MA455 MANIFOLDS

$$\left\{ \begin{array}{c} \text{POST MORTEM ON THE 2004-2005 EXAM} \\ \text{RU READY FOR NEXT YEAR'S COURSE} \end{array} \right\}$$

Marking both this year's and last year's exams was nerve-racking. Both papers began with questions on transversality and on the notion of the derivative of a smooth map, and these were the worst-answered of the lot! Marking them first, I was afraid the average would be disastrously low. But marks on the later questions were much higher, thank goodness.

This surprised me a lot. I don't think that transversality is such a hard topic. But in fact the difficulties with transversality did not really concern theory; rather they were in concrete algebraic manipulation. Part (i)(a) of question 1 came down to the following:

> Let W be a vector space, and let U and V be subspaces of W. In the product space $W \times W$, let D be the diagonal, $D = \{(w, w) : w \in W\}$. Show that
>
> $$U + V = W \quad \text{if and only if} \quad (U \times V) + D = W \times W.$$

Part (i)(d) of question 1 came down to the following:

> Let U, V and W be vector spaces, and let $S : U \to W$ and $T : V \to W$ be linear maps. Let
>
> $$X = \{(u, v) \in U \times V : S(u) = T(v)\}.$$
>
> Let $p : X \to U$ be the restriction of the usual projection $U \times V \to U$. Show that
> (a) p is injective if and only if T is injective.
>
> Now suppose also that $S(U) + T(V) = W$. Show that
> (b) p is surjective if and only if T is surjective.

Try these questions now! Neither involves much more than basic set theory.

The other thing that people did badly on was the derivative of a smooth map. This is the key definition in the course MA225 Differentiation, but clearly many people only felt comfortable with the notion of derivative they learned about at A-level. The derivative of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^p$ at a point $x \in \mathbb{R}^n$ is not a number, but a linear map $\mathbb{R}^n \to \mathbb{R}^p$, which depends on x. If you write its matrix with respect to the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^p$, the entry in row i and column j is the partial derivative of the i'th component f_i with respect to x_j, $\frac{\partial f_i}{\partial x_j}$, evaluated at x. But in many cases it is not practicable to calculate all these partial derivatives, and you have to make use of the notion of the derivative itself, as a linear map. The simplest way to describe the derivative $d_x f$ is to give a formula for $d_x f(\hat{x})$, where x is the point we are calulating the derivative and $\hat{x}$ is the vector to which we are applying it. By definition

$$d_x f(\hat{x}) = \lim_{h \to 0} \frac{f(x + h\hat{x}) - f(x)}{h}, \tag{1}$$

and in many cases it is possible to calculate this explicitly. Incidentally, the statement that f is differentiable at x is essentially just the statement that the limit in (1) depends *linearly* on $\hat{x}$. It is really important that you have a good understanding of this notion. It is one of those ideas that ultimately makes mathematicss simpler rather than more complicated, although it may cost some effort to master it at first. It ties together into a single concept all $n \times p$ first order partials of f, and, for example, makes remembering the chain rule absolutely straightforward — whereas a statement of the chain rule purely in terms of partial derivatives is quite complicated and easy to memorise wrongly. It plays a central role in the subject of differentiable manifolds.

Here are some exercises on the derivative:

1) Find the derivative at (x, y) of the map $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ sending (x, y) to the vector product $x \times y$. 'Find the derivative' means: give an explicit formula for $d_{(x,y)}f(a, b)$. Important: write your answer as a sum of vector products.

2) Ditto, for the map $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ sending (x, y) to the scalar product $x_1y_1 + \cdots + x_ny_n$. Important: write your answer as a sum of scalar products.

3) Find the derivative of the map $\mathrm{Mat}_{2\times2}(\mathbb{R}) \to \mathbb{R}$ sending the 2×2 matrix M to $\det M$. Here $\mathrm{Mat}_{2\times2}(\mathbb{R})$ is the vector space of all 2×2 matrices. Important: write $d_A \det(B)$ as a sum of determinants.

4) Find the derivative of the map $\det : \mathrm{Mat}_{3\times3}(\mathbb{R}) \to \mathbb{R}$ sending the 3×3 matrix M to $\det M$. Essential: write $d_A \det(B)$ as a sum of determinants.

5) What is the kernel of $d_I \det$, where I is the identity matrix?

6) On the basis of (3) and (4), guess a formula for $d_A \det(B)$ for the general case where we consider $\det : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$. Can you prove your formula?

7) Let

$$p : \mathrm{Mat}_{n\times p}(\mathbb{R}) \times \mathrm{Mat}_{p\times q}(\mathbb{R}) \to \mathrm{Mat}_{n\times q}(\mathbb{R})$$

be the product map, $p(X, Y) = XY$. Write down a formula for

$$d_{(X,Y)}p(A, B).$$

8) Inside $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is the open set $\mathrm{Gl}_n(\mathbb{R})$ consisting of invertible matrices. Consider the *inversion map* $i : \mathrm{Gl}_n(\mathbb{R}) \to \mathrm{Gl}_n(\mathbb{R})$ sending A to A^{-1}.
i) Find $d_I i$ when $n = 2$ (where I is the identity matrix). That is, find an expressions for $d_I i(A)$, where A is a matrix. *You must write your anwer in terms of matrices.* How does this compare with the case $n = 1$?
ii) What is the answer for $n = 3$? For general n? Guesswork, on the basis of examples, is as good a route as any for arriving at the answer, but of course a proof is eventually required.
iii) There is a very simple proof which makes clever use of the the chain rule, by applying it to the composition

$$\mathrm{Gl}_n(\mathbb{R}) \xrightarrow{j} \mathrm{Gl}_n(\mathbb{R}) \times \mathrm{Gl}_n(\mathbb{R}) \xrightarrow{i\times \mathrm{id}} \mathrm{Gl}_n(\mathbb{R}) \times \mathrm{Gl}_n(\mathbb{R}) \xrightarrow{p} \mathrm{Gl}_n(\mathbb{R}), \tag{2}$$

where $j(X) = (X, X)$ and $i \times \mathrm{id}$ is the map $(X, Y) \mapsto (X^{-1}, Y)$. The composite of the maps in (2) maps everything to I,

$$X \mapsto (X, X) \xrightarrow{i \times 1} (X^{-1}, X) \xrightarrow{p} X^{-1}X = I$$

so its derivative at any point (matrix) $X \in \mathrm{Gl}_n(\mathbb{R})$ is zero. Here p is the product map of Exercise (7) above. Use the chain rule, and your knowledge of $d_{(X,Y)}p$, to obtain (and indeed prove) a simple formula for $d_I i(A)$ and, more generally, for $d_X i(A)$. How does your answer to the last part compare with the case $n = 1$? What is the crucial difference?

8. Consider the map $f : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}\mathbb{R}$ sending $X \to X^3$. What is $d_X f(A)$? Be careful! Find your answer by *carefully* computing the limit (1).

9. The *exponential map* for $n \times n$ matrices is defined by the same formula as when $n = 1$:

$$\exp(X) = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots.$$

Find $d_X \exp(A)$. Find $d_I \exp(A)$.

At a more basic level, it is also extremely important that you have a clear idea of how to write down the matrix of a linear map $L : V \to W$ with respect to chosen bases of the vector spaces V and W.

8) Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and $B \in \mathrm{Mat}_{p\times p}(\mathbb{R})$. Define linear maps

$$R_A : \mathrm{Mat}_{n\times p}(\mathbb{R}) \to \mathrm{Mat}_{n\times p}(\mathbb{R})$$

and

$$L_B : \mathrm{Mat}_{n\times p}(\mathbb{R}) \to \mathrm{Mat}_{n\times p}(\mathbb{R})$$

by $R_A(X) = XA$, $L_B(X) = BX$.
i) In the case where $n = 2$ and $p = 1$, choose a basis for $\mathrm{Mat}_{n\times p}(\mathbb{R})$ and write down the matrices of R_A and L_B with respect to this basis.
ii) Ditto when $n = p = 2$.
iii) What is the relation between the determinants of the matrices you have found in (i) and (ii) and $\det A$ and $\det B$?
iv) Find the answer to (iii) for general n, p.
v) Suppose that $\det A = 0$. What is the kernel of R_A? If $\det B = 0$, what is the kernel of L_B?

Final piece of advice: in 1965 John Milnor wrote a book on Manifolds, called *Topology from the Differentiable Viewpoint.* It is still in print, because it is so short and lucid. The first 19 pages make an excellent introduction to the course MA455 Manifolds. If you can get hold of a copy, spend some time this summer reading it.

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