Complex monodromy and changing real pictures

Thomas Cooper and David Mond,
Mathematics Institute, University of Warwick

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1 Introduction

The purpose of this note is to generalise, and give a more illuminating proof, of a theorem of [13] (Theorem 1.1 below). Before stating it, we provide some introductory information. Consider the following two sequences of pictures: in each we see a 1-parameter family $X_{\mathbb{R},t}$ of real algebraic hypersurfaces,

Figure 1: Bifurcation of real algebraic sets

which undergoes a bifurcation when the parameter t is equal to 0. Note that in both (i)(a) and (b), and in (ii)(b), the surface $X_{\mathbb{R},t}$ has a purely 1-dimensional part, which we have indicated with a dotted line, and that in (i)(b) we have drawn a curve vertically along the middle of the surface to make clearer the way it passes through itself. The reader will observe that (a), $X_{\mathbb{R},t}$ is homotopically a 2-sphere when $t > 0$ and a 0-sphere when $t < 0$, while in (b), $X_{\mathbb{R},t}$ is a homotopy 1-sphere both for $t < 0$ and $t > 0$.

Such sequences are typical in singularity theory; each is in fact the family of algebraic closures of images of a versal deformation of a codimension 1 singularity of mapping.

Now suppose that the complexification $X_{\mathbb{C},t}$ is a homotopy n-sphere. In [13] the second author pointed out that it follows that $X_{\mathbb{R},t}$ is a homotopy sphere for $t \neq 0$ (allowing the empty set as a -1-sphere). Indeed, in the local situation, or globally in the weighted homogeneous case, there are well-defined integers k_+ and k_- between -1 and n such that $X_{\mathbb{R},t} \simeq S^{k_+}$ for $t > 0$ and $X_{\mathbb{R},t} \simeq S^{k_-}$ for $t < 0$.

We describe $X_{\mathbb{R},t}$ for $t \in \mathbb{R} - 0$ as "good" if the homotopy dimension of $X_{\mathbb{R},t}$ is equal to n. In this case the inclusion $X_{\mathbb{R},t} \hookrightarrow X_t$ is a homotopy equivalence ([13], 1.1).

Our main result is

Theorem 1.1 *Let $X \subseteq \mathbb{C}^{n+1} \times \mathbb{C}$ be a hypersurface and let X_t be the fibre of X over $t \in \mathbb{C}$, with respect to the projection p onto the last component. Suppose that*

1. *p has an isolated singularity at $(0,0)$;*
2. *X is weighted homogenenous (with respect to a system of positive weights);*
3. *X has a real defining equation;*
4. *for $t \neq 0$, the rank of $H_n(X_t; \mathbb{Z})$ is equal to 1.*
5. *for $t > 0$, the rank of $H_n(X_{\mathbb{R},t}; \mathbb{Z})$ is equal to 1 (i.e. $X_{\mathbb{R},1}$ is good),*

then the homological monodromy of the fibration of $X - X_0$ over $\mathbb{C} - 0$ is just multiplication by $(-1)^\Delta$, where $\Delta = k_+ - k_-$ is the change in homotopy-dimension of $X_{\mathbb{R},t}$ as t passes through 0.

Our definition of "isolated singularity" is as follows: $f : X \to \mathbb{C}$ has an isolated singularity at x if for all x' in some punctured neighbourhood of x, there is a germ of vector field $v \in \theta_{X,x'}$ such that $v \cdot f \neq 0$. Equivalently, the ideal $J_X(f)$ of $\mathcal{O}_{X,x}$, obtained by differentiating f along all vector fields in θ_X, contains a power of the maximal ideal $m_{X,x}$. This is the definition used, for example, in [1], but is more restrictive than Lê's notion of isolated singularity, as described in [8].

Theorem 1.1 is just Theorem 5.2 of [13], except that we have removed one of the hypotheses: that the quasihomogeneous monodromy of the fibration $(X - X_0) \to (\mathbb{C} - 0)$ be real. In fact we show below that this is forced by the hypothesis that X_1 be good. The proof given in [13] was purely combinatorial, and gave no geometrical insight. This defect is remedied here.

2 Proofs

First, we reduce the theorem to a comparison of the Morse theory of a function on $\mathbb{C}^{n+1}$ and the Morse theory of its restriction to the real subspace.

Take coordinates $x_1, ..., x_{n+1}, t$ on $\mathbb{C}^{n+1} \times \mathbb{C}$, let $G : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$ be a reduced defining equation for X, and let $J_{rel}G = (\partial G/\partial x_1, ..., \partial G/\partial x_{n+1})$ be the relative Jacobian ideal of G. The following lemma concerns the polar curve of the equation G with respect to the linear form t. It is well-known (see e.g [4]); we include a proof because it introduces some ideas we shall use later on.

Lemma 2.1 *Suppose that the projection $p : X = G^{-1}(0) \to \mathbb{C}$ given by $p(x,t) = t$ has an isolated singularity at $(0,0)$. Then*
i) $V(J_{rel}G) - X$ is at most 1-dimensional
and
ii) the projection $cl(V(J_{rel}G) - X) \to \mathbb{C}$ is finite.

Proof If $(0,0) \neq (x,t) \in X$, there exists $\xi \in \theta_{X,(x,t)}$ such that $\xi \cdot p \neq 0$. The vector field ξ is the restriction to X of an ambient vector field, which we may also call ξ, and since $\xi \cdot p \neq 0$, the $\partial/\partial t$ component of ξ is non-zero at (x,t). Since ξ is tangent to X, we have $\xi \cdot G \in (G)$, and it follows that $\partial G/\partial t \in J_{rel}G + (G)$ at (x,t).

Now suppose that $(0,0 \neq (x,t) \in cl(V(J_{rel}G) - X) \cap X$. Then there is an analytic curve-germ $\gamma : (\mathbb{C}, 0) \to (V(J_{rel}G), (x,t))$ such that $G(\gamma(s)) \neq 0$ for $s \neq 0$. We have $d(G \circ \gamma)/ds = \sum_{i=1}^{n+1} d\gamma_i/ds(\partial G/\partial x_i) \circ \gamma + d\gamma_{n+2}/ds(\partial G/\partial t) \circ \gamma = d\gamma_{n+2}/ds(\partial G/\partial t) \circ \gamma$ (since $\gamma(s) \in V(J_{rel}G)$ for all s). Thus, by the preceding paragraph, $d(G \circ \gamma)/ds \in (G \circ \gamma)$. Since $G(\gamma(0)) = 0$, this is absurd.

It follows that the intersection of $cl(V(J_{rel}G) - X)$ with the hypersurface X is contained in $\{(0,0\}$, i.e. is at most 0-dimensional. Part (i) follows.

Part (ii) follows also: for were it false, there would be a curve $\gamma : (\mathbb{C}, 0) \to (cl(V(J_{rel}(G) - X)) \cap \mathbb{C}^{n+1} \times \{0\}$, with $\gamma(s) \neq (0,0)$ for $s \neq 0$. Clearly the image of such a curve must lie in X; but each point of its image is the limit point of a curve in $V(J_{rel}G) - X$, and this now contradicts (i). $\square$

Corollary 2.2 *With the hypotheses of Lemma 2.1, suppose in addition that rank $H^n(X_t, \mathbb{Z}) = 1$ for $t \neq 0$. Then for $t \neq 0$, the function $g_t = G(\ ,t)$ has a single critical point P_t off its zero-locus X_t. Furthermore, this critical point is non-degenerate, and lies in $\mathbb{R}^{n+1}$ if G is real.*

Proof By Lemma 2.1, g_t has only a finite number of critical points off X_t. A Morse-theoretic argument due to Siersma ([16]) now shows that X_t is homotopy-equivalent to a wedge of n-spheres, whose number is equal to the sum of the Milnor numbers of g_t at its critical points off X_t. Thus, there can be only one such point, and it must be non-degenerate. If in addition G is real, the complex conjugate of P_t is also critical; it follows that P_t is conjugation-invariant.$\square$

The proof of Theorem 1.1 is based on the Morse theory of the critical point whose existence and uniqueness we have just proved. We need

Lemma 2.3 *Suppose g is a complex analytic function with a non-degenerate critical point at P, and $g(P) \neq 0$. Then*

1. *if $\mathbf{v} \in T_P\mathbb{R}^{2n+2}$ is an eigenvector of $d^2 \mid g \mid (P)$ with eigenvalue λ, then $i\mathbf{v}$ is an eigenvector with eigenvalue $-\lambda$,*
2. *the index of the Hessian of $\mid g \mid$ at P is equal to $n+1$, and*
3. *the negative eigenspace of $d^2 \mid g \mid (P)$ spans $\mathbb{C}^{n+1}$ over $\mathbb{C}$.*

Proof Modulo higher-order terms, we have $g(P + \mathbf{z}) = p + iq + \mathbf{z^t}[\mathbf{a} + i\mathbf{b}]\mathbf{z}$, where $p + iq = g(P)$ and $[\mathbf{a} + i\mathbf{b}]$ is the matrix of $(1/2)d^2g(P)$, written as a sum of real and imaginary matrices. With $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$, we therefore have

$$|g|^2(P+\mathbf{x}+i\mathbf{y}) = p^2+2p\{\mathbf{x}^t\mathbf{a}\mathbf{x}-\mathbf{y}^t\mathbf{a}\mathbf{y}-2\mathbf{x}^t\mathbf{b}\mathbf{y}\}+q^2+2q\{\mathbf{x}^t\mathbf{b}\mathbf{x}-\mathbf{y}^t\mathbf{b}\mathbf{y}+2\mathbf{x}^t\mathbf{a}\mathbf{y}\},$$

again modulo higher-order terms. It follows that

$$d^2|g|^2(P)(\mathbf{x}+i\mathbf{y},\mathbf{x}+i\mathbf{y}) = 4p\{\mathbf{x}^t\mathbf{a}\mathbf{x}-\mathbf{y}^t\mathbf{a}\mathbf{y}-2\mathbf{x}^t\mathbf{b}\mathbf{y}\}+4q\{\mathbf{x}^t\mathbf{b}\mathbf{x}-\mathbf{y}^t\mathbf{b}\mathbf{y}+2\mathbf{x}^t\mathbf{a}\mathbf{y}\}.$$

From this it is easy to check that $d^2|g|^2(P)(i\mathbf{z},i\mathbf{z}) = -d^2|g|^2(P)(\mathbf{z},\mathbf{z})$. Since $d^2|g|^2(P) = 2|g(P)|d^2|g|(P)$, (i) follows.

The remaining two statements are obvious consequences of (i). $\square$

Lemma 2.4 *Suppose that $P \in \mathbb{R}^{n+1}$ and that $g \in \mathcal{O}_{\mathbb{C}^{n+1},P}$ is real, and let $g_{\mathbb{R}}$ denote its restriction to $\mathbb{R}^{n+1}$. Suppose also that g has a non-degenerate critical point at P, and that $g(P) > 0$. Let $S(\mid g \mid, P)$ and $S(g_{\mathbb{R}}, P)$ denote the stable manifolds of (the gradient flow of) $\|g\|$ and $g_{\mathbb{R}}$ at P. Then*

$$S(g_{\mathbb{R}}, P) = S(\|g\|, P) \cap \mathbb{R}^{n+1}.$$

Proof Since g is a real function (has real coefficients in its power series expansion), $\mid g \mid$ is invariant with respect to complex conjugation, which we think of as an $\mathbb{R}$-linear automorphism τ on $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. Hence $\nabla \mid g \mid$ is τ-equivariant, and in particular τ-invariant on $Fix(\tau) = \mathbb{R}^{n+1}$. In other words, $\nabla \mid g \mid$ is tangent to $\mathbb{R}^{n+1}$. It follows that $\nabla \mid g \mid_{|\mathbb{R}^{n+1}} = \nabla \mid g_{\mathbb{R}} \mid = \nabla g_{\mathbb{R}}$ (the second equality because $g_{\mathbb{R}} = \mid g_{\mathbb{R}} \mid$ in the neighbourhood of P). The lemma follows. $\square$

To simplify notation, from now on we consider the fibres $X_{\mathbb{R},-1}$ and $X_{\mathbb{R},1}$ rather that $X_{\mathbb{R},t}$ for $t < 0$ and $t < 0$. There is no loss of generality in so doing.

Corollary 2.5 *Suppose that*

1. *the projection $p : X = G^{-1}(0) \to \mathbb{C}$ given by $p(x,t) = t$ has isolated singularity;*
2. *rank $H^n(X_t; \mathbb{Z}) = 1$ for $t \neq 0$;*
3. *G is real.*

Let $\ell = dim\ S(\mid g \mid, P_\epsilon) \cap \mathbb{R}^{n+1} - 1$ where $\epsilon = \pm 1$. Then $X_{\mathbb{R},\epsilon}$ is a homotopy ℓ-sphere.

Proof Clearly $g_\epsilon(P_\epsilon) \in \mathbb{R} - 0$. Suppose first that $g_\epsilon(P_\epsilon) > 0$. Let B be a large ball in $\mathbb{R}^{n+1}$, centred at 0 and containing P_ϵ. By Morse theory, B is obtained from $X_{\mathbb{R},\epsilon}$ by gluing in an k-cell, where $k+1 = \dim S(g_\epsilon, P_\epsilon)$. Since B is contractible, $X_{\mathbb{R},\epsilon}$ is homotopy-equivalent to the boundary of this cell. By the lemma, $k = \ell$.

If $g_\epsilon(P_\epsilon) < 0$, then the preceding analysis applies to $-g_\epsilon$. $\square$

We now recall first the construction of the weighted homogeneous monodromy. Suppose that X has real defining equation $G(z_1, ..., z_{n+1}, t)$, which is homogeneous of degree d with respect to weights $w_1, ..., w_{n+1}, w$. Write $g_t(z_1, ..., z_{n+1}) = G(z_1, .., z_{n+1}, t)$, so that g_t defines X_t. Then

$$g_{te^{il}}(e^{ilw_1/w}z_1, ..., e^{ilw_{n+1}/w}z_{n+1}) = e^{ild/w}g_t(z_1, ..., z_{n+1}),$$

so the linear isomorphism A_l whose matrix, with respect to the standard basis, is $\mathrm{diag}(e^{ilw_1/w}, ..., e^{ilw_{n+1}/w})$, maps $X_t = {g_t}^{-1}(0)$ diffeomorphically to $X_{e^{ilt}}$. In particular $A_{2\pi}$ induces the geometric monodromy of the fibration $(X - X_0) \to (\mathbb{C} - 0)$.

Proposition 2.6 *In the situation of Theorem 1.1, each entry in the matrix of $A_{2\pi}$ is equal to ± 1.*

Proof By replacing G by $-G$ if necessary, we may assume that $g(P_1) > 0$. Because $X_{\mathbb{R},1}$ is good, the critical point of $g_{\mathbb{R},1}$ at P_1 is a local maximum, with stable manifold of dimension $n+1$. On the other hand, the stable manifold of $\mid g_1 \mid$ at P_1 also has dimension equal to $n+1$, by Lemma 2.3. It follows from Lemma 2.4 that $S(\|g_1\|, P_1))$ coincides with $S(g_1, P_1)$.

Now $g_t \circ A_{2\pi} = e^{2i\pi d/w}g_t$, so because P_t is the unique critical point with non-zero critical value, $A_{2\pi}(P_t) = P_t$, and indeed $e^{2i\pi d/w} = 1$. It follows also that $A_{2\pi}(S(\|g_1\|, P_1)) = S(\|g_1\|, P_1)$, and thus that $A_{2\pi}(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1}$. Hence $A_{2\pi}$ is real. Since all of the eigenvalues of $A_{2\pi}$ are roots of unity, each must be equal to ± 1. $\square$

Remark 2.7 It should be noted that when *rank* $H_n(X_t; \mathbb{Z}) > 1$, the quasi-homogenenous monodromy $A_{2\pi}$ is not in general real.

From the coincidence of $S(g_1, P_1)$ and $S(|\ g_1\ |, P_1)$ in the previous proposition, it follows that the inclusion $X_{\mathbb{R},1} \hookrightarrow X_1$ is a homotopy equivalence (see Theorem 1.1 of [13]). In particular, it induces an isomorphism on n'th homology. Since the inclusion $X_{\mathbb{R},1} \hookrightarrow X_1$ is equivariant with respect to the action of $A_{2\pi}$, it follows that homological information about the the complex monodromy can be read off from the action of $A_{2\pi}$ on $X_{\mathbb{R},1}$.

Proposition 2.8 *Let $X \subseteq \mathbb{C}^{n+1} \times \mathbb{C}$ be a hypersurface, and suppose that hypotheses 1, 2 and 4 of Theorem 1.1 hold. Then the homological monodromy $h_* : H_n(X_1; \mathbb{Z}) \to H_n(X_1; \mathbb{Z})$ of the fibration of X over $\mathbb{C} - \{0\}$ is just multiplication by $det A_{2\pi}$.*

Proof The quasihomogeneous monodromy $A_{2\pi}$ maps V to itself, and so induces a real linear automorphism of V, which we will denote by $A_{2\pi\ |V}$. Since X_1 is homotopy equivalent to the boundary of an n-cell in $S(|\ g_1\ |, P_1)$, by a homotopy equivalence which is equivariant with respect to the quasihomogeneous monodromy, it follows that h_* is multiplication by 1 if $A_{2\pi\ |V}$ preserves orientation and -1 if it reverses it. That is, $h_* = 1$ if $det(A_{2\pi\ |V}) > 0$ and $h_* = -1$ if $det(A_{2\pi\ |V}) < 0$. Since by Lemma 2.3 V spans $\mathbb{C}^{n+1}$ over $\mathbb{C}$, a real basis for V is also a complex basis for $\mathbb{C}^{n+1}$. It follows that $det(A_{2\pi\ |V}) = det(A_{2\pi})$. All the eigenvalues of $A_{2\pi}$ are roots of unity, and so as $det(A_{2\pi})$ is real, it must be equal to ± 1. The result follows. □

Proof of Theorem 1.1 The "semi-monodromy" A_π is the diagonal matrix with entry in the j-th place equal to $\exp(i\pi w_j/w)$. Clearly $A_\pi(X_1) = X_{-1}$. Since A_π is an isometry on the underlying real space, it follows that
$A_\pi(S(\|g_1\|, P_1) = S(\|g_{-1}\|, P_{-1})$. We now apply Lemma 2.4. There are two cases.

First, if $g_{-1}(P_{-1}) > 0$ then $X_{\mathbb{R},-1}$ is homotopy equivalent to the boundary of a neighbourhood of P_{-1} in $S(g_{\mathbb{R},-1}, P_{-1})$. By Lemma 2.4, $S(g_{\mathbb{R},-1}, P_{-1}) = S(|\ g_{-1}\ |, P_{-1}) \cap \mathbb{R}^{n+1} = A_\pi(S(|\ g_1\ |, P_1)) \cap \mathbb{R}^{n+1}$. Hence the dimension of $S(g_{\mathbb{R},-1}, P_{-1})$ is equal to the number of real diagonal entries in A_π, and k_- is 1 less than this.

If $g_{-1}(P_{-1}) < 0$ then $X_{\mathbb{R},-1}$ is homotopy equivalent to the boundary of a neighbourhood of P_{-1} in $S(-g_{\mathbb{R},-1}, P_{-1})$. By Lemma 2.4, $S(-g_{\mathbb{R},-1}, P_{-1}) = S(|\ g_{-1}\ |, P_{-1}) \cap \mathbb{R}^{n+1}$, and once again k_- is 1 less than the number of real diagonal entries in A_π.

On the other hand, since $A_{2\pi} = A_\pi^2$ is real, the remaining diagonal entries of A_π are all square roots of -1. It follows that $\det(A_{2\pi}) = (-1)^{n+1-(k_-+1)}$, and this proves the theorem. $\square$

The fact that all of the entries of A_π are fourth roots of unity, which was proved in Proposition 2.6, turns out to have another interesting consequence:

Theorem 2.9 *: Suppose that the hypotheses of Theorem 1.1 hold, and that Y is another real form of X, also quasihomogeneous with the same weights and degrees. Let ℓ_+ and ℓ_- be the homotopy dimension of $Y_{\mathbb{R},t}$ for $t > 0$ and $t < 0$ respectively. Then $\ell_+ - \ell_- \equiv k_+ - k_- (mod\, 2)$, and so in particular the parity of the change in homotopy dimension determines the monodromy of the complex fibration over $\mathbb{C} - 0$.*

Proof Let $e_1, \ldots, e_{n+1}$ be the standard basis of $\mathbb{C}^{n+1}$. Then the involutions τ (complex conjugation) and $A_{2\pi}$ each send $\mathbb{C}\langle e_i \rangle$ to itself (for all i) and their actions on $\mathbb{C}\langle e_i \rangle$ commute. Therefore $A_{2\pi}$ and τ commute.

Let S_i denote $T_{P_i}S(\mid g_i \mid; P_i)$ (for $i = 1, -1$). We have already seen that τ and $A_{2\pi}$ each act on both of these spaces, so both are representations of $\langle A_{2\pi}, \tau \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore S_1 and S_{-1} each decompose into four subspaces corresponding to the four irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

For $i = \pm 1$, let $S_i = U_i \oplus V_i$ be the decomposition of S_i into the invariant and anti-invariant subspaces under the action of $A_{2\pi}$. Similarly let $S_i = R_i \oplus I_i$ be the decomposition of S_i into the invariant and anti-invariant subspaces under the action of τ (i.e., the real and imaginary parts). Then our four subspaces of S_i are $U_i \cap R_i$, $U_i \cap I_i$, $V_i \cap R_i$ and $V_i \cap I_i$.

Notice that $A_\pi \colon S_1 \cong S_{-1}$. Also $A_\pi(U_i) \subseteq U_{-i}$ and $A_\pi(V_i) \subseteq V_{-i}$. Furthermore if $x \in U_i$ then $x = \sum \lambda_j e_j$ where $w_j/w \notin \mathbb{Z} \Rightarrow \lambda_j = 0$ so $x \in R_i \Rightarrow A_\pi(x) \in R_{-i}$. Therefore $A_\pi(U_i \cap R_i) \subseteq R_{-i}$. Similarly $A_\pi(U_i \cap I_i) \subseteq I_{-i}$, $A_\pi(V_i \cap R_i) \subseteq I_{-i}$ and $A_\pi(V_i \cap I_i) \subseteq R_{-i}$. Putting these facts together we see that

$$A_\pi \colon U_1 \cap R_1 \cong U_{-1} \cap R_{-1} \qquad A_\pi \colon U_1 \cap I_1 \cong U_{-1} \cap I_{-1}$$

$$A_\pi \colon V_1 \cap R_1 \cong V_{-1} \cap I_{-1} \qquad A_\pi \colon V_1 \cap I_1 \cong V_{-1} \cap R_{-1}.$$

Now $\ell_+ - \ell_- = (\dim R_1 - 1) - (\dim R_{-1} - 1) = \dim R_1 - \dim R_{-1} = [\dim(U_1 \cap R_1) + \dim(V_1 \cap R_1)] - [\dim(U_{-1} \cap R_{-1}) + \dim(V_{-1} \cap R_{-1})] = \dim(U_1 \cap R_1) + \dim(V_1 \cap R_1) - \dim(U_1 \cap R_1) - \dim(V_1 \cap I_1) = \dim(V_1 \cap R_1) - \dim(V_1 \cap I_1)$.

This is congruent modulo 2 to $\dim(V_1 \cap R_1) + \dim(V_1 \cap I_1) = \dim V_1$. This is the number of -1's along the diagonal in $A_{2\pi}$. Therefore $\det A_{2\pi} = (-1)^{l_+ - l_-}$ $\square$

Remark 2.10 That the quasihomogeneous monodromy $A_{2\pi}$ be real is equivalent to having $2w_j/w \in \mathbb{Z}$ for all j. Evidently this is a rather strong requirement. However, as we have seen, it is forced by the existence of a good real form, and in the class of examples described below, a good real form seems always to exist.

Let $f : \mathbb{C}^s, S \to \mathbb{C}^{n+1}, 0$ $(s \geq n)$ be a map-germ of finite $\mathcal{A}_e$-codimension, and let $f_t : U_t \to \mathbb{C}^{n+1}$ be a stable perturbation of f. Then the discriminant of f_t (or image, when $s = n$) has the homotopy type of a wedge of n-spheres ([3], [12]). The number of these is the *discriminant Milnor number* of f (or *image Milnor number* when $s = n$). It is greater than or equal to the $\mathcal{A}_e$-codimension provided $s \geq n+1$ and $(s, n+1)$ are nice dimensions in the sense of J.Mather ([11]), with equality holding in the quasihomogeneous case ([3]). The same relation holds also when $(s, n+1) = (1, 2)$ ([14]) or $(2, 3)$ ([7],[12], and conjecturally in all cases $(n, n+1)$ which are in Mather's range of nice dimensions. The first author has shown in [2] that all singularities $f : \mathbb{C}^n, S \to \mathbb{C}^{n+1}, 0$ with image Milnor number 1, and whose component mono-germs have kernel rank at most 1, are quasihomogeneous and have real forms with good perturbations.

In all other examples of germs f with discriminant (or image) Milnor number equal to 1 for which the answer is known, there is a real form with a good perturbation (see e.g. [13]).

3 Generators of the vanishing homology

Suppose that $X_{\mathbb{R},1}$ is a good real form, and let U be the connected component of the complement of $X_{\mathbb{R},1}$ in $\mathbb{R}^{n+1}$ containing the unique critical point of the defining equation g_t with non-zero critical level. Denote the topological adherence of U by ∂U.

Lemma 3.1 *U is bounded.*

Proof Let B_R denote the ball in $\mathbb{R}^{n+1}$ of radius R, centred at 0, and let S_R denote its boundary. All level sets of $|g_{\mathbb{R},0}|$ are stratified transverse to S_R, and therefore for parameter values t sufficiently close to 0, all level sets of $|g_{\mathbb{R},t}|$ are stratified transverse to S_R. Since the unique critical point P_t of $g_{\mathbb{R},t}$ lying off $g_{\mathbb{R},t}^{-1}(0)$ tends to 0 as t tends to 0, it follows that for small t, $P_t \in B_R$. Let U_t be the connected component of the complement of $X_{\mathbb{R},t}$ containing P_t. We claim that provided $t > 0$ is small enough then $U_t \subset B_R$. To see this, observe that U is foliated by level sets $C_\eta = g_{\mathbb{R},t}^{-1}(\eta) \cap U$, for η between 0 and $g_{\mathbb{R},t}(P_t)$. Each set C_η is a sphere, since each is the image under the downward gradient flow of $g_{\mathbb{R},t}$ of a level set near the non-degenerate local maximum P_t. If any C_η meets S_R, let $\eta_0 = sup\{\eta \mid C_\eta \cap S_R \neq \emptyset\}$. Then C_{η_0} fails to be transverse to S_R, contradicting the statement made earlier. Therefore all C_η are contained in the interior of S_R. Since U_t is bounded, so is $U = U_1$. $\square$

Proposition 3.2 *Suppose that X_1 is good. Then $A_\pi(X_{\mathbb{R},1}) \cap \mathbb{R}^{n+1}$ is a deformation-retract of $X_{\mathbb{R},-1}$.*

Proof Recall that the quasihomogenenous monodromy $A_{2\pi}$ is represented in the standard basis of $\mathbb{C}^{n+1}$ by a diagonal matrix with diagonal entries ± 1. Let V_+ and V_- be the positive and negative eigenspaces of the restriction of $A_{2\pi}$ to $\mathbb{R}^{n+1}$. The semimonodromy A_π is the diagonal matrix with ± 1 in place of each positive entry in $A_{2\pi}$ and $\pm i$ in place of each negative entry. We have the following relation between the germs at P_{-1} of stable manifolds:

$$S(|g_{\mathbb{R},-1}|, P_{-1}) = S(|g_{-1}|, P_{-1}) \cap \mathbb{R}^{n+1}$$

$$= A_\pi(S(|g_1|, P_1) \cap \mathbb{R}^{n+1} = A_\pi(\mathbb{R}^{n+1}) \cap \mathbb{R}^{n+1} = V_+.$$

Now observe that $g_{-1\ |V_+}$ has unique critical point P_{-1} off its zero-locus. For $g_{\mathbb{R},-1}$ is invariant under $A_{2\pi}$, so that P_{-1} must lie in V_+, and moreover $dg_{\mathbb{R},-1\ |V_-} = 0$, so that any critical point of $g_{-1\ |V_+}$ is also a critical point of $g_{\mathbb{R},-1}$. Note that P_{-1} is a local maximum of the restriction of $|g_{\mathbb{R},-1}|$ to V_+. It follows by Lemma 3.1 that P_{-1} lies in a bounded connected component U' of the complement in V_+ of $X_{\mathbb{R},-1}$, and moreover that U' is foliated by its intersection with the level sets of $|g_{\mathbb{R},-1\ |V_+}|$, which are spheres of dimension k_-.

Let $X^\epsilon_{\mathbb{R},-1} = g_{\mathbb{R},-1}^{-1}([-\epsilon, \epsilon])$. Note that for small ϵ (less than $g_{-1}(P_{-1})$, $X^\epsilon_{\mathbb{R},-1}$ is a smooth manifold with boundary. By classical Morse theory, it is rendered contractible by gluing in a ball along the sphere $|g_{\mathbb{R},-1}|^{-1}(\epsilon) \cap$

$V_+ \cap U'$. The same is true of $X^\epsilon_{\mathbb{R},-1} \cap V_+$. It follows that the inclusion $X^\epsilon_{\mathbb{R},-1} \cap V_+ \hookrightarrow X^\epsilon_{\mathbb{R},-1}$ is a homotopy equivalence. As $X_{\mathbb{R},-1}$ can be given a Whitney stratification, and P_{-1} is the only critical point of $g_{\mathbb{R},-1}$ off $X_{\mathbb{R},-1}$, it follows that for $\epsilon > 0$ small enough there is a deformation-retraction $X^\epsilon_{\mathbb{R},-1} \to X_{\mathbb{R},-1}$ (see [5], Chapter 2). Similarly, $X_{\mathbb{R},-1} \cap V_+$ is a deformation-retract of $X^\epsilon_{\mathbb{R},-1} \cap V_+$. By composition of homotopy-equivalences, this shows that the inclusion $X_{\mathbb{R},-1} \cap V_+ \hookrightarrow X_{\mathbb{R},-1}$ is a homotopy equivalence, so that the former space is a deformation-retract of the latter (because the pair $(X\mathbb{R}, -1, X_{\mathbb{R},-1} \cap V_+)$ is algebraic and therefore triangulable, it is an NDR pair, see [17] page 25.) Finally, $X_{\mathbb{R},-1} \cap V_+ = A_\pi(X_{\mathbb{R},1} \cap \mathbb{R}^{n+1})$. $\square$

We illustrate the proposition in an example in Figure 2, which shows the bifurcation "birth of two triple points". For fixed value of the parameter t, the surface consists of three immersed surfaces, parametrised by

$$\begin{cases} (x,y) & \to & (x,y,y) \\ (x,y) & \to & 9x,y,-y) \\ (x,y) & \to & (x,y,t-x^2) \end{cases}$$

The family is weighted homogenenous with respect to (target) weights $w_1 = 1, w_2 = 2, w_3 = 2, w = 2$.
FIGURE 2 HERE
The matrix A_π inducing the weighted homogenenous semi-monodromy is

$$\begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We leave to the reader to check that for $t \neq 0$, the unique critical point of the defining equation $g_t(X_1, X_2, X_3) = (X_2 - X_3)(X_2 + X_3)(X_3 + X_1^2 - t)$ lying off the surface is the point $(0, 0, 2t/3)$.

In (b), which is a good real picture, the surface is shown opaque. The three immersed surfaces meet in two triple points. In (a) the surfaces are shown transparent. There are no triple points. In (a) we have highlighted a triangle contained in $A_\pi(X_{\mathbb{R},1}) \cap \mathbb{R}^{n+1}$ with a thick black line, in the centre of the picture; the remainder of $A_\pi(X_{\mathbb{R},1}) \cap \mathbb{R}^{n+1}$ consists of the continuation of the three sides of the triangle. The triangle caries the vanishing homology of $X_{\mathbb{R},-1}$; it is in fact equal to $A_\pi(\partial U) \cap \mathbb{R}^{n+1}$.

We conjecture

Conjecture 3.3 *With the assumptions and notation of this section, we have*

1. ∂U *is a topological* n*-sphere,*
2. ∂U *is a deformation-retract of* $X_{\mathbb{R},1}$,
3. $A_\pi(\partial U) \cap \mathbb{R}^{n+1}$ *is a topological* k_-*-sphere, and*
4. $A_\pi(\partial U) \cap \mathbb{R}^{n+1}$ *is a deformation-retract of* $X_{\mathbb{R},-1}$.

4 Real images and their closure

Suppose that instead of real algebraic sets, we are interested in the images of (proper) real algebraic maps $\mathbb{R}^n \to \mathbb{R}^{n+1}$. These are of course merely semi-algebraic. The Zariski closure of the real image in general contains lower dimensional pieces which may contribute non-trivially to its homology. For example, in Figure 1(i) we see the Zariski closure $\overline{f_t(\mathbb{R}^2)}$ of the image of $f_t(x,y) = (x, y^2, y^3 + x^2y + ty)$ for $t = -1$ in (a) and for $t = 1$ in (b). In (b), the singular curve of the surface is of course an algebraic set, and continues beyond the surface itself. However, it is clear in this case that $f_1(\mathbb{R}^2) \hookrightarrow \overline{f_1(\mathbb{R}^2)}$ is a homotopy equivalence. On the other hand, in (a) this last is not the case: $f_{-1}(\mathbb{R}^2)$ is contractible, whereas its Zariski closure is a homotopy 0-sphere.

Conjecture 4.1 *If* $\overline{f_t(\mathbb{R}^n)}$ *is a good real picture, then the inclusion* $f_t(\mathbb{R}^n) \hookrightarrow \overline{f_t(\mathbb{R}^n)}$ *is a homotopy-equivalence.*

Since the complement of $f_1(\mathbb{R}^2)$ in $\overline{f_1(\mathbb{R}^2)}$ is of dimension less than n, it is easy to see that the inclusion of the former in the latter induces an isomorphism on n'th homology.

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