Finding Equations

Corollary 1.29 of the Lecture Notes for the Manifolds Course states:

If $M^m \subset N^n$ are manifolds then for each point $x \in M$, there is a neighbourhood U of x in N and a smooth map $g : U \to \mathbb{R}^{n-m}$ such that 0 is a regular value of g, and $M \cap U = g^{-1}(0)$.

The components of a map g with this property are called *regular equations* for M in U.

How does one find regular equations? It can be very hard. Here I give some relatively easy examples and exercises. Mostly, finding equations is a question of paying careful attention to the description of the submanifold M for which one wants to find equations.

Example 0.1. (1) Let $(0,0) \neq (a,b) \in \mathbb{R}^2$, and let

$$M = \{(u,v) \in \mathbb{R}^2 : (x,y) \text{ is orthogonal to } (a,b)\}.$$

It is easy to find an equation for M:

$$(x,y) \in M \quad \Leftrightarrow \quad (a,b) \cdot (x,y) = 0 \quad \Leftrightarrow \quad ax + by = 0.$$

The map $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = ax + by$ is a regular equation for M in all of $\mathbb{R}^2$, because the matrix $[d_{(x,y)}g]$ is equal to $[a\ b]$ (as g is linear, its is equal to its own derivative at every point), and $[a\ b]$ defines a surjective linear map (remember that we are assuming $(a,b) \neq (0,0)$.)

(2) Now let $M = \{(u,v) \in \mathbb{R}^2 : (u,v) \text{ is parallel to } (x,y)\}$. How to find equation(s) for M?

First approach The vector $(-b,a)$ is orthogonal to (a,b), so (x,y) is parallel to (a,b) if it is orthogonal to $(-b,a)$. Thus, $M = \{(x,y) : -bx + ay = 0\}$ and we can take $g(x,y) = -bx + ay$. Similar reasoning to (i) shows g is a regular equation for M in all of $\mathbb{R}^2$.

Second approach Two vectors in $\mathbb{R}^2$ are parallel if they are linearly dependent. Linear dependence of two vectors in $\mathbb{R}^2$ (and in general of n vectors in $\mathbb{R}^n$) is detected by the vanishing of a determinant:

$$M = \{(x,y) : \det \begin{pmatrix} a & b \\ x & y \end{pmatrix} = 0\}.$$

As the determinant here is equal to $-bx + ay$, we have arrived at the same conclusion as with the first approach. Perhaps this approach is better. It's easier to see how to generalise it to higher dimensions.

(3) Let

$$M = \{(a,b,x,y) \in \mathbb{R}^4 : (a,b) \neq (0,0),\ (x,y) \text{ is parallel to } (a,b)\}.$$

Is M a manifold? The condition for membership is the same as before: that the determinant should be zero. But now we are treating a and b as variables as well. So if we take

$$g(a,b,x,y) = \det \begin{pmatrix} a & b \\ x & y \end{pmatrix}$$

as before, is g a regular equation for M? Notice that in the definition of M we stipulate that $(a,b) \neq (0,0)$, so we only need to think of the behaviour of g in $U := (\mathbb{R}^2 \setminus \{(,0)\}) \times \mathbb{R}^2$. We have

$$[d_{(a,b,x,y)}g] = [y, -x, -b, a]$$

and this is a surjective linear map for every $(a,b,x,y) \in U$. In fact we see that it is surjective if $(a,b,x,y) \neq (0,0,0,0)$, so g defines a manifold in $\mathbb{R}^4 \setminus \{0\}$.

(4) In all of the examples we've looked at so far, one equation was enough. If M has codimension greater than 1 then more equations will be needed. Let $0 \neq (a,b,c) \in \mathbb{R}^3$, and let

$$M = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \text{ is parallel to } (a,b,c)\}.$$

Clearly M is a line in 3-space, so will need two equations in the neighbourhood of every point. But which two equations? The condition that the two vectors be parallel is equivalent to the the matrix

$$\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$$

having rank 1. This means that all of its 2×2 minors must vanish. But it has three 2×2 minors. Which two should we choose? The answer is that it depends on the vector (a,b,c). If $(a,b,c) = (1,2,3)$ then the three minors, in the order $\det(\text{col } 2, \text{col } 3), \det(\text{col } 1, \text{col } 3), \det(\text{col } 1, \text{col } 2)$, are

$$2z - 3y, z - 3x, y - 2x$$

and in fact *any* two of these will do, as if both are zero then so is the third. For example

$$y - 2x = -\frac{1}{3}(2z - 3y) + \frac{2}{3}(z - 3x).$$

Exercise 0.2. (1) Assume $(a,b,c) \neq 0$ and let

$$M = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \text{ is parallel to } (a,b,c)\}.$$

Show that if $a \neq 0$ then $\det(\text{col } 2, \text{col } 3)$ can be written as a linear combination of $\det(\text{col } 1, \text{col } 2)$ and $\det(\text{col } 1, \text{col } 3)$, and deduce that M has, as regular equations, $\det(\text{col } 1, \text{col } 3)$ and $\det(\text{col } 1, \text{col } 2)$. Show that if $a = 0$ then these are no longer regular equations for M. Find regular equations if $b \neq 0$, and if $c \neq 0$.

(2) Let

$$M = \{(a,b,c,x,y,z) \in \mathbb{R}^6 : \operatorname{rank} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} = 1\}.$$

Find regular equations for M. Note that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq M,$$

so we only need be concerned with equations in $\mathbb{R}^6 \setminus \{0\}$. Note also that by (i), there is no set of regular equations which is good for all of $\mathbb{R}^6 \setminus \{0\}$. Instead, you will have to find different sets of regular equations for different open subsets of $\mathbb{R}^6 \setminus \{0\}$.

Example 0.3. (1) Let $\gamma(t) = (\cos t, \sin t, t)$. The image of γ is a helix. It's easy to find regular equations for it: since $\gamma_3(t) = t$, we can take the equations

$$x_1 - \cos x_3, \quad x_2 - \sin x_3. \tag{1}$$

(2) Now let $\sigma(t) = (\cos t, \sin t, t^3)$. The image of σ is once again a manifold (sketch it!). But now finding regular equations is not so easy. Near any point on the curve where $x_3 \neq 0$, we can recover the parameter t as $(x_3)^{\frac{1}{3}}$, and then take, as equations

$$x_1 - \cos (x_3)^{\frac{1}{3}}, \quad x_2 - \sin (x_3)^{\frac{1}{3}}. \tag{2}$$

These are smooth functions on $\mathbb{R}^3 \smallsetminus \{x_3 = 0\}$, and give regular equations there (Exercise 0.4(1). But what about points where $x_3 = 0$? The function $(x_3)^{\frac{1}{3}}$ is not smooth at $x_3 = 0$. There is such a point on the curve, namely $\sigma(0) = (1, 0, 0)$. Fortunately when t is near 0 we can recover it from $\sin t$, using the arcsin function, which defines a diffeomorphism $(-1, 1) \to (-\pi/2, \pi/2)$. Thus in the neighbourhood of $(1, 0, 0)$ we can use the equations

$$x_1 - \cos(\arcsin x_2), \quad x_3 - (\arcsin x_2)^3. \tag{3}$$

Exercise 0.4. (1) Show that the equations (1), (2) and (3) are all regular equations (for the manifolds in question) where they claim to be.

(2) Is the image of the parametrisation $\rho(t) = (\cos t, \sin t, t^2)$ a manifold? If it is, find regular equations for it.

(3) In Example 0.3(1) and 0.3(2), which theorem(s) from the course can be used to show, directly from properties of the parametrisation γ (i.e. without using equations), that the images of γ and of σ are manifolds?

(4) Find regular equations for the image of the map $\mathbb{R} \to \mathbb{R}^3$ given by $f(t) = (\cos t, \sin t, \cos t \sin t)$.

(5) Find equations for the image of the map $\mathbb{R} \to \mathbb{R}^3$ given by $g(t) = (\cosh t, \sinh t, e^t)$.

(6) Find regular equations for the set

$$\{(a, b, c, d, w, x, y, z) \in \mathbb{R}^8 \smallsetminus \{0\} : \text{rank} \begin{pmatrix} a & b & c & d \\ w & x & y & z \end{pmatrix} = 1\}.$$

(7) For each fixed $k \leq p \leq n$, the set

$$\{A \in \text{Mat}_{p \times n}(\mathbb{R}) : \text{rank } A = k\}$$

is a manifold of codimension $(p - k)(n - k)$. Hints for a proof can be found in Exercises I number 18. Exercises 0.2(2) and 0.4(4) above give alternative proofs for the case of matrices of rank 1 in the space of 2×3 matrices, and matrices of rank 1 in the space of 2×4 matrices.
To do: Find regular equations for this manifold when $p = 3, n = 3, k = 1$.