 0$, and in particular is independent of the coefficients p, q, r. Suppose $L \in \mathcal{L}_1 \setminus \Delta_1$ has equation ℓ, and let ℓ_0 be obtained from ℓ by setting to zero the coefficients of U, V and W. Then for each t, the hyperplane $L_t = \{(1-t)\ell_0 + t\ell = 0\}$ also lies outside Δ_1, and it follows from Mather's Lemma (and finite determinacy) that the linear interpolation between ℓ and the map ℓ_0 is $\mathcal{K}(E)$-trivial. Hence the corresponding family of immersions $\mathbb{C}^6 \to \mathbb{C}^7$ is $\mathcal{K}_E$-trivial. This completes the proof. $\square$

The Lemma tells us we can forget the family $\mathcal{L}_1$ of all linear maps $\mathbb{C}^7 \to \mathbb{C}$ and consider only those of the form $kA + lB + mC + nD$; we denote the family of these by $\mathcal{L}$.

Remark 2.3 The equation for the discriminant Δ may also be obtained by the following alternative method, which also relies on *Macaulay*, but which is primarily topological rather than deformation-theoretic.

Suppose that L is a hyperplane in $\mathbb{C}^7$, with defining equation $h_L(A, \ldots, W) = \mathbf{u_L} \cdot (A, \ldots, W) = 0$, and let G be a reduced defining equation for the image E of F. If the $\mathcal{A}_e$-codimension of $F_{|L}$ is 1, then the family of normal translations of L, $L_t := \{t\mathbf{u_L}\} + L$, $t \in (\mathbb{C}, 0)$, induces an $\mathcal{A}_e$-versal deformation of $F_{|L}$. This follows from the fact that for a deformation of a codimension 1 germ, versality is equivalent to infinitesimal non-triviality. By a Morse-theoretic argument of Dirk Siersma ([12]), the image of $F_{|L_t}$, i.e. $E \cap L_t$, has the homotopy type of a wedge of spheres, whose number, the *image Milnor number* of $F_{|L}$, $\mu_I(F_{|L})$), is equal to the sum of the Milnor numbers of the (isolated) singular points of $G_{|L}$ which move off the zero locus $E \cap L_t$ as t moves off 0. Let us call these points *indicator points.* For each value of t, the number of indicator points is finite; as t varies, they form a curve. In fact this curve is simply the polar curve of G with respect to the linear form h_L; its equations can easily be obtained as the generators of the transporter ideal $J_L : G$, where J_L is the ideal of maximal minors of the 2×7 jacobian matrix of the map (G, h_L).

It is worth noting that $V(J_L : G)$ evidently *contains* all the indicator points. However, *a priori* it is not clear that every zero of $J_L : G$ (off $\{t = 0\}$) is an indicator point. Nevertheless, this is the case. For provided $t \neq 0$, each germ of $F_{|L_t}$ is right-left stable. In Mather's nice dimensions (which contain $(5,6)$, cf [8]), every stable germ is weighted homogeneous in appropriate coordinates, and so its image has a weighted homogeneous defining equation. It follows that $E \cap L_t$ is locally weighted homogeneous at each point, and thus $G_{|L_t} \in J_{|L_t}$ at any point where $G = 0$. Since, for $t \neq 0$, the deformation induced by varying t is right-left trivial, it follows that $G \in J_L$ at any point where $G = 0$ and $t \neq 0$. Hence the set of indicator points coincides with $V(J_L : G)$.

All of this can be done simultaneously for the family of all hyperplanes $L = \{kA + lB + mC + nD = 0\}$. We work in $\mathbb{C}[A, \ldots, W, k, \ldots, n]$. Let $h(A, \ldots, n) = kA + lB + mC + nD$, let J be the ideal of maximal minors of the jacobian of (G, h) with respect to the variables $A, \ldots, W$, and let

$$\mathcal{I}_0 := J : G.$$

Evidently $\mathcal{I}_0$ has a (k,l,m,n)-primary component, corresponding to the absent hyperplane for which $k = l = m = n = 0$, which we remove by using a transporter ideal. We find that

$$\mathcal{I} = \mathcal{I}_0 : (k,l,m,n)$$

has no (k,l,m,n)-primary component, and is thus the ideal we require. It is equal to

$$(Ak - 1/3Bl + 1/3Cm - Dn, 4Bm - 4Ck - 3Dl, 3Am + 4Bn - 4Cl, B^2 - 3W, C^2 - 3U, 8BC - 3AD + 12V).$$

We claim that $V(\mathcal{I}) \to \mathcal{L}$ restricts to a locally trivial fibration over $\mathcal{L} \setminus \Delta$, with fibre over $L \in \mathcal{L} \setminus \Delta$ equal to the set of indicator points of the family $F_{|L_t}$, where $L_t = \{t\mathbf{u_L}\} + L$ is the normal translation of L. In fact this follows immediately from what we showed in the preceding paragraph: for $t \neq 0$ and $L \notin \Delta$, the deformation induced by varying L is locally trivial, and so the fibrewise equality

$$\{\text{indicator points}\} = V(J_L : G)$$

gives a relative equality

$$\{\text{relative indicator points}\} = V(J : G) = V(\mathcal{I}).$$

Now $F_{|L}$ has image Milnor number 1 if and only if the curve of indicator points in the deformation $F_{|L_t}$ is non-singular and transverse to L. We find the ideal of the set of points where this fails, by taking the determinant of the jacobian matrix of the pair $(g_1, \ldots, g_6, h)$, where $g_1, \ldots, g_6$ generate $\mathcal{I}$. Up to a scalar factor, this (thanks to *Macaulay*) gives the equation f that we found above,

$$f(k,l,m,n) = 8kl^3 + 3l^2m^2 + 24klmn + 8m^3n - 16k^2n^2.$$

We note as soon as we assign numerical values to the parameters k, l, m and n for which this equation does not vanish, thus determining a hyperplane $L \notin \Delta$, then the specialisation $\mathcal{I}_L$ of the ideal $\mathcal{I}$ defines a smooth curve $\mathbb{C}^7$ (the curve of indicator points), transverse to L and to its normal translates $L_t = \{t\mathbf{u_L}\} + L$. At any point P of this curve,

$$\mathcal{O}_{\mathbb{C}^7,P}/\mathcal{I}_L + (h_L - t) \simeq \mathcal{O}_{L_t,P}/J_{G|L_t};$$

non-singularity of the curve implies, by Siersma's result, that $\mu_I(F_{|L})) = 1$. This conclusion retrospectively justifies the approach we adopt in this remark.

We record this conclusion as:

Proposition 2.4 *Any map-germ* $(\mathbb{C}^5,0) \to (\mathbb{C}^6,0)$ *with* $\mathcal{A}_e$*-codimension equal to 1 and Boardman type* Σ^2 *has image Milnor number* μ_I *equal to 1.* $\square$

Remark 2.5 If L_λ is any path in $\mathcal{L} \setminus \Delta$ then the argument of Lemma 2.2 shows that the family $F_{|L_\lambda}$ is $\mathcal{A}$-trivial. The ideas just used prove also

Proposition 2.6 *If* $F_{|L}$ *has* $\mathcal{A}_e$*-codimension 1 then it is 2-determined for* $\mathcal{A}$*-equivalence.*

Proof Since $f := F_{|L}$ has $\mathcal{A}_e$-codimension 1, and there is no stable germ in the same $\mathcal{K}$-orbit, its $\mathcal{A}$-orbit (in any jet space) must be open in its $\mathcal{K}$-orbit, and the tangent spaces to the two orbits coincide at each point of the former. The latter tangent space is easily shown to contain $m^2_{\mathbb{C}^5,0}\theta(f)$. The result now follows by Mather's Lemma. $\square$

3 The complement of the real discriminant $\Delta_{\mathbb{R}}$

To determine a list of normal forms for $\mathcal{A}_e$-codimension 1 germs $(\mathbb{R}^5,0) \to (\mathbb{R}^6,0)$ of type Σ^2, we have to select one real point from each component of $\mathbb{P}(\mathcal{L} \setminus \Delta)$. This can be done as follows: within each component of $S^3 \setminus \Delta_{\mathbb{R}}$, the function $f_{|\ S^3}$ achieves at least one local maximum or local minimum. The critical points of $f_{|\ S^3}$ are the zeros of the ideal I of 2×2 minors of the jacobian matrix of (f, r^2), where r^2 is the usual distance-squared function. To eliminate the components lying in Δ we calculate the transporter ideal $I : f$. Once again by a *Macaulay* calculation, we find that this is equal to

$$(km - 4/7lm, lm - 7/4ln, k^2 - 1/3l^2 + 1/3m^2 - n^2).$$

This ideal defines a curve consisting of the eight lines through $(0,0,0,0)$, shown in the left hand column of table 1. The second column shows the signs of the eigenvalues of the Hessian of $f_{|\ S^3}$ at the points where these lines meet the sphere. The third column shows information obtained in the next section: it is the dimension of the sphere to which the real section $L_t \cap E \cap \mathbb{R}^6$ is homotopy-equivalent for $t \neq 0$. By a Morse-theoretic argument given in the next section, this is one less than the index displayed in the third column of Table 2.

Line	Eigenvalues	Homotopy dimension of $L_t \cap E \cap \mathbb{R}^6$
$\mathrm{Sp}\{(1,\sqrt{3},0,0)\} \cap S^3$	$- \ - \ -$	4
$\mathrm{Sp}\{(1,-\sqrt{3},0,0)\} \cap S^3$	$+ \ + \ -$	3
$\mathrm{Sp}\{(0,0,\sqrt{3},1)\} \cap S^3$	$- \ - \ -$	4
$\mathrm{Sp}\{(0,0,-\sqrt{3},1)\} \cap S^3$	$+ \ + \ -$	3
$\mathrm{Sp}\{(1,0,0,1)\} \cap S^3$	$+ \ + \ +$	3
$\mathrm{Sp}(1,0,0,-1)\} \cap S^3$	$+ \ + \ +$	3
$\mathrm{Sp}\{(4,7,7,4)\} \cap S^3$	$+ \ - \ -$	4
$\mathrm{Sp}\{(4,7,-7,-4)\} \cap S^3$	$+ \ - \ -$	4

Table 1: Critical points of $f_{|\ S^3}$

To obtain a representative of each $\mathcal{A}$- equivalence class we need take only those $F_{|L}$ for which L corresponds to one of the lines at which $f_{|\ S^3}$ has a local maximum or minimum: $L = \{A+\sqrt{3}B=0\}$, $L=\{\sqrt{3}C+D=0\}$, $L=\{A+D=0\}$ and $L=\{A-D=0\}$. In fact F has an obvious symmetry permuting x and y in the source and U and W in the target, and permuting also A and D, and B and C in both source and target. This symmetry interchanges the first and second of the hyperplanes just listed, and therefore we do not need to consider both.

4 The topology of real perturbations

Let $L_{\mathbb{R}} \subset \mathbb{R}^7$ be a hyperplane (containing 0), let $\mathbf{u_L} \neq 0$ be orthogonal to $L_{\mathbb{R}}$, and let $L_{\mathbb{R},t}$ be the affine translate of $L_{\mathbb{R}}$ by $t\mathbf{u_L}$. If $F_{|L_{\mathbb{R}}}$ has $\mathcal{A}_e$-codimension 1, then the family $F_{\mathbb{R}|L_{\mathbb{R},t}}$ is an $\mathcal{A}_e$- versal deformation of $F_{\mathbb{R}|L_{\mathbb{R}}}$.

From 2.4 it follows (*cf* [3]) that $L_{\mathbb{R},t}\cap E_{\mathbb{R}}$ is homotopy-equivalent to a d-sphere, for some d between 0 and 5. For by Siersma's result, $F_{|L_t}$ has just one indicator point; this must therefore be real (otherwise its complex conjugate would also be an indicator point), and now $L_{\mathbb{R},t}\cap E_{\mathbb{R}}$ is homotopy-equivalent to the boundary of a k-cell, where k is the index of $G_{\mathbb{R}|L_{\mathbb{R},t}}$ at the indicator point. The indicator point can be determined by substituting the values of k,l,m and n into the expressions for the generators of the ideal $\mathcal{I}$ listed in Remark 2.3. The Hessian of $G_{|\ L_t}$ at the indicator point can be computed as P^tH_GP, where H_G is the Hessian of G and P is a 7×6 matrix whose columns form an orthonormal basis for L.

The results of this computation are displayed in Table 2. In principle we should display the signs of the eigenvalues of the Hessian of $G_{|\ L_t}$ at the indicator point for two values of t, one positive and one negative. However, there is an obvious symmetry which renders this unnecessary: if (A,B,C,D,U,V,W) is the indicator point for L_t then $(-A,-B,-C,-D,U,V,W)$ is the indicator point for L_{-t}, and the Hessian matrices of G at these two indicator points are the same.

Equation of L	Ideal of indicator point	Index
$A+\sqrt{3}B$	$(\sqrt{3}A-B,C,D,U,V,3W-B^2)$	5
$A-\sqrt{3}B$	$(\sqrt{3}A+B,C,D,U,V,3W-B^2)$	4
$\sqrt{3}C+D$	$(\sqrt{3}D-C,A,B,W,V,3U-C^2)$	5
$\sqrt{3}C-D$	$(\sqrt{3}D+C,A,B,W,V,3U-C^2)$	4
$A+D$	$(B,C,A-D,D^2-4V,W,U)$	4
$A-D$	$(B,C,A+D,D^2+4V,W,U)$	4
$4A+7B+7C+4D$	$(B-7/4D,A-D,C-B,D^2-48/49W,U-W,V+86/49W)$	5
$4A+7B-7C-4D$	$(B+7/4D,A+D,C+B,D^2-48/49W,U-W,V-86/49W)$	5

Table 2: Indicator points and Morse indices

5 Concluding Remarks

This counterexample reveals a sharp difference between singularities of corank 1 and of higher corank. The main theorem of [4] states that *every* $\mathcal{A}_e$-codimension 1 multi-germ $(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$

($n \geq p-1$, (n,p) nice dimensions), in which each component has corank ≤ 1, has a real form with a good real perturbation, and thus a "good real picture". It is clear from our example here that good real pictures cannot be expected for germs of corank greater than 1.

6 Appendix: Geometry of the discriminant Δ

Jan Stevens has pointed out to us that the discriminant Δ of Section 3 is the affine cone over the tangent developable surface of the rational normal curve (a twisted cubic) in $\mathbb{P}^3$ parametrised by

$$(u,v) \mapsto (u^3, -2uv^2, 2u^2v, -v^3).$$

For it is easy to check that the singular locus of Δ is the cone over this curve, and then to check that Δ coincides with the tangent developable. The tangent developable has a cuspidal edge along the curve, and is otherwise smooth. The map $\mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^3$ parametrising the real part of the tangent developable is therefore a homeomorphism onto its image, and from this it follows by a straightforward duality argument that the complement of $\Delta_{\mathbb{R}}$ has just two connected components. Thus, it is no surprise that the eight lines in the complement of $\Delta_{\mathbb{R}}$ that we listed in Section 3, and the eight hyperplanes in $L \subset \mathbb{R}^7$ that they correspond to, should give rise to just two inequivalent real codimension 1 map-germs $F_{|L}$.

It also turns out that Δ is a free divisor. This is easy to check, using Saito's criterion (*cf* [11]): a *Macaulay* calculation shows that the module $\mathrm{Der}(\log \Delta)$ of vector fields on $\mathcal{L}$ which are tangent to Δ is generated by the vector fields $\chi_1, \ldots, \chi_4$ whose coefficients in the basis $\partial/\partial k, \partial/\partial l, \partial/\partial m, \partial/\partial n$ form the columns of the matrix

$$M = \begin{pmatrix} k & -3/4k & 0 & -3/4m \\ l & 1/4l & -m & -n \\ m & -1/4m & k & l \\ n & 3/4n & 3/4l & 0 \end{pmatrix},$$

and the determinant of this matrix is a reduced defining equation of Δ. We note for future reference that $\chi_1 \cdot f = 4f$ and $\chi_i \cdot f = 0$ for $i \neq 1$.

The free divisor Δ is unusual in that all of the generating vector fields are linear. This makes it particularly easy to calculate the cohomology of the complement: from its geometrical description, Δ is clearly locally quasihomogeneous, and thus we can use the theorem of [2]:

Theorem 6.1 *If $D \subset \mathbb{C}^n$ is a locally quasihomogenenous free divisor, then integration of forms along cycles defines an isomorphism*

$$h^q(\Gamma(\mathbb{C}^n, \Omega^\bullet(\log \Delta))) \simeq H^q(\mathbb{C}^n \setminus \Delta; \mathbb{C}).$$

□

Via a contracting homotopy defined by the Lie derivative with respect to the Euler vector field, the complex $\Omega^\bullet(\log \Delta)_0$ is quasi-isomorphic to its subcomplex of weight zero. As all the generating vector fields are linear, they have weight zero, and so dually the generators $\omega_1, \omega_2, \omega_3, \omega_4$ of $\Omega^1(\log \Delta)$ (determined by the property that $\omega_i(\chi_j) = \delta_{i,j}$) also have weight zero. Thus the weight zero subalgebra of $\Omega^\bullet(\log\Delta)$ is the free exterior algebra over $\mathbb{C}$ on generators $\omega_1,\ldots,\omega_4$. As soon as we determine the coefficients $\Gamma^i_{j,k}$ in

$$d\omega_i = \sum_{j<k}\Gamma^i_{j,k}\omega_j\wedge\omega_k$$

which must be simply complex numbers, determination of $H^*(\mathbb{C}^4\setminus\Delta)$ is reduced to linear algebra. In fact we have

$$\begin{aligned} d\omega_1 &= 0\\ d\omega_2 &= -\omega_3\wedge\omega_4\\ d\omega_3 &= \tfrac{1}{2}\,\omega_2\wedge\omega_3\\ d\omega_4 &= -\tfrac{1}{2}\,\omega_2\wedge\omega_4\end{aligned}$$

and thus

$$\begin{aligned} d(\omega_1\wedge\omega_2) &= \omega_1\wedge\omega_3\wedge\omega_4 & d(\omega_2\wedge\omega_3) &= 0\\ d(\omega_1\wedge\omega_3) &= -\tfrac{1}{2}\omega_1\wedge\omega_2\wedge\omega_3 & d(\omega_2\wedge\omega_4) &= 0\\ d(\omega_1\wedge\omega_4) &= \tfrac{1}{2}\omega_1\wedge\omega_2\wedge\omega_4 & d(\omega_3\wedge\omega_4) &= 0\end{aligned}$$

and

$$d(\omega_i\wedge\omega_j\wedge\omega_k) = 0$$

for all i,j,k. Hence

Proposition 6.2

$$H^q(\mathcal{L}\setminus\Delta;\mathbb{C}) = \begin{cases}\mathbb{C} & \text{if } q=0\\ \mathbb{C},\ \textit{generated by } \omega_1 & \text{if } q=1\\ 0 & \text{if } q=2\\ \mathbb{C},\ \textit{generated by } \omega_2\wedge\omega_3\wedge\omega_4 & \text{if } q=3\\ \mathbb{C},\ \textit{generated by } \omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4 & \text{if } q=4\end{cases}$$

The equality $d\omega_2 = -\omega_3\wedge\omega_4$ shows that the family $f:\mathbb{C}^4\to\mathbb{C}$ is a *logarithmic contact bundle*. For, as with any free divisor $D=\{f=0\}$, the wedge product of all the generators of $\Omega^1(\log D)$ is equal to the generator of the top-dimensional module of logarithmic differential forms. Since $\omega_1 = df/f$ can be taken as one of these generators, this shows that the wedge product of the remaining generators of $\Omega^1(\log D)$ generates the relative dualising module ω_f (for it can be written as the "wedge division" $dz_1\wedge\cdots\wedge dz_n/df$) and thus its restriction to the regular fibres is a holomorphic volume form. Since in our example, $\omega_2\wedge d\omega_2 = -\omega_2\wedge\omega_3\wedge\omega_4$, we have

Proposition 6.3 *The restriction of ω_2 to each fibre of f is a contact form, and so $f:\mathbb{C}^4\to\mathbb{C}$ is a logarithmic contact bundle.* □

We note that a similar phenomenon can never occur for a function with isolated singularity, in any dimension. For if $f:(\mathbb{C}^{2n},0)\to(\mathbb{C},0)$ has isolated singularity at 0 and $D=f^{-1}(0)$, then

$$\Omega^1(\log D) = \Omega^1_{\mathbb{C}^{2n}} + \mathcal{O}_{\mathbb{C}^{2n}}\cdot df/f,$$

and so the restriction to the fibre of any logarithmic 1-form ω is actually the restriction of a regular form. Hence for any logarithmic 1-form ω, $df\wedge\omega\wedge(d\omega)^{\wedge n-1}\subset df\wedge\Omega^{2n-2}_{\mathbb{C}^{2n}}$; since

$\omega_f/df\wedge\Omega^{2n}_{\mathbb{C}^{2n+1}}$ has positive dimension $\tau(D)$ if f is singular, no element of $df\wedge\Omega^{2n}$ can generate ω_f.

We do not know whether this property of Δ, or the fact that it is a free divisor, can be deduced from the singularity theory in which it is embedded.

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