Cohomology of the complement of a free divisor*

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Abstract

We prove that if D is a "strongly quasihomogeneous" free divisor in the Stein manifold X, and U is its complement, then the de Rham cohomology of U can be computed as the cohomology of the complex of meromorphic differential forms on X with logarithmic poles along D, with exterior derivative. The class of strongly quasihomogeneous free divisors, introduced here, includes free hyperplane arrangements and the discriminants of stable mappings in Mather's nice dimensions (and in particular the discriminants of Coxeter groups).

1 Introduction

The purpose of this paper is to prove the following theorem:

Theorem 1.1 *Let D be a strongly quasihomogeneous free divisor in the complex manifold X, let U be the complement of D in X, and let $j : U \to X$ be inclusion. Then the natural morphism from the complex $\Omega^\bullet_X(\log D)$ of differential forms with logarithmic poles along D to $\mathbf{R}j_*\mathbf{C}_U$ is a quasi-isomorphism.*

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The natural morphism referred to here is simply the de Rham morphism, which at presheaf level is just the map sending $\omega \in \Gamma(V, \Omega^p(\log D))$ to $\int \omega : H_p(V - V \cap D, \mathbf{C}) \to \mathbf{C}$.

This result should be contrasted with Grothendieck's Comparison Theorem ([4]):

Theorem 1.2 *If D is a divisor in the complex manifold X, then the de Rham morphism*

$$\Omega_X^\bullet(*D) \to \mathbf{R}j_*(\mathbf{C}_U)$$

is a quasi-isomorphism. $\square$

Here $\Omega_X^\bullet(*D)$ is the complex of meromorphic differential forms on X with meromorphic poles (of arbitrary order) along D; $\Omega_X^\bullet(\log D)$ is the subcomplex consisting of forms ω such that both ω and $d\omega$ have at most a first-order pole along D. Logarithmic differential forms were first defined in full generality by Kyogi Saito in [12], where the notion of *free divisor* was also introduced. A hypersurface $D \subseteq X$ is a *free divisor* if the $\mathcal{O}_X$-module $\mathrm{Der}(\log D)$ of (ambient) vector fields which are tangent to D at each of its smooth points, is locally free. A smooth hypersurface is of course a free divisor, as is the union of the coordinate hyperplanes in $\mathbf{C}^t$: in the latter case, $\mathrm{Der}(\log D)$ is freely generated over $\mathcal{O}$ by the vector fields $x_1\partial/\partial x_1, \ldots, x_t\partial/\partial x_t$. The determinant of the matrix of coefficients of the generators of $\mathrm{Der}(\log D)$ in this example is actually an equation for D; this is always the case for a free divisor, and characterises them (this is "Saito's criterion"; see[12], 1.8).

The notion of *strong quasihomogeneity* is a slight extension of that of quasihomogeneity. We shall use the term *weighted homogeneous* in the strict sense: a divisor $D \subset \mathbf{C}^t$ is weighted homogeneous if it has a defining equation h which is homogeneous with respect to *positive* weights w_i.

Definition 1.3 *A divisor D in a t-dimensional complex manifold X is strongly quasihomogeneous if at each point $q \in D$, there are local coordinates $(U; x_1, \ldots, x_t)$ centred at q (i.e. with $x_i(q) = 0$ for i=1,...,t) with respect to which $D \cap U$ is weighted homogeneous. Similarly, the polynomial $f(x_1, \ldots, x_t)$ is strongly quasihomogeneous if $f^{-1}(0)$ is strongly quasihomogeneous.*

For an example of a weighted homogeneous divisor which is not strongly quasihomogeneous, consider the surface D in 3-space given by $f(x, y, z) =$

$x^5z + x^3y^3 + y^5z = 0$. Clearly D is weighted homogeneous, and thus its germ at each point $q \neq 0$ is analytically trivial (i.e. a product) along the $\mathbf{C}^*$- orbit through q. By Lemma 2.3, if the germ (D, q) were isomorphic to a weighted homogeneous germ $(D_1, 0)$ then (D, q) would be isomorphic to a product $(\mathbf{C}, 0) \times (D', 0)$ with D' weighted homogeneous. We show that this cannot be the case at any point $q = (0, 0, z_0)$ with $z_0 \neq 0$. The singular locus of D is the z-axis, and so any product structure in the neighbourhood of a point on this axis must have the axis as a factor. However, if S is a smooth surface meeting the z-axis transversely at q, then the germ at q of $f_{|S}$ is not in its own jacobian ideal (this can be easily checked by pulling back the equation f via an embedding $(\mathbf{C}^2, 0) \to (\mathbf{C}^3, q)$.) It follows that the germ of $S \cap D$ at q is not weighted homogeneous with respect to any system of coordinates centred at q.

Natural examples of strongly quasihomogeneous free divisors include the union of reflecting hyperplanes of a complex reflection group, and indeed any free hyperplane arrangement (see e.g [11], Chapter 4). Also if X and Y are complex manifolds with $\dim X \geq \dim Y$, and $f : X \to Y$ is a stable mapping, then its discriminant $D(f)$ is a free divisor; if moreover ($\dim X$,$\dim Y$) are nice dimensions in the sense of J. Mather, then $D(f)$ is also strongly quasihomogeneous. This is discussed in more detail below.

Theorem 1.1 has the following corollaries:

Corollary 1.4 *If X is a Stein manifold, $D \subseteq X$ is a strongly quasihomogeneous free divisor, and U is its complement, then*

$$H^k(U; \mathbf{C}) \simeq h^k(\Gamma(X, \Omega^\bullet(logD))).$$

(here we use the lower case h^p to denote the cohomology of a complex).
Proof By a standard argument, $H^p(U; \mathbf{C}) = \mathbf{H}^p(X, \mathbf{R}j_*(\mathbf{C}_U))$, where $\mathbf{H}$ means hypercohomology. By theorem 1.1,

$$\mathbf{H}^p(X, \mathbf{R}j_*(\mathbf{C}_U)) \simeq \mathbf{H}^p(X, \Omega^\bullet(logD)).$$

Finally, because X is a Stein space, we have

$$\mathbf{H}^p(X, \Omega^\bullet(logD)) \simeq h^p(\Gamma(X, \Omega^\bullet(logD))).$$

□

In particular,

Corollary 1.5 *Let X and Y be complex manifolds of dimensions n and t, with $n \geq t$ and (n,t) nice dimensions, and suppose in addition Y is Stein. If $f : X \to Y$ is a locally stable map (i.e. each of its multi-germs is stable) with discriminant D, then for all $k \geq 0$,*

$$h^k(\Gamma(Y, \Omega^\bullet_Y(log D))) = H^k(Y - D; \mathbf{C}).$$

Proof i) D is a free divisor. Since this is a local question, it is enough to consider, for each $y \in D$, the germ

$$f : (X, f^{-1}(y) \cap \Sigma) \to (Y, y)$$

where Σ is the critical set of f. Freeness of (D, y) was proved first by Saito (when the germ of f is a deformation of a hypersurface singularity) and then by Looijenga in ([6], 6.13) in the case that the germ of f is a versal deformation of an isolated complete intersection singularity (ICIS). It is well-known (and easy to prove) that if $n \geq t$ then f is stable if and only if $f^{-1}(0)$ is an ICIS and f is a versal deformation of $f^{-1}(0)$.
ii) in the nice dimensions, every stable map-germ is quasihomogeneous, in approriate coordinates (see below). □

Corollary 1.6 *If $D \subset \mathbf{C}^t$ is the discriminant of a Coxeter group, then for all $k \geq 0$,*

$$h^k(\Gamma(\mathbf{C}^t, \Omega^\bullet_{\mathbf{C}^t}(log D))) = H^k(\mathbf{C}^t - D; \mathbf{C}).$$

Proof It was shown by Brieskorn (cf. [1]) that D is the discriminant of the versal deformation of the isolated hypersurface singularity which bears the name of the group. Thus, as described in the preceding proof, D is a free divisor. All of the discriminants of the Coxeter groups are quasihomogeneous; moreover, at each point $x \neq 0$, D is the product of the discriminant of a Coxeter group and a smooth factor (if irreducible at x) or the union of several such, meeting transversely. Strong quasihomogeneity follows. □

It is conjectured by H. Terao and S. Yuzvinsky that for hyperplane arrangements in affine spaces, the algebraic version of Corollary 1.4 holds even without the hypothesis of freeness:

Conjecture 1.7 *Let A be an arrangement of affine hyperplanes in $\mathbf{C}^t$. Let $\Omega^\bullet_{alg}(\log A)$ be the complex of germs of rational differential forms on $\mathbf{C}^t$ with logarithmic poles along the arrangement, with exterior derivative d. Then*

$$h^p(\Gamma(\mathbf{C}^t, \Omega^\bullet_{alg}(\log A))) = H^p(\mathbf{C}^t - A; \mathbf{C})$$

for $p \geq 0$.

They have proved the following partial affirmative answers:
$p = 0, 1, t$,
$t = 1, 2, 3$,
A is in general position, and
A is (central) generic.

The nice dimensions were defined by Mather in [8] as those dimensions (n, t) such that for any two smooth manifolds N and T of dimensions n and t, stable mappings are dense in the space of all proper smooth maps $N \to T$ (with its Whitney topology). The nice dimensions can also be described in complex analytic terms as those dimensions (n, t) such that every map-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^t, 0)$ of finite $\mathcal{A}_e$- codimension (i.e. isolated instability) has a 1-parameter deformation $\{f_\lambda\}_{\lambda \in (\mathbf{C},0)}$ such that f_λ is stable for $\lambda \neq 0$.

Mather determined exactly which dimensions were nice in [9]. In fact niceness reduces to a purely local question on the existence of moduli for contact equivalence, and the computations in [9] involved listing stable map-germs in the nice dimensions. All of the normal forms in the list are weighted homogeneous. All stable germs in the nice dimensions are obtained from the list (up to biholomorphic coordinate change) by the use of one or both of the following steps:

a) trivial unfolding: take f_0 from the list and define $f : (\mathbf{C}^n \times \mathbf{C}^\ell, 0) \to (\mathbf{C}^p \times \mathbf{C}^\ell, 0)$ by $f(x, u) = (f_0(x), u)$;

b) (in the case of stable multigerms), by putting together trivial unfoldings of germs from the list, with different domains, in such a way that the isosingular loci of the different germs meet in general position in the target [7].

Moreover, the quasihomogeneity of all stable germs actually characterises the nice dimensions, a fact which surely deserves some explanation (which we cannot give).

That D is *strongly* quasihomogeneous in the nice dimensions follows from what we have just said: for at each point, D is the discriminant of a stable map-germ.

We do not know if the theorem is true for all free divisors.

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2 Proofs

The proof of the main result relies on an easy technical lemma due to Naruki:

Lemma 2.1 *Suppose that $\mathcal{O}_{\mathbf{C}^t,0}$ (or $\mathbf{C}[x_1,\ldots,x_t]$) is filtered by giving each variable x_i the weight w_i, and let $\chi = \Sigma w_i x_i \partial/\partial x_i$ be the associated Euler vector field (i.e. associated infinitesimal generator of the $\mathbf{C}^*$- action). If ω is a homogeneous differential form, then*

$$L_\chi(\omega) = wt(\omega)\omega.$$

(Here $L_\chi(\omega) = d\iota_\chi(\omega) + \iota_\chi(d\omega)$ is the Lie derivative of ω with respect to χ.)

Proof Straightforward calculation. □

If ω is logarithmic then so is $\iota_\chi(\omega)$. Thus, the Lie derivative can be used to provide a contracting homotopy on the subcomplex $\Omega^\bullet(\log D)_0^w$ of the stalk complex $\Omega^\bullet(\log D)_0$ consisting of all germs of forms of fixed weight $w \neq 0$, showing it to be acyclic. It follows that

$$\Omega^\bullet(\log D)_0^0 \hookrightarrow \Omega^\bullet(\log D)_0,$$

is a quasi-isomorphism. For if $\omega \in \Omega^p(\log D)$ is closed, then so is each of its homogeneous parts. Writing $\omega = \sum_j \omega_j$, where ω_j is homogeneous of weight j, we then have $\omega_j = \frac{1}{j}L_\chi(\omega_j) = d(\frac{1}{j}\iota_\chi\omega_j)$ for $j \neq 0$, so $\omega = \omega_0 + d\sigma$, where $\sigma = \iota_\chi \sum_{j\neq 0}\frac{1}{j}\omega_j$. There is no problem with convergence of the infinite sum, so that σ really is a well-defined logarithmic form. Hence the classes of ω and ω_0 in $h^p(\Omega^\bullet(\log D)_0)$ are equal.

A similar argument shows that

$$\Omega^\bullet(*D)^0_0 \hookrightarrow \Omega^\bullet(*D)_0,$$

is a quasi-isomorphism, where $\Omega^\bullet(*D)_0$ is the complex of germs at 0 of meromorphic forms with poles of arbitrary order along D.

For future use, we record some further easy consequences of this argument.

Lemma 2.2 *Let X be a complex analytic manifold of dimension n and let $D \subseteq X$ be a divisor. Let $\widetilde{X} := X \times \mathbf{C}$, let $\pi : \widetilde{X} \to X$ and $q : \widetilde{X} \to \mathbf{C}$ be the projections, let $\widetilde{D} := D \times \mathbf{C}$ and let $H := X \times \{0\}$. Then the following hold:*

i) The natural morphism $\pi^{-1}\Omega^\bullet_X(logD) \to \Omega^\bullet_{\widetilde{X}}(log\widetilde{D})$ is a quasi-isomorphism.

ii) The natural morphism $\pi^{-1}\Omega^\bullet_X(*D) \to \Omega^\bullet_{\widetilde{X}}(*\widetilde{D})$ is a quasi-isomorphism.

iii) The inclusion $\Omega^\bullet_{\widetilde{X}}(log\widetilde{D} \cup H) \hookrightarrow \Omega^\bullet_{\widetilde{X}}(log\widetilde{D})(*H)$ is a quasi-isomorphism.

iv) $D \subset X$ is free if and only $\widetilde{D} \subset \widetilde{X}$ is free. Moreover, in this case $\Omega^1_{\widetilde{X}}(log\widetilde{D}) = \pi^*\Omega^1_X(logD) \oplus \mathcal{O}_{\widetilde{X}} \cdot dt$.

v) $D \subset X$ is free if and only if $\widetilde{D} \cup H$ is free. Moreover, in this case $\Omega^1_{\widetilde{X}}(log\widetilde{D} \cup H) = \pi^*\Omega^1_X(logD) \oplus \mathcal{O}_{\widetilde{X}} \cdot \frac{dt}{t}$.

Proof

(i), (ii) and (iii): by giving t weight 1, and local coordinates on X weight 0, we ensure that the complex on the left hand side of each arrow is the weight 0 part of the complex on the right.

(iv) This is an easy consequence of Saito's criterion. Suppose first that $\widetilde{D}$ is free. Since $\partial/\partial t \in \mathrm{Der}(log\widetilde{D})$, we can choose a basis $\xi_1, \ldots, \xi_n, \partial/\partial t$ for $\mathrm{Der}(log\widetilde{D})_{(x_0,t_0)}$ whose first n members have no $\partial/\partial t$ component. By Saito's criterion, the determinant of their matrix of coefficients (with respect to some coordinate system on X) is a local defining equation h for $\widetilde{D}$. The restrictions of the ξ_i to a slice $X \times \{t_0\}$ are in $\mathrm{Der}(logD)_{x_0}$, and the determinant of their matrix of coefficients is the restriction of h to $X \times \{t_0\}$. It is thus a local equation for D. This proves that D is free. The converse is still easier. The second assertion does not in fact require the hypothesis of freeness, but its proof in the free case is an obvious consequence of the dual version of Saito's criterion: if g is a defining equation for a free divisor D at x_0, then $\omega_1, \dots, \omega_n$ generate $\Omega^1_X(log D)_{x_0}$ over $\mathcal{O}_{X,x_0}$ if and only if $\frac{1}{g}\omega_1 \wedge \cdots \wedge \omega_n$ is a generator of Ω^n_{X,x_0}. It follows from this criterion (applied to D and to $\widetilde{D}$) that the $\mathcal{O}_{\widetilde{X}}$-modules $\pi^*\Omega^1_X(log D) \oplus \mathcal{O}_{\widetilde{X}} \cdot dt$ and $\Omega^1_{\widetilde{X}}(\log \widetilde{D})$ are equal.

(v) The proof here is similar to the proof of (iv). The only change is that at a point $(x, 0) \in \widetilde{D} \cup H$ we take a basis of $\mathrm{Der}(log \widetilde{D} \cup H)$ whose last element is $t\partial/\partial t$ instead of $\partial/\partial t$. $\square$

The main theorem is proved below by induction on the dimension of the divisor: in outline, we use Proposition 2.4 (below) to show that we may assume the result of Theorem 1.1 true in a punctured neighbourhood of any point; the problem is then reduced to a calculation in local cohomology, where once again we make use of Lemma 1.1. First we need

Lemma 2.3 *Let $D \subseteq \mathbf{C}^t$ be a weighted homogeneous divisor, and suppose that there is a vector field ξ tangent to D with $\xi(0) \neq 0$. Then $(D, 0)$ is isomorphic to a product $(D' \times \mathbf{C}, (0,0))$, with D' weighted homogeneous.*

Proof

We preface the proof by noting that (as an easy argument shows) an isomorphism of germs of singular hypersurfaces always extends to an isomorphism of germs of ambient vector spaces; this fact is used implicitly in what follows, since the weighted homogeneity of a divisor D is actually a property of the pair $(\mathbf{C}^t, D)$.

The proof of the lemma is very simple: clearly ξ must be transverse to one of the coordinate hyperplanes in some neighbourhood of 0. Reordering the coordinates, we may suppose this to be $\mathbf{C}^{t-1} \times \{0\}$. The integral flow of ξ then defines an isomorphism from $D \cap (\mathbf{C}^{t-1} \times \{0\}) \times V$ (where V is some neighbourhood of 0 in $\mathbf{C}$) onto some neighbourhood of 0 in D. Evidently the divisor $D \cap (\mathbf{C}^{t-1} \times \{0\})$ is weighted homogeneous. $\square$

Now we have

Proposition 2.4 *Let X be a complex manifold of dimension t, let D be a strongly quasihomogeneous divisor in X, and let $p \in D$. Then there is an* *open neighbourhood U of p such that for each $q \in U \cap D, q \neq p$, the germ of pair (X, D, q) is isomorphic to a product $(\mathbf{C}^{t-1} \times \mathbf{C}, D' \times \mathbf{C}, (0,0))$, where D' is a strongly quasihomogeneous divisor.*

Proof By definition of strong quasihomogeneity, there are coordinates in some neighbourhood U of p, in which p is the origin and D has a weighted homogeneous defining equation. Let $0 \neq q \in D \cap U$. Again by definition of strong quasihomogeneity, there is a neighbourhood $V \subseteq U$ of q and an analytic isomorphism

$$\phi : (D \cap V, q) \rightarrow (V_1 \cap D_1, 0),$$

where D_1 is weighted homogeneous and V_1 is some neighbourhood of 0 in $\mathbf{C}^t$. Let χ be the infinitesimal generator of the $\mathbf{C}^*$- action on U. Then χ is tangent to D, and $\chi(q) \neq 0$; hence $\phi_*(\chi)$ is tangent to D_1, and $\phi_*(\chi)(0) \neq 0$. It follows from Lemma 2.3 that there is some neighbourhood $V_2 \subseteq V_1$ of 0 and an isomorphism

$$\psi : D_1 \cap V_2 \rightarrow (D' \cap W_1) \times W_2$$

with $\psi(0) = (0,0)$, where W_1 and W_2 are neighbourhoods of 0 in $\mathbf{C}^{t-1}$ and $\mathbf{C}$ respectively, and D' is weighted homogeneous.

It remains to show that D' is strongly quasihomogeneous. Let $0 \neq q'' \in D'$, and let $a = \psi^{-1}(q'', 0)$. Clearly D_1 is strongly quasihomogeneous, so it follows by the argument of the previous paragraph that (as germs)

$$(D_1, a) \simeq (D_2 \times \mathbf{C}, (0,0))$$

for some weighted homogeneous divisor D_2. Hence we have

$$(D' \times \mathbf{C}, (q'', 0)) \simeq (D_2 \times \mathbf{C}, (0,0));$$

the cancellation property for germs of reduced analytic spaces [3] now implies that $(D', q'') \simeq (D_2, 0)$, showing that D' is strongly quasihomogeneous. $\square$

Before proving Theorem 1.1, we need one further preparatory result.

Lemma 2.5 *Let X be a complex manifold, $D \subseteq X$ a divisor and $j : U \to X$ the inclusion of the complement of D in X. The following properties are equivalent:*

a) The natural morphism $\Omega^\bullet_X(\log D) \to \mathbf{R}j_*\mathbf{C}_U$ is a quasi-isomorphism.

b) For each Stein open set $V \subseteq X$ (or indeed each sufficiently small Stein open set), and for each integer $p \geq 0$, the natural morphism

$$h^p(\Gamma(V, \Omega^\bullet(\log D))) \to h^p(\Gamma(V - D, \Omega^\bullet_U)) = H^p(V - D; \mathbf{C})$$

is an isomorphism.

Proof By the Poincaré Lemma, property (a) is equivalent to the inclusion of complexes

$$\Omega^\bullet(\log D) \hookrightarrow j_*\Omega^\bullet_U \ (= \mathbf{R}j_*(\Omega^\bullet_U))$$

being a quasi-isomorphism. Suppose that (a) holds. Taking hypercohomology over an open set $V \subset X$, we have an isomorphism

$$\mathbf{H}^p(V, \Omega^\bullet_X(\log D)) \simeq \mathbf{H}^p(V, \mathbf{R}j_*\Omega^\bullet_U) = \mathbf{H}^p(V - D, \Omega^\bullet_U)$$

for each $p \geq 0$. If V is Stein, then $V - D$ is also, and as the $\Omega^i_X(\log D)$ and Ω^i_U are coherent (see [12]), we have

$$\mathbf{H}^p(V, \Omega^\bullet_X(\log D)) \simeq h^p(\Gamma(V, \Omega^\bullet_X(\log D))),$$

$$\mathbf{H}^p(V - D, \Omega^\bullet_U) \simeq h^p(\Gamma(V - D, \Omega^\bullet_U)),$$

from which (b) follows.

Now suppose that (b) holds. Evidently the morphism of (a) is a quasi-isomorphism outside D. So let $x \in D$. We must show that the natural morphism

$$h^p(\Omega^\bullet_X(\log D)_x) \to h^p((j_*\Omega^\bullet_U)_x)$$

is an isomorphism for each $p \geq 0$. As taking the homology of a complex commutes with inductive limits, the first term is the inductive limit of $h^p(\Gamma(V, \Omega^\bullet_X(\log D)))$ and the second is the inductive limit of $h^p(\Gamma(V-D, \Omega^\bullet_U))$ where V runs through a fundamental system of neighbourhoods (which we may take to be Stein) of x, and thus (a) follows. □

Proof of Theorem 1.1

We use induction on dimension. If $t = 1$, then D is a collection of isolated points and the result is obvious (especially bearing in mind 2.1). Now suppose $t > 1$, and let $p_0 \in D$. Let U be a neighbourhood of p_0 as in the statement of Proposition 2.4. Then at any point $q \in D \cap U$ with $q \neq p_0$, D is locally a product $\mathbf{C} \times D'$, where D' is a strongly quasihomogeneous free divisor of dimension $t-2$. The induction hypothesis, applied to D', together with Lemma 2.2, implies that the result holds also for $\mathbf{C} \times D'$, and hence $\Omega^\bullet(\log D)_q \to \mathbf{R}j_*(\mathbf{C}_U)_q$ is a quasi-isomorphism.

We now have to prove that the same holds at p_0. By lemma 2.5 it is enough to show that for any open polycylinder $V \subseteq U$ centred at p_0, the de Rham morphism defines an isomorphism $h^p(\Gamma(V, \Omega^\bullet(\log D))) \to h^p(\Gamma(V - V \cap D; \Omega^\bullet_U))$ for $p \geq 0$.

To do this, we compare four spectral sequences. Take local coordinates $x_1, \ldots, x_t$ centred at p_0, as in Definition 1.3. Let $V_i = V \cap \{x_i \neq 0\}$, and let $V_i' = V_i - V_i \cap D$. Then $\{V_i\}_{i=1,\ldots,t}$ and $\{V_i'\}_{i=1,\ldots,t}$ are Stein open covers of $V - 0$ and $V - V \cap D$ respectively. Consider the double complexes

$$K^{p,q} = \bigoplus_{1 \leq i_0 < \cdots < i_q \leq t} \Gamma(\bigcap_{j=0}^{q} V_{i_j}, \Omega^p(\log D)),$$

$$\tilde{K}^{p,q} = \bigoplus_{1 \leq i_0 < \cdots < i_q \leq t} \Gamma(\bigcap_{j=0}^{q} V'_{i_j}, \Omega^p),$$

each of which comes equipped with Čech differential $\check{d}$ and exterior derivative d. The restriction morphism $\rho_0^{\bullet,\bullet} : K^{\bullet,\bullet} \to \tilde{K}^{\bullet,\bullet}$ commutes with both differentials; hence, it induces morphisms of the spectral sequences arising from the standard filtrations on the total complexes of these double complexes. Denoting the spectral sequences for $K^\bullet$ by $'E$ and $''E$, and those for $\tilde{K}^\bullet$ by $'\tilde{E}$ and $''\tilde{E}$, we have

$$'E_1^{p,q} = h^p(K^{\bullet,q}) = \bigoplus_{1 \leq i_0 < \cdots < i_q \leq t} h^p(\Gamma(\bigcap_{j=0}^{q} V_{i_j}, \Omega^\bullet(\log D)))$$

$$'\tilde{E}_1^{p,q} = \bigoplus_{1 \leq i_0 < \cdots < i_q \leq t} h^p(\Gamma(\bigcap_{j=0}^{q} V'_{i_j}, \Omega^\bullet)) = \bigoplus_{1 \leq i_0 < \cdots < i_q \leq t} H^p(\bigcap_{j=0}^{q} V'_{i_j}; \mathbf{C})$$

$$''E_1^{p,q} = h^q(K^{p,\bullet}) = H^q(V - 0, \Omega^p(\log D))$$

and

$$''\tilde{E}_1^{p,q} = h^q(\tilde{K}^{p,\bullet}) = H^q(V - V \cap D, \Omega^p).$$

By induction hypothesis, the morphism $'\rho_1^{p,q} : {}'E_1^{p,q} \to {}'\tilde{E}_1^{p,q}$ is an isomorphism for all p, q. Hence, $'\rho_\infty^{p,q}$ is an isomorphism also.

On the other hand, the spectral sequences $''E$ and $''\tilde{E}$ both collapse at E_2. For $''\tilde{E}$, this is clear: $V - V \cap D$ is a Stein space and Ω^p is coherent, so that $''\tilde{E}_1^{p,q} = 0$ for $q > 0$. For $''E$, more is required. It is here that we make use of the quasihomogeneity of D.

Because $\Omega^p(\log D)$ is a locally free $\mathcal{O}$-module, there is an isomorphism $\phi_p : \mathcal{O}_V^{d_p} \to \Omega_V^p(\log D)$, which carries over to an isomorphism

$$\tilde{\phi}_p : H^{t-1}(V - 0, \mathcal{O}_V)^{d_p} \to H^{t-1}(V - 0, \Omega_V^p(\log D)).$$

By standard facts about depth and local cohomology (cf. [5]), we have $H^q(V - 0, \mathcal{O}) = 0$ for $q \neq 0, t - 1$, and $H^{t-1}(V - 0, \mathcal{O})$ is the quotient of $\Gamma(\cap_{i=1}^t V_i, \mathcal{O})$ by the subspace B^{t-1} of sections of the form $\sum_{i=1}^t f_{i\ |V^0}$, where $f_i \in \Gamma(\cap_{j \neq i} V_j, \mathcal{O})$. Let $V^0 = \cap_{i=1}^t V_i$. Elements of $\Gamma(V^0, \mathcal{O})$ can be represented by Laurent series

$$\sum_{i_1, \ldots, i_t \in \mathbf{Z}} a_{i_1, \ldots, i_t} x_1^{i_1} \cdots x_t^{i_t}$$

converging in V^0, and B^{t-1} consists of precisely those series in which all coefficients $a_{i_1, \ldots, i_t}$ in which all i_j are negative vanish. It follows that $H^{t-1}(V - 0, \mathcal{O})$ is isomorphic to the space of all Laurent series $\sum a_{i_1, \ldots, i_t} x_1^{i_1} \cdots x_t^{i_t}$, converging on V^0, in which $a_{i_1, \ldots, i_t} = 0$ if not all i_j are negative. In fact all such series converge throughout $\cap_{i=1}^t (\mathbf{C}^t - \{x_i = 0\})$, and so $H^{t-1}(V - 0, \mathcal{O}) = H^{t-1}(\mathbf{C}^t - 0, \mathcal{O})$ and is independent of the open set V.

Note that each term in $H^{t-1}(V - 0, \mathcal{O})$ has filtration less than or equal to $-\sum_{i=1}^t w_i$. This fact will play a crucial role in the proof.

Let $\omega_1^{(p)}, \ldots, \omega_{d_p}^{(p)}$ be a free homogeneous basis of $\Omega^p(\log D)$. The elements of $H^{t-1}(V - 0, \Omega^p(\log D))$ are of course equivalence classes of meromorphic differential forms on V with poles along $\bigcup\{x_i = 0\}$. However, via the isomorphism $\tilde{\phi}_p : H^{t-1}(V - 0, \mathcal{O})^{d_p} \to H^{t-1}(V - 0, \Omega^p(\log D))$, they can be viewed as linear combinations of the $\omega_i^{(p)}$ with coefficients in $H^{t-1}(V - 0, \mathcal{O})$. We will now use the fact that $H^{t-1}(V - 0, \mathcal{O})$ has filtration less than or equal to $-\sum_{i=1}^t w_i$, to show that $''E_2^{p,q} = 0$ if $q > 0$.

Lemma 2.6 *If D is a weighted homogeneous divisor on positive weights $w_1, \ldots, w_t$, then for each p, $\Gamma(V, \Omega^p(\log D))$ has a free $\Gamma(V, \mathcal{O}_{\mathbf{C}^t})$-basis consisting of homogeneous generators of weight less than $\sum w_i$.*

Proof Let $\xi_1, \ldots, \xi_t$ be homogeneous generators of $\mathrm{Der}(\log D)_0$, and let $\omega_1, \ldots, \omega_t \in \Omega^1(\log D)_0$ be the dual basis. Then weight(ω_i) = -weight(ξ_i) and as homogeneous basis for $\Omega^p(\log D)_0$ we have all exterior products $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq t$. Let h be a weighted homogeneous defining equation for D. Since by Saito's criterion $\xi_1 \wedge \cdots \wedge \xi_t = h \cdot \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_t$ (up to a non-zero scalar multiple) it follows that $\sum_{i=1}^t \text{weight } \xi_i = \text{weight } h - \sum_{i=1}^t w_i$; this proves the lemma for the case $p = t$, since h is of positive weight. It also shows that it is possible to order the ξ_i so that the coefficient of $\partial/\partial x_i$ in ξ_i is not zero. Hence weight$(\xi_i) > -w_i$. Since all the w_i are positive, this proves the lemma for $p < t$. $\square$

From the lemma and the discussion immediately preceding it, it follows that with respect to the filtration on $H^{t-1}(V - 0, \Omega^p(\log D))$ coming from the $\mathbf{C}^*$-action, $H^{t-1}(V - 0, \Omega^p(\log D))$ has negative filtration, for sufficiently small V. Now the exterior derivative d, which represents the differential d_1 of the spectral sequence $''E$, is homogeneous of degree 0. It follows by Lemma 2.1, that the complex $H^{t-1}(V - 0, \Omega^\bullet(\log D))$ is acyclic. For any element ω of $H^{t-1}(V - 0, \Omega^p(\log(D))$ can be represented by a sum $\sum_j f_j \omega_j^{(p)}$ where the f_j are Laurent series in monomials $x_1^{i_1} \cdots x_t^{i_t}$ with all $i_j < 0$. By Lemma 2.6, such a form can be written as a convergent series $\sum_{j<0} \omega_{(j)}$, where $\omega_{(j)}$ is a homogeneous logarithmic form of weight j. If such a form is closed, then so is each homogeneous part $\omega_{(j)}$; since by Lemma 2.1 $d\iota_\chi(\omega_{(j)}) = L_\chi(\omega_{(j)}) = j\omega_{(j)}$, and $\sum_{j<0} \frac{1}{j}\omega_{(j)}$ converges on V^0, it follows that $\omega = d(\iota_\chi(\sum_{j<0} \frac{1}{j}\omega_{(j)})$ is exact.

Thus, the only non-zero terms in $''E_2$ are $''E_2^{p,0} = h^p(\Gamma(V - 0, \Omega^\bullet(\log D)))$. Finally, because $t > 1$, $\Gamma(V - 0, \Omega^\bullet(\log D)) = \Gamma(V, \Omega^\bullet(\log D))$. We conclude that $''E_1^{p,q} \Rightarrow h^{p+q}(\Gamma(V, \Omega^\bullet(\log D))$.

Now it follows that $'E_1^{p,q} \Rightarrow h^{p+q}(\Gamma(V, \Omega^\bullet(\log D)))$ also; the isomorphism $\rho_1^{p,q} : {}'E_1^{p,q} \to {}'\tilde{E}_1^{p,q}$, passing to the limit of the spectral sequence, thus defines an isomorphism

$$h^p(\Gamma(V, \Omega^\bullet(\log D))) \simeq h^p(\Gamma(V - V \cap D, \Omega^\bullet)) = H^p(V - V \cap D; \mathbf{C}).$$

This proves the theorem. $\square$

From the analytic corollaries 1.4 an algebraic version follows in the case that D is a strongly quasihomogenenous weighted homogenenous free divisor in $\mathbf{C}^t$. For in this case the complex $\Gamma(\mathbf{C}^t, \Omega^\bullet(\log D))$ is quasi-isomorphic to its subcomplex of weight zero, which is also the weight zero subcomplex of the corresponding complex $\Gamma(\mathbf{C}^t, \Omega^\bullet_{alg}(\log D))$ of rational differential forms with logarithmic poles along D. Now this latter complex is graded (and not just filtered) by weight, and so the argument used to prove the quasi-isomorphism in the analytic case can also be used in the algebraic case: if $\omega \in \Gamma(\mathbf{C}^t, \Omega^q_{alg}(\log D))$ is closed and ω_0 is its weight 0 part, then $\omega - \omega_0 \in \Gamma(\mathbf{C}^t, \Omega^q_{alg}(\log D))$, and is exact. Hence, as a corollary of Theorem 1.1, we have

Corollary 2.7 *Let $D \subset \mathbf{C}^t$ be a strongly quasihomogenenous, weighted homogeneneous free divisor. Then for all $p \geq 0$, the de Rham morphism induces an isomorphism*

$$h^p(\Gamma(\mathbf{C}^t, \Omega^\bullet_{alg}(\log D))) \simeq H^p(\mathbf{C}^t - D; \mathbf{C}). \quad \square$$

To prove the algebraic result in greater generality (e.g. for a strongly quasihomogenenous free algebraic divisor in $\mathbf{C}^t$ which is not necessarily weighted homogeneneous)), another argument will be needed.

Such a theorem could be proved by an argument following Grothendieck's proof of his comparison theorem (1.2 above) in [5], involving compactification of $\mathbf{C}^t$ and an application of GAGA. It would be sufficient to show that it is possible to choose a compactification $\overline{X}$ of $X = \mathbf{C}^t$ such that

(CC1) If $\overline{D}$ is the closure of D and $H_\infty = \overline{X} - X$, then for all $p \in \overline{D} \cap H_\infty$ there exist local coordinates $x_1, ..., x_m, x_{m+1}, ..., x_{m+r}$ centred on p, with respect to which $\overline{D}$ has equation $h(x_1, ..., x_m) = 0$, and H_∞ is the normal crossing divisor $\{x_{m+1} \cdots x_{m+r} = 0\}$.

From this it follows, by repeated application of Lemma 2.2 that

(CC2) $\Omega^\bullet_{\overline{X}}(\log \overline{D} \cup H_\infty) \hookrightarrow \Omega^\bullet_{\overline{X}}(\log \overline{D})(*H_\infty)$is a quasi-isomorphism,
and

(CC3) $\overline{D} \cup H_\infty$is free.

Moreover,

(CC4) $\overline{D}\cup H_\infty$is strongly quasihomogenenous,

for $\overline{D}$ is strongly quasi-homogeneous outside H_∞, and therefore at points of H_∞ also, because the equation h is independent of the last r coordinates; H_∞ is strongly quasihomogenenous since it is a normal crossing divisor, and the strong quasi-homogeneneity of $\overline{D}\cup H_\infty$ follows, by "logarithmic transversality" of $\overline{D}$ and H_∞.

Now write $U = X - D = \overline{X} - (\overline{D}\cup H_\infty)$, and let $j : U\to X$, $\overline{j} : U\to\overline{X}$ and $\sigma : X\to\overline{X}$ denote inclusions. The symbol "$\simeq$" between two complexes denotes quasi-isomorphism; between vector spaces it denotes isomorphism.

Assuming that $\overline{X}$ has these properties, we have

$$\Gamma(X,\Omega^\bullet_{X,alg}(\log D)) \simeq \mathbf{R}\Gamma(X,\Omega^\bullet_{X,alg}(\log D))$$

(since X is affine)

$$\simeq \mathbf{R}\Gamma(\overline{X},\mathbf{R}\sigma_*\Omega^\bullet_{X,alg}(\log D))$$

(by composition of derived functors)

$$\simeq \mathbf{R}\Gamma(\overline{X},\Omega^\bullet_{\overline{X},alg}(log\overline{D})(*H_\infty))$$

$$\simeq \mathbf{R}\Gamma(\overline{X},\Omega^\bullet_{\overline{X}}(log\overline{D})(*H_\infty))$$

(by GAGA)

$$\simeq \mathbf{R}\Gamma(\overline{X},\Omega^\bullet_{\overline{X}}(log\overline{D}\cup H_\infty))$$

(by (CC2))

$$\simeq \mathbf{R}\overline{j}_*\mathbf{C}_{\overline{U}},$$

(by (CC3), (CC4) and Theorem 1.1). So

$$h^p(\Gamma(X,\Omega^\bullet(\log D))) \simeq H^p(U;\mathbf{C}).$$

We close this section with a conjecture:

Conjecture 2.8 *For every free divisor $D\subset\mathbf{C}^t$, it is possible to find a compactification $\overline{X}$ such that (CC1) holds.*

Note that when D is a homogenenous divisor in $\mathbf{C}^t$, the usual compactification $\mathbf{C}^t\hookrightarrow\mathbf{P}^t$ satisfies (CC1). We note that proof of the conjecture seems distinctly more feasible in the special case where D is an affine hyperplane arrangement.

We hope to return to this topic in a future paper.

3 Another application of Lie derivatives

The argument with contracting homotopies can be used to give an amusing proof of the following well-known proposition:

Proposition 3.1 *If $H = \{h_1 \cdots h_r = 0\}$ (with each h_r a linear form) is a divisor in $\mathbf{C}^t$ with normal crossings and U is its complement, then the inclusions*

$$\bigwedge^{\bullet} \Sigma_{j=1}^r \mathbf{C}\frac{dh_j}{h_j} \hookrightarrow \Omega^\bullet(\log H) \hookrightarrow \Omega^\bullet(*H)$$

are quasi-isomorphisms.

Here $\bigwedge^\bullet \Sigma\mathbf{C}\frac{dh_j}{h_j}$ is the "Brieskorn complex", the exterior sub-algebra of $\Omega^\bullet_{\mathbf{C}^t}$ generated over $\mathbf{C}$ by the forms $\frac{dh_j}{h_j}$; Brieskorn showed in [2] that for any hyperplane arrangement H with defining equation $h = h_1 \cdots h_r$ (with each h_i affine), the corresponding complex $\bigwedge^\bullet \Sigma\mathbf{C}\frac{dh_i}{h_i}$ (on which d is of course trivial) is quasi-isomorphic to $\mathbf{R}j_*\mathbf{C}_U$; thus $\bigwedge^\bullet \Sigma\mathbf{C}\frac{dh_i}{h_i} \simeq H^\bullet(U;\mathbf{C})$.

From now on we will denote $\bigwedge^p \Sigma\mathbf{C}\frac{dh_j}{h_j}$ by B^p.

Proof of 3.1 At a general point in H we may take local coordinates so that $H = H' \times \mathbf{C}^\ell$, where $H' \subseteq \mathbf{C}^d$ is the union of the coordinate hyperplanes $\{x_i = 0\}$. Evidently the result will follow if we can show it for H', so we assume $H = H'$ and $\ell = 0$. The key point now is that H has an infinite number of different $\mathbf{C}^*$-actions: its defining equation $h(x) = x_1 \cdots x_t$ is weighted homogeneous with respect to every set of positive weights $(w_1, \ldots, w_t)$ for the variables $x_1, \ldots, x_t$. The proposition will be proved by showing that the Brieskorn complex $B^\bullet$ is equal to the intersection, over all sets of positive weights $(w_1, \ldots, w_t)$, of the weight zero complex $\Omega^\bullet(*H)^0_0$.

First, it is clear that $B^\bullet$ is contained in this intersection. For the opposite inclusion, consider first the weights $w_1 = \cdots = w_t = 1$. Then any form $\omega \in \Omega^p(*H)^0_0$ must be a sum of monomials of the form

$$\frac{dx_{i_1} \wedge \cdots \wedge dx_{i_p}}{x_{j_1} \cdots x_{j_p}},$$

where $i_1 < i_2 < \cdots < i_p$ and $j_1 \leq j_2 \leq \cdots \leq j_p$. In order that this form have weight 0 with respect to the weighting $(w_1, \ldots, w_t)$ it is necessary and sufficient that

$$w_{i_1} + \cdots + w_{i_p} = w_{j_1} + \cdots + w_{j_p},$$

and now it is evident that this will hold for *all* sets of positive weights if and only if $i_1 = j_1, \ldots, i_p = j_p$. Hence ω is a sum of terms of the form

$$\frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_p}}{x_{i_p}}$$

and belongs to the Brieskorn complex. $\square$

In fact

Proposition 3.2 *If $H = \{(x_1, \ldots, x_t) \in \mathbf{C}^t : x_1 \cdots x_t = 0\}$ then $B^\bullet$ is the weight 0 part of $\Omega^\bullet(\log H)$ with respect to the standard weighting $w_1 = \cdots = w_t = 1$.*

Proof Suppose that $\omega \in \Omega^p(\log H)^0_0$ contains a term

$$\omega_{i,j} = \frac{dx_{i_1} \wedge \cdots \wedge dx_{i_p}}{x_{j_1} \cdots x_{j_p}}$$

where $i_1 < .. < i_p$ and $j_1 \leq \cdots \leq j_p$. By definition of logarithmic form, $h\omega$ is regular (where $h = x_1 \cdots x_t$), so there can be no repetitions among the j_k; also for each k, $\frac{dx_k}{x_k} \wedge \omega$ is the wedge of logarithmic forms and therefore logarithmic, (see [12], 1.3) so $h\frac{dx_k}{x_k} \wedge \omega$ is regular. Now $\frac{dx_k}{x_k} \wedge \omega_{i,j}$ is non-zero if and only if k is not among the indices $i_1, \ldots, i_p$; in this case the fact that $h\frac{dx_k}{x_k}\omega_{i,j}$ is regular means that k is not among the indices $j_1, \ldots, j_p$ either. That is, $\{1, \ldots, t\} - \{i_1, \ldots, i_p\} = \{1, \ldots, t\} - \{j_1, \ldots, j_p\}$ so $\{i_1, \ldots, i_p\} = \{j_1, \ldots, j_p\}$ and $\omega_{i,j}$ belongs to the Brieskorn complex. $\square$

In the case of a general hyperplane arrangement, $B^\bullet$ is strictly smaller than $\Omega^\bullet(\log H)^0_0$. This can be seen very easily by counting dimensions. For example, consider the case of A_k, where H is the union of the $(k-1)$-planes $x_i = x_j$ in the space $V = \{(x_0, \ldots, x_k) \in \mathbf{C}^{k+1} : x_0 + \cdots + x_k = 0\}$. The basic vector fields, which generate $\mathrm{Der}(\log H)$, can be found as the gradients of the generators $\sigma_2, \ldots, \sigma_{k+1}$ of the ring of invariant functions. With respect to the weighting on $\mathbf{C}[V]$ induced by the weighting on $\mathbf{C}^{k+1}$ with all variables having weight 1, their weights are thus $0, \ldots, k-1$. Let $\omega_0, \ldots, \omega_{k-1} \in \Omega^1(\log H)$ be the $\mathcal{O}_{V,0}$-dual basis (so that $\omega_i(\nabla\sigma_{j+1}) = \delta_{i,j}$). Then ω_i has weight $-i$.

It follows that $\dim_{\mathbf{C}}(\Omega^1(\log H)^0_0) = c(k,0) + c(k,1) + \cdots + c(k,k-1) = c(k{+}1,k{-}1)$, where $c(p,q) = (p{+}q{-}1)C(p{-}1)$ is the number of monomials of degree q in p variables. When $k > 3$, it is thus strictly greater than $\dim_{\mathbf{C}} B^1$, which is equal to the number of reflecting hyperplanes, $(k+1)C2$.

From this it follows (and it is easy to see directly) that the exterior derivative d is not identically 0 on $\Omega^\bullet(\log H)^0_0$, whereas of course it is on $B^\bullet$.

We do not know of a direct proof that in the case of a free hyperplane arrangement H, $B^\bullet \hookrightarrow \Omega^\bullet(\log H)$ is a quasi-isomorphism, although of course it follows by putting together Brieskorn's result with our Theorem 1.1.

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