F-manifolds from composed functions

Ignacio de Gregorio and David Mond

1 Introduction

In this note we show how to endow the base of a versal deformation of a composite singularity with an F-manifold structure, as defined by Hertling and Manin in [9], and in particular with a pointwise, integrable multiplication on the tangent bundle. This is closely related to, but not a special case of, K.Saito's construction of a Frobenius manifold structure on the base-space of a versal deformation of a function with isolated critical point.

Let us clarify the notion of versality we are concerned with. Consider a function $f : (Y, y_0) \to \mathbf{C}$ and a map-germ $F : (X, x_0) \to (Y, y_0)$. Deformations of F give rise to some, but not all, of the possible deformations of the composite $f \circ F$. In this context, a deformation $\mathscr{F}$ of F is versal if, up to the usual notion of equivalence, it contains all the deformations of $f \circ F$ which can be achieved by deforming F. A precise definition is given below. For now, we point out that even when $f \circ F$ has non-isolated singularity, F may have a finite-dimensional versal deformation in the sense being considered. Exactly this is the case in Damon's theory of almost free divisors, [4].

The ideas of the previous paragraph are made precise by means of the subgroup $\mathcal{K}_f$ of the contact group $\mathcal{K}$, acting on the space of map-germs $(X, x_0) \to (Y, y_0)$. We give the definition at the start of Section 2. The $\mathcal{K}_f$-equivalence of F_1 and F_2 equivalence implies (though it is not implied by) right-equivalence of the composites $f \circ F_1$ and $f \circ F_2$. A $\mathcal{K}_f$-miniversal deformation $\mathscr{F} : (X \times S, (x_0, 0)) \to (Y, y_0)$ of $F : (X, x_0) \to (Y, y_0)$ can be constructed by the usual procedures of singularity theory; loosely speaking, the tangent space $T_0 S$ is isomorphic to the quotient $T^1_{\mathcal{K}_f} F$ of the space $\theta(F)$ of infinitesimal deformations of F by the tangent space to the $\mathcal{K}_f$-orbit of F.

If $f : (\mathbf{C}^n, 0) \to \mathbf{C}$ is a germ of function with isolated singularity, $T^1 f := \mathcal{O}_{\mathbf{C}^n} / J_f$ is naturally a ring, and it is this that Saito uses to define the multiplicative structure on the tangent bundle of the base of a versal deformation of f. It turns out (see Section 2) that $T^1_{\mathcal{K}_f} F$ is equal to

$$\frac{\theta(F)}{tF(\theta_X) + F^*(\mathrm{Der}(-\log f))}$$

and does not appear to carry a multiplicative structure. However, contraction with $F^*(df)$ defines an $\mathcal{O}_X$-linear epimorphism

$$\theta(F) = F^*(\theta_Y) \to F^*(J_f)$$

which passes to the quotient to give a natural $\mathcal{O}_X$-linear epimorphism

$$T^1_{\mathcal{K}_f} F \to \frac{F^*(J_f)}{J_{f \circ F}},$$

whose target does have a multiplicative structure. Once again, let $\mathscr{F} : (X \times S, (x_0, 0)) \to (Y, y_0)$ be a $\mathcal{K}_f$-versal deformation of F. Write $F_s(x) = \mathscr{F}(x, s)$. A multiplicative structure on TS is therefore defined at those points $s \in S$ such that for all $x \in \operatorname{supp} T^1_{\mathcal{K}_f} F_s$, the epimorphism

$$\left(T^1_{\mathcal{K}_f} F_s\right)_x \longrightarrow \left(\frac{F_s^*(J_f)}{J_{f \circ F_s}}\right)_x \tag{1}$$

is injective. From Proposition 3.3 (due to Jim Damon) it follows that under certain rather weak conditions on the divisor $E := f^{-1}(0)$,

$(*)$ there is a proper analytic subset B of S, such that for $s \in S - B$,

$$\operatorname{supp} T^1_{\mathcal{K}_f} F_s \cap V(F_s^*(J_f)) = \emptyset.$$

If $s \in S - B$, the image of (1) is all of $\mathcal{O}_{X,x}/J_{f \circ F_s}$. Less obviously, if $s \in S - B$ the morphism of (1) is also injective (Lemma 4.3 of [6]). We will use (1) and the relative Kodaira-Spencer map of the deformation to endow $T(S-B)$ with a well-defined multiplication with unit. In Section 3 we prove a transversality lemma which allows us to show that for s in a certain non-empty open set in S, the critical points of $f \circ F_s$ off $F_s^{-1}(E)$ are generically non-degenerate and that the critical values are generically pairwise distinct.

To transfer the multiplicative structure to the tangent sheaf of the base, the relative Kodaira-Spencer map

$$\theta_S \to \pi_*(T^1_{\mathcal{K}_f/S}\mathscr{F})$$

must be an isomorphism, and in particular $T^1_{\mathcal{K}_f/S}\mathscr{F}$ must be free over the base. Section 4 recalls the arguments given in [5] and [6] to prove freeness in three cases: where $E := f^{-1}(0)$ is a free divisor, and where $\dim X$ is equal to $m_0 := \dim Y - \dim E_{\mathrm{S}ing}$ or to $m_0 - 1$.

If $\dim X \geq m_0$, the generic fibres $D_s = F_s^{-1}(E)$ will contain singularities; they are only *partial* smoothings of $D := F^{-1}(E)$. The most extreme case is where E is a (singular) free divisor. In this case, D_s, like E, will be singular in codimension 1. Nevertheless, in all cases every fibre D_s has the homotopy type of a wedge of spheres of middle dimension. This is because we have "triviality at the boundary" (recall that under the assumption that $\operatorname{supp} T^1_{\mathcal{K}_f} = \{0\}$, F is transverse to E away from 0), and thus the vanishing homology of D_s is accounted for by the isolated critical points which move off D as s moves away from 0 (cf [11]). The number of spheres in this wedge for a generic parameter-value s is called by Damon the "singular Milnor number" of D. We will denote it by $\mu_E(F)$. In Section 3, we show, following the argument of [5] Section 5, that if

1. $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $\mathcal{O}_S$, and
2. condition $(*)$ holds,

then $\mu_E(F) = \dim_{\mathbf{C}} T^1_{\mathcal{K}_f} F$. Thus, in these favourable circumstances, our F-manifold $S - B$ supports a locally trivial holomorphic fibration whose fibre D_s has homology concentrated in middle dimension, where its rank is equal to the dimension of S. We expect that using the Gauss-Manin connection on this fibration it will be possible to go some way further towards endowing S with a Frobenius structure, but here we do not attempt this.

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2 Background and notation

Throughout this paper, X and Y will denote the germs $(\mathbf{C}^m, 0)$ and $(\mathbf{C}^n, 0)$ respectively. We consider a fixed map $f : Y \to \mathbf{C}$, and classify map-germs $X \to Y$ as follows: $F_1 : X \to Y$ and $F_2 : X \to Y$ are $\mathcal{K}_f$-equivalent if there exists a germ of diffeomorphism $\Phi : X \times Y \to X \times Y$ such that

1. Φ covers a diffeomorphism $\phi : X \to X$ (i.e. there is a germ of diffeomorphism $\phi : X \to X$ such that $\pi_X \circ \Phi = \phi \circ \pi_X$)
2. Φ preserves the level sets of f; more precisely, $f \circ \pi_Y \circ \Phi = f \circ \pi_Y$, and
3. $\Phi(\mathrm{graph}(F_1)) = \mathrm{graph}(F_2)$.

Observe that $\mathcal{K}_f$ contains the group $\mathscr{R}$ of right-equivalences. One calculates that the extended tangent space to the group action on a germ $F : X \to Y$ is

$$T\mathcal{K}_f F = tF(\theta_X) + F^*(\mathrm{Der}(-\log f))$$

where $\mathrm{Der}(-\log f)$ is the $\mathcal{O}_Y$-module of germs of vector-fields tangent to all the level sets of f. We denote the quotient $\theta(F)/T\mathcal{K}_f F$ by $T^1_{\mathcal{K}_f} F$. It is easy to show that

$$F_1 \sim_{\mathcal{K}_f} F_2 \quad \Rightarrow f \circ F_1 \sim_{\mathscr{R}} f \circ F_2,$$

but the converse does not always hold.

The group $\mathcal{K}_f$ is geometric, in the sense of Damon [1], and so the usual properties hold; in particular, if $T^1_{\mathcal{K}_f} F$ has finite length then a deformation $\mathscr{F} : X \times U \to Y$ of F is versal if the initial velocities $\partial \mathscr{F} / \partial s_i|_{s=0}$ generate $T^1_{\mathcal{K}_f} F$ over $\mathbf{C}$, and miniversal if they form a basis.

Closely related to $\mathcal{K}_f$ is the group $\mathcal{K}_E$, introduced by Damon in [2], in which part 2 of the definition above is weakened to the requirement that Φ preserve only the level set $X \times E$ of $f \circ \pi_Y$. It is an immediate consequence of Nakayama's lemma that

$$\mathrm{supp}\, T^1_{\mathcal{K}_E} F = \{x \in X : F \not\pitchfork E \text{ at } x\},$$

where transversality to E is understood to mean transversality to the distribution $\mathrm{Der}(-\log E)$. Damon showed in [3] that if

$$\begin{array}{ccc} Y_0 & \xrightarrow{H} & Y_1 \\ \uparrow & & \uparrow F \\ X_0 & \xrightarrow{h} & X_1 \end{array}$$

is a fibre square in which H is a right-left stable map-germ with discriminant (or image, if $\dim Y_0 < \dim Y_1$) equal to E, and $F \pitchfork H$, then

$$T^1_{\mathcal{K}_E} F \simeq T^1_{h:X_0 \to X_1} := \frac{\theta(h)}{th(\theta_{X_0}) + \omega h(\theta_{X_1})}.$$

Our F-manifold structure is therefore closely related to the theory of right-left equivalence of map-germs, and of right-left versal unfoldings.

3 Transversality

To prove a number of properties of $\mathcal{K}_f$-versal deformations, we will use a local transversality lemma. Recall that $X^{(r)}$ is the subset of the r-fold cartesian product X^r consisting of r-tuples of pairwise distinct points, that ${}_rJ^k(X,Y)$ is the restriction of $(J^k(X,Y))^r$ to $X^{(r)}$, and that

$$ {}_rj^k_x\mathscr{F} : X^{(r)} \times S \to {}_rJ^k(X,Y), $$

the *relative r-fold multi-jet extension map*, is defined by

$$ {}_rj^k_x\mathscr{F}(x_1,\ldots,x_r,s) = \big(j^kF_s(x_1),\ldots,j^kF_s(x_r)\big). $$

Lemma 3.1. *Let $W \subset {}_rJ^k(X,Y)$ be a $\mathcal{K}_f$ invariant submanifold. If $\mathscr{F} : X\times S \to Y$ is a $\mathcal{K}_f$-versal deformation of a germ $F : X \to Y$, then ${}_rj^k_x\mathscr{F} \pitchfork W$.*

Proof. It is possible to find a deformation $\widetilde{\mathscr{F}} : X \times S \times U \to Y$ of $\mathscr{F}$ such that ${}_rj^k_x\widetilde{\mathscr{F}} : X \times S \times U \to J^k(X,Y)$ is transverse to W. For example, identifying X and Y with $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively, we take as U the space of polynomial maps $p : \mathbf{C}^m \to \mathbf{C}^n$ with each component of degree $\leq N$, and define

$$ \widetilde{\mathscr{F}}(x,s,p) = \mathscr{F}(x,s) + p(x). $$

If N is sufficiently large then ${}_rj^k_x\widetilde{\mathscr{F}}$ is a submersion, and in particular transverse to W.

As $\mathscr{F}$ is $\mathcal{K}_f$-versal, so is $\widetilde{\mathscr{F}}$. Versality of $\mathscr{F}$ implies that $\widetilde{\mathscr{F}}$ is $\mathcal{K}_{f\mathrm{un}}$-equivalent to the deformation $i^*\mathscr{F}$ induced from $\mathscr{F}$ by some map of base-spaces $i : S \times U \to S$. Versality of $\widetilde{\mathscr{F}}$ implies that i is a submersion, and in particular locally surjective. As W is $\mathcal{K}_f$-invariant, the transversality of $j^k_x\widetilde{\mathscr{F}}$ to W implies that $j^k_xi^*(\mathscr{F}) : X \times S \times U \to J^k(X,Y)$ is also transverse to W. But $j^k_xi^*(\mathscr{F}) = (j^k_x\mathscr{F})\circ(\mathrm{id}_X \times i)$. As $\mathrm{id}_X \times i$ is surjective, it follows that $j^k_x\mathscr{F}$ is transverse to W. $\square$

In what follows, when we consider a map $F : X \to Y$, we write $D = V(f\circ F)$. If $\mathscr{F} : X \times S \to Y$ is a deformation of F, we write $F_s(x) := \mathscr{F}(x,s)$, $\mathscr{D} := \mathscr{F}^{-1}(E)$ and $D_s = F_s^{-1}(E)$.

From our transversality lemma we derive first a statement about the behaviour of perturbations F_s of F off the zero set D_s of $f\circ F_s$. The reason we want this is that typically, D_s will have non-isolated singularities.

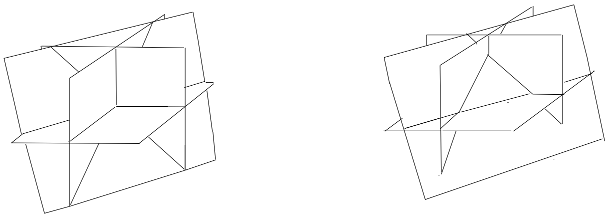
Here is a simple example. Define $f : \mathbf{C}^4 \to \mathbf{C}$ by $f(y_1,y_2,y_3,y_4) = y_1y_2y_3y_4$, let $E = f^{-1}(0)$ and let $F : \mathbf{C}^3 \to \mathbf{C}^4$ be given by $F(x_1,x_2,x_3) = (x_1,x_2,x_3,x_1+x_2+x_3)$. $\mathrm{Der}(-\log E)$ is well known to be generated by the vector fields $y_i\partial/\partial y_i$ for $i = 1,\ldots,4$, and $\mathrm{Der}(-\log f)$ consists of all linear combinations $\sum a_iy_i\partial/\partial y_i$ where $\sum_i a_i = 0$. Thus $T\mathcal{K}_fF$ is generated over $\mathcal{O}_X$ by

$$ \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_4},\ \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_4},\ \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_4} $$

and by

$$ x_1\frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial y_2},\ x_1\frac{\partial}{\partial y_1} - x_3\frac{\partial}{\partial y_3},\ x_1\frac{\partial}{\partial y_1} - (x_1+x_2+x_3)\frac{\partial}{\partial y_4} $$

The quotient $T^1_{\mathcal{K}_f}F$ has length 1, and is generated by the class of $\partial/\partial y_4$. The two drawings below show the real part of $D = F^{-1}(E)$ and $D_s = F_s^{-1}(E)$, where F_s is the deformation $F_s = F + s\partial/\partial y_4$, for $s < 0$. Both surfaces have non-isolated singularities; the defining equation $x_1x_2x_3(x_1+x_2+x_3+s) = 0$ of the second also has an isolated singularity at $(-s/4,-s/4,-s/4)$ with Milnor number 1, inside the chamber which has opened up as s moves away from zero.



Proposition 3.2. *If $\mathscr{F} : X \times S \to Y$ is a $\mathcal{K}_f$-miniversal deformation of F, then*
(i) the variety $\sum^{rel}_{f\circ\mathscr{F}}$ defined by the ideal $J^{rel}_{f\circ\mathscr{F}} := (\{\partial(f \circ \mathscr{F})/\partial x_i : i = 1, \ldots, m\})$ is non-singular off $\mathscr{D}$.
Moreover, there are analytic hypersurfaces B_1 and B_2 of S such that
(ii) for $s \in S - B_1$, each critical point of $f\circ F_s$ off D_s is non-degenerate, and
(iii) for $s \in S - B_2$, the values of $f\circ F_s$ at these critical points are all distinct.

Proof. Apply 3.1 taking as W the submanifold of $J^1(X,Y)$ consisting of 1-jets (x, y, A) with the property that $y \notin E$ and the image of the linear map A lies in $\ker d_y f$. For each $y \notin E$, $\ker d_y f$ has dimension $n-1$, and so the space of admissible matrices A has dimension $m(n-1)$. Thus the codimension of W in $J^1(X,Y)$ is m. Clearly $\sum^{\rm rel}_{f\circ\mathscr{F}-\mathscr{D}} = (j^1_x f \circ \mathscr{F})^{-1}(W)$ and is therefore smooth by 3.1. This proves (i).

Let $W_1 \subset J^2(X,Y)$ be the set consisting of jets $j^2H(x)$ such that $H(x) \notin E$, $d_x(f\circ H) = 0$ and the Hessian determinant of $f\circ H$ vanishes at x. Although not a manifold, W_1 is an analytic set and can be stratified. Its open stratum has codimension $m+1$. The set B_1 is the closure of $\pi_S\big((j^2\mathscr{F})^{-1}(W_1)\big)$.

The projection $\pi : \sum^{\rm rel}_{f\circ\mathscr{F}} - \mathscr{D} \to S$ is finite. If it were not, then for some $s \in S$, $f\circ F_s$ would have a non-isolated singularity off D_s. But the length of $T^1_{\mathcal{K}_f} F_s$ is upper semi-continuous, and for $s = 0$ it is finite. Thus, B_1 is a hypersurface.

To ensure that the critical values of the critical points off the zero level are all distinct, let $W_2 \subset {}_2J^1(X,Y)$ be the submanifold consisting of jets $(x_1, y_1, A_1, x_2, y_2, A_2)$ such that $f(y_1) = f(y_2) \neq 0$, and $d_{y_i} f\circ A_i = 0$ for $i = 1, 2$. As $\mathcal{K}_f$ leaves the level sets of f unchanged, W_2 is indeed $\mathcal{K}_f$-invariant. The codimension of W_2 in ${}_2J^1(X,Y)$ is $2m+1$, so transversality of ${}_2j^1_x\mathscr{F}$ to W_2 means that the set $(x_1, x_2, s) \in X^{(2)} \times S$ such that x_1, x_2 are critical points of $f\circ F_s$ not in D_s and with equal critical values, is empty or has dimension $\dim S - 1$. In particular, the closure B_2 of its projection to S is a hypersurface (or empty), and if s is not in B_2 then the values of $f\circ F_s$ at its critical points off D_s are pairwise distinct. $\square$

A divisor E is *holonomic* at x if the logarithmic partition of E is locally finite (and thus a stratification) in some neighbourhood U of E. Holonomicity is an analytic condition, and thus the set of points where it fails is an analytic subset of E. We say that E is *holonomic in codimension* k if this subset has codimension at least $k+1$.

In similar vein, E is *Euler-homogeneous* at x if there is a germ of vector field χ at x such that
(i) $\chi(x) = 0$, and
(ii) $\chi \cdot f = f$ (where f is a reduced equation for E).
Note that this property is independent of the choice of defining equation f. We say that E is *strongly Euler-homogeneous in codimension* k if there is a Whitney stratification of E such that E is Euler-homogeneous at every point of each stratum of codimension $\leq k$.

Proposition 3.3. *(J.N.Damon, [4]) Suppose that $E = V(f)$ is holonomic and strongly Euler-homogeneous in codimension m. Then there is a proper analytic subset Δ of the semi-universal base-space S of $\mathscr{F}$ such that if $s \notin \Delta$,*

$$\operatorname{supp}\left(T^1_{\mathcal{K}_f} F_s\right) \cap\ D_s = \emptyset.$$

Proof. Let $\mathscr{S} = \{E_\alpha\}$ be a Whitney stratification of E, in which each stratum of codimension $\leq m$ is logarithmic. Any $\mathcal{K}_f$-versal deformation of F is logarithmically transverse to E, and thus is transverse to $\mathscr{S}$. By Sard's Theorem, the set Δ_α of critical values in S of the projection $\mathscr{F}^{-1}(E_\alpha) \to S$ has measure zero, and so also does $\Delta := \bigcup_\alpha \Delta_\alpha$. If $s \notin \Delta$ then F_s is transverse to each stratum E_α of $\mathscr{S}$. Since E is holonomic in codimension m, this means that F_s meets only holonomic strata of E, and so in fact F_s is logarithmically transverse to E itself.

For $s \in S - \Delta$, we have

$$d_x F_s(T_x X) + T^{\log}_{F_s(x)} E = T_{F_s(x)} Y \tag{2}$$

for all $x \in F_s^{-1}(E)$. Let be a germ of Euler vector field, vanishing at $F_s(x)$, such that $\chi \cdot f = f$. Then $\operatorname{Der}(-\log E)_{F_s(x)} = \langle \chi \rangle_{F_s(x)} + \operatorname{Der}(-\log f)_{F_s(x)}$, where $\langle \chi \rangle$ is the $\mathcal{O}_Y$ module generated by χ. Since χ vanishes at $F_s(x)$, from (2) we obtain

$$d_x F_s(T_x X) + \operatorname{Der}(-\log f)(F_s(x)) = T_{F_s(x)} Y \tag{3}$$

and now Nakayama's Lemma gives $(T\mathcal{K}_f F_s)_x = \theta(F_s)_x$. □

Corollary 3.4. *With the condition of Proposition 3.3, suppose that $\mathscr{F} : X \times S \to Y$ is a $\mathcal{K}_f$-versal deformation of $F : X \to Y$. If also $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over S, then the generic fibre $D_s := F_s^{-1}(E)$ has the homotopy-type of a wedge of $\tau = \dim S$ spheres of dimension $m - 1$.*

Proof. By a theorem of Siersma, ([11]), D_s has the homotopy type of a wedge of $(m-1)$-spheres, whose number is equal to the sum of the Milnor numbers of the critical points of $f \circ F_s$ which move off D_s as s moves off 0. By the hypothesis of freeness, we have

$$\tau := \dim_{\mathbf{C}} T^1_{\mathcal{K}_f} F = \sum_i \dim_{\mathbf{C}} (T^1_{\mathcal{K}_f} F_s)_{x_i}$$

where the x_i are the points of the support of $T^1_{\mathcal{K}_f} F_s$. These fall into two subsets: those in D_s and those outside it. If $s \in S - \Delta$, the first subset is empty.

If $x_i \notin D_s$ then the morphism (1)

$$F^*(df) : (T^1_{\mathcal{K}_f} F_s)_{x_i} \to \mathcal{O}_{X,x_i}/J_{f \circ F_s}$$

is an isomorphism. It follows that if $s \notin \Delta$, all points of $\operatorname{supp} T^1_{\mathcal{K}_f} F_s$ are outside D_s and moreover

$$\tau = \sum_i \dim_{\mathbf{C}} \mathcal{O}_{X.x_i}/J_{f \circ F_s}.$$

□

Let

$$B = \{s \in S : \operatorname{supp}\bigl(T^1_{\mathcal{K}_f} F_s\bigr) \cap V\bigl(\mathscr{F}^*(J_f)\bigr) \neq \emptyset\}.$$

Since $V\bigl(F_s^*(J_f)\bigr) \subset D_s$, (3.3) assures us that if $V(f)$ is holonomic and strongly Euler homogeneous in codimension m, then B is a proper analytic subset of S.

4 When is the relative T^1 free over the base?

Suppose that $\mathcal{O}_Y/J_f$ is Cohen-Macaulay, and let m_0 be the codimension of $V(J_f)$ in Y. Let $\mathscr{F} : X \times S \to Y$ be a deformation of a germ $F : X \to Y$ for which $\operatorname{supp} T^1_{\mathcal{K}_f} F = \{0\}$. There are three cases where we can show that the relative module $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over S. These are

1. where f is a defining equation for a free divisor E, such that $f \in J_f$.
2. where $\dim X = m_0$, and
3. where $\dim X = m_0 - 1$.

Proofs of all three are straightforward, and may be found, for example, in [5], [6], though for completeness we sketch them here.

1. Free divisors $\operatorname{Der}(-\log f)$ is a direct summand of the free module $\operatorname{Der}(-\log E)$, with complementary summand generated by a vector field χ such that $\chi \cdot f = f$, and so is free on $n-1$ generators. Thus, the presentation

$$\theta_{X\times S/S} \oplus \mathscr{F}^*(\operatorname{Der}(-\log f)) \to \theta(\mathscr{F}) \to T^1_{\mathcal{K}_f/S}\mathscr{F} \to 0$$

can be read as

$$\mathcal{O}^m_{X\times S} \oplus \mathcal{O}^{n-1}_{X\times S} \to \mathcal{O}^n_{X\times S} \to T^1_{\mathcal{K}_f/S}\mathscr{F} \to 0. \tag{4}$$

Since $\operatorname{supp} T^1_{\mathcal{K}_f/S}\mathscr{F}$ is finite over S, $\dim T^1_{\mathcal{K}_f/S}\mathscr{F} \leq \dim S$; on the other hand, from (4) it follows that the codimension of $\operatorname{supp} T^1_{\mathcal{K}_f/S}\mathscr{F}$ is no greater than $\dim X$. Thus $\dim T^1_{\mathcal{K}_f/S}\mathscr{F} = \dim S$, and now from (4) it follows, by the theorem of Eagon-Northcott, that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ has depth equal to $\dim S$. Since it is finite over $\mathcal{O}_S$, it is a free $\mathcal{O}_S$-module.

2. The cases $m = m_0$ and $m = m_0 - 1$

Lemma 4.1. *If $m \leq m_0$ and $T^1_{\mathcal{K}_f}F$ has finite length then $f \circ F$ has an isolated singularity.*

Proof. First, because $\operatorname{supp} T^1_{\mathcal{K}_f}F = \{0\}$, the restriction of F to $X - \{0\}$ is transverse to every level set of f. We have $\dim X \leq \operatorname{codim} V(J_f) \leq \operatorname{codim} E_\alpha$ for each stratum E_α of any Whitney stratification of E contained in $V(J_f)$, and so $F^{-1}(E_\alpha)$ must consist of isolated points. Thus the germ of $F^{-1}(V(J_f))$ consists at most of $\{0\}$. At every point $x \notin F^{-1}(V(J_f))$, the transversality of F to the level set of f through $F(x)$ means that x is not a critical point of $f \circ F$. □

The multiplicity, μ, of the critical point of $f \circ F$ is preserved in any deformation. When $m = m_0$, the exact sequence of Corollary 1.3 of [6] reduces to

$$0 \to T^1_{\mathcal{K}_f}F \to \mathcal{O}_X/J_{f\circ F} \to \mathcal{O}_X/F^*(J_f) \to 0. \tag{5}$$

The lengths of the second and third non-trivial terms in this short exact sequence are conserved (the latter because $\mathcal{O}_Y/J_f$ is Cohen-Macaulay), and hence so is the length of the first. This implies that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $\mathcal{O}_S$.

When $m = m_0 - 1$, the exact sequence acquires an extra term, and becomes

$$0 \to \operatorname{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_X) \to T^1_{\mathcal{K}_f}F \to \mathcal{O}_X/J_{f\circ F} \to \mathcal{O}_X/F^*(J_f) \to 0 \tag{6}$$

An easy argument ([6] Lemma 4.3(i)) shows that the lengths of the modules $\operatorname{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_X)$ and $\mathcal{O}_X/F^*(J_f)$ are equal, so that the length of $T^1_{\mathcal{K}_f}F$ is equal to μ. As μ is conserved, so is the length of $T^1_{\mathcal{K}_f}F$, and so once again $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $\mathcal{O}_S$.

5 Multiplication on the tangent bundle of the base

Let $\mathscr{F}$ be a $\mathcal{K}_f$-miniversal deformation of some germ $F\colon X \to Y$ for which $T^1_{\mathcal{K}_f}F$ has finite length. Suppose that E is strongly Euler-homogeneous and holonomic in codimension m, and that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over S. By Prop. 3.3, there is a proper analytic subset B of the base space S, such that for $s \in S - B$, $\operatorname{supp} T^1_{\mathcal{K}_f}F_s$ does not meet $V\big(F_s^*(J_f)\big)$. For such s,

$$(T^1_{\mathcal{K}_f}F_s)_x \simeq \mathcal{O}_{X,x}/J_{f\circ F_s}$$

for each $x \in \operatorname{supp} T^1_{\mathcal{K}_f}F_s$, and indeed

$$\pi_*(T^1_{\mathcal{K}_f/S}\mathscr{F}) \simeq \pi_*(\mathcal{O}_{(X\times S)-\mathscr{D}}/J^{\mathrm{rel}}_{f\circ\mathscr{F}}). \tag{7}$$

Because $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over S, the Kodaira-Spencer map gives an isomorphism of free sheaves

$$\theta_S \simeq \pi_*(T^1_{\mathcal{K}_f/S}\mathscr{F})$$

on all of S. Composing this with the isomorphism (7) we get an isomorphism

$$\theta_{S-B} \simeq \pi_*(\mathcal{O}_{(X\times S)-\mathscr{D}}/J^{\mathrm{rel}}_{f\circ\mathscr{F}})\big|_{S-B} \tag{8}$$

and it is this that we use to define a multiplication on the tangent sheaf, just as in the case of deformations of isolated hypersurface singularities.

Hertling and Manin introduced the notion of F-manifold in [9]. A complex manifold with an associative and commutative multiplication $\star$ on the tangent bundle is called an F-manifold if:

1. *(unity)* there exists a *global* vector field e such that $e \star u = u$ for any $u \in \theta_M$ and,
2. *(integrability)* $\mathrm{Lie}_{u\star v}(\star) = u \star \mathrm{Lie}_v(\star) + \mathrm{Lie}_u(\star) \star v$ for any $u, v \in \theta_M$.

The main consequence of this definition is the integrability of multiplicative subbundles of TM, namely, if in a neigbourhood U of a point $p \in M$ we can decompose TU as a sum $A \oplus B$ of multiplicatively closed subbundles with unity, then A and B are integrable.

An *Euler vector field* E for M is defined by the condition

$$\mathrm{Lie}_E(\star) = \star$$

Theorem 5.1. *The complement $S - B$ with the multiplication induced from (8) is an F-manifold with Euler vector field E_S coming from the class of $f \circ \mathscr{F}$ in $\mathcal{O}_{(X\times S)-\mathscr{D}}/J^{rel}_{f\circ\mathscr{F}}$ via the isomorphism*

$$\theta_{S-B} \simeq \pi_*\big(\mathcal{O}_{(X\times S)-\mathscr{D}}/J^{rel}_{f\circ\mathscr{F}}\big).$$

Proof. It is enough to show that the integrability condition holds in an open and dense subset of $S-B$. According to Prop. 3.2, there exists a proper analytic subvariety B_1 such that for $s \in S - B_1$, the composite $f \circ F_s$ has only non-degenerate critical points. In a neigbourhood $U \subset S - B$ of such a point, the integrability condition is equivalent to the image L of the map

$$\operatorname{supp}\, T^1_{K_f}\mathscr{F} \ni (x,s) \mapsto d_{(x,s)}(f\circ\mathscr{F}) \in T^*_sS \tag{9}$$

being a Lagrangian subvariety of T^*S (see [8], Th. 3.2). If α denotes the canonical 1-form on T^*S and $p\colon T^*S \to S$ the projection, it is easy to check that the diagram

$$\begin{tikzcd}
\pi_*(T^1_{K_f/S}\mathscr{F}) \arrow[rr] & & p_*\mathcal{O}_L \\
& \theta_S \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

is commutative. The homomorphism on the right hand side is given by evaluation, so that it can also be expressed as $u \mapsto \alpha(\tilde{u})$ where $\tilde{u}$ is a lift of $u \in \theta_S$ to θ_{T^*S}. Hence $\alpha|_L$ is the relative differential of $(f \circ \mathscr{F})$ when thought of as a map on L via the identification (9). It follows that $\alpha|_L$ is exact and hence closed, so that L is Lagrangian.
The statement about the Euler vector field is an easy calculation that we leave to the reader (see [8], Th. 3.3.). □

Remark 5.2. It follows that the critical values of $f \circ \mathscr{F}$ are local coordinates around a generic point in the base. For a point s where $\operatorname{supp} T^1_{\mathcal{K}_f} F_s$ consists of m different points, the algebra T_sS decomposes in 1-dimensional subalgebras with unity. Hence there exist coordinates $(u_1, \ldots, u_m)$ such that $(\partial/\partial_i) \star (\partial/\partial_j) = \delta_{ij}\partial/\partial_i$. These special coordinates are known as *canonical coordinates.* Writing the Euler vector field E in these coordinates and using the fact that $d(f \circ \mathcal{F}) = \alpha|_L$, we see that the canonical coordinates coincide, up to a constant, with the critical values of $f \circ \mathscr{F}$.

6 Morphisms of F-manifolds

In the cases where $f \circ F$ has an isolated singularity, we can compare its $\mathcal{R}$-miniversal deformation with the $\mathcal{K}_f$-versal deformation $\mathscr{F}$ of F. Suppose that $f \circ \mathscr{F}$, thought of as a deformation of $f \circ F$, is (up to $\mathcal{R}_{\text{e-un}}$-equivalence) induced from some other, say $G : X \times W \to \mathbf{C}$. Then we have a fiber square

$$\begin{tikzcd}
X \times S \arrow[rr, "\Phi"] \arrow[dr, "f\circ\mathscr{F}"'] \arrow[dd, "\pi_S"'] & & X \times W \arrow[dl, "G"] \arrow[dd, "\pi_W"] \\
& \mathbf{C} & \\
S \arrow[rr, "i"'] & & W
\end{tikzcd} \tag{10}$$

where Φ is the $\mathcal{R}_{\text{e-un}}$-equivalence, of the form

$$\Phi(x, s) = (\varphi(x, s), i(s)) = (\varphi_s(x), i(s)),$$

and i is the inducing map from the base-space of $f \circ \mathscr{F}$ to the base space of G.

Lemma 6.1. *There is a commutative diagram*

$$\begin{tikzcd}
(\pi_S)_*(T^1_{\mathcal{K}_f/S}\mathscr{F}) \arrow[r] & (\pi_S)_*(\mathcal{O}_{X\times S}/J^{rel}_{f\circ\mathscr{F}}) \\
\theta_S \arrow[u] \arrow[r, "ti"] & \theta(i) \arrow[u]
\end{tikzcd} \tag{11}$$

The vertical arrow on the left hand side is the Kodaira-Spencer map of $\mathscr{F}$ as a $\mathcal{K}_f$-deformation whereas the one on the right hand side is the pull-back (by a morphism explained below) of that of G as an $\mathcal{R}$-deformation of $f \circ F$. The right hand vertical morphism is an isomorphism when G is miniversal.

Proof. The morphism of the top row is just $\mathscr{F}^*(df)$, defined on a section $\xi = \sum_j \xi_j \partial/\partial y_j$ of $\theta(\mathscr{F})$ by

$$\sum_{i=1}^{n} \xi_j \partial/\partial y_j \mapsto \sum_j \big((\partial f/\partial y_j) \circ \mathscr{F}\big)\xi_j$$

and passing to the quotient as before. The vertical morphism of the right is defined on the generators $\partial/\partial w_j$ by

$$(\partial/\partial w_j) \mapsto \big(\partial G/\partial w_j\big) \circ \Phi$$

and extended by $\mathcal{O}_S$-linearity. It is straightforward to check that the diagram commutes.
Recall that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ and $\mathcal{O}_{X\times S}/J^{\mathrm{rel}}_{f\circ\mathscr{F}}$ are both finite over S, so that in particular

$$\pi_*\big(\mathcal{O}_{X\times S}/J^{\mathrm{rel}}_{f\circ\mathscr{F}}\big)_s \simeq \bigoplus_x \mathcal{O}_{X\times S,(x,s)}/J^{\mathrm{rel}}_{f\circ\mathscr{F}}.$$

Reducing the right-hand vertical morphism modulo the maximal ideal $m_{S,s}$, we obtain the reduced Kodaira Spencer map of G at $i(s)$, or rather, its translation by the isomorphism φ_s^* (composition with φ_s), as shown in the diagram

$$\begin{tikzcd} \bigoplus_x\big(\mathcal{O}_{X,x}/J_{f\circ F_s}\big) & \\ & \bigoplus_x\big(\mathcal{O}_{X,\varphi_s(x)}/J_{g_{i(s)}}\big) \arrow[ul] \\ T_{i(s)}W \arrow[uu] \arrow[ur] & \end{tikzcd} \tag{12}$$

in which the northeast arrow is the reduced Kodaira-Spencer map of G at s and the northwest arrow is φ_s^*. Miniversality of G implies that the northeast arrow in (12) is an isomorphism. Hence, so is the right hand vertical map in (11), since its source and target are both free modules of rank $\mu(f\circ F)$ and its reduction modulo $m_{S,s}$ is an isomorphism. □

Proposition 6.2. *Assume that G in (10) is an $\mathcal{R}_e$-miniversal deformation of the isolated singularity $f\circ F$, and that $\mathscr{F}$ is a $\mathcal{K}_f$-miniversal deformation of F. Then*

1. *if $m=m_0$, i is a local immersion into the discriminant Δ_G of G,*
2. *if $m=m_0-1$, the critical locus $\mathcal{C}$ of i is $\pi_S(V(\mathscr{F}^*J_f))$ and $i\colon S-\mathcal{C}\to T-\Delta$ is an unramified covering.*

Proof. In each of the two cases, since G is a $\mathcal{R}_{\mathrm{e}}$-miniversal deformation, the vertical arrows of (11) are both isomorphisms. The exact sequences (6) and (5) show that the support of the cokernel of ti is exactly the projection to S of supp $\mathcal{O}_{X\times S}/\mathscr{F}^*J_f$. As this last module is Cohen-Macaulay and supported inside $\mathscr{D}=V(f\circ\mathscr{F})$ we see that $D_s=V(f\circ F_s)$ is singular. Hence the set of values where i is not submersive is contained in Δ_G.
To conclude, the exact sequence (6) says that i is a local immersion in the case $m=m_0$, whereas for $m=m_0-1$, (5) says the critical locus $\mathcal{C}$ is the projection by π_S of $\operatorname{supp}\operatorname{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f,\mathcal{O}_{X\times S})$ and hence equal to $\pi_S(V(\mathscr{F}^*J_f))$. □

Remark 6.3. Let $G: X\times T\to\mathbf{C}$ be a miniversal deformation of the composite function $f\circ F$. The Kodaira-Spencer map $\rho_G\colon\theta_T\to\mathcal{O}_{X\times T}/J^{\mathrm{rel}}_G$ defines Saito's F-manifold structure on T (see e.g. [8] Chapter 5), and hence on $\theta(i)=\theta_T\otimes\mathcal{O}_S$. The Euler vector field E_T of this F-manifold is given by the class of G in the relative Jacobian algebra. The mapping i respects the multiplication and Euler vector field in the sense that

$$\begin{aligned} ti(u\star v) &= ti(u)\star ti(v)\\ ti(E_S) &= E_T\circ i \end{aligned} \tag{13}$$

Thus, the restriction of i to $S-B$ is a morphism of F-manifolds.

Note that in the case $m=m_0$ the multiplication in TS is defined even at the points of B, since now $\operatorname{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f,\mathcal{O}_X)$ vanishes and $(T^1_{\mathcal{K}_f}F_s)_x$ can be identified with $\big(F_s^*(J_f)/J_{f\circ F}\big)_x$ for each $x\in\operatorname{supp}T^1_{\mathcal{K}_f}F_s$. Nevertheless, since the multiplication at a point $s\in B$ lacks a unit, the unit vector field is not defined at s, and we cannot refer to all of S as an F-manifold. Similarly, the Euler field on $S-B$, which corresponds to the class of $f\circ\mathscr{F}$ in $\pi_*\big(\mathcal{O}_{X\times S}/J_{f\circ\mathscr{F}}\big)$ under the identifications

$$\theta_{S-B}\simeq\pi_*\big(T^1_{\mathcal{K}_f/S}\mathscr{F}\big)\simeq\pi_*\big(\mathcal{O}_{(X\times S)-\mathscr{D}}/J_{f\circ\mathscr{F}}\big),$$

is no longer defined at points of B where the class of $f\circ\mathscr{F}$ does not lie in $\pi_*\big(\mathscr{F}^*(J_f)/J^{\mathrm{rel}}_{f\circ\mathscr{F}}\big)$.

Remark 6.4. The statement (2) in 6.2 implies a conjecture in [7]. Let $F\colon \mathbf{C}^2\to\mathrm{Sym}_n$ be a family of $n\times n$-symmetric matrices and $f\colon\mathrm{Sym}_n\to\mathbf{C}$ the determinant. Then the subvariety $\sum\subset S$ corresponding to values of the parameter space for which F_s intersects the set of matrices of corank at least 2 is $(\pi_S)_*V(\mathscr{F}^*J_f)$ and hence coincides with $\mathcal{C}$.

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