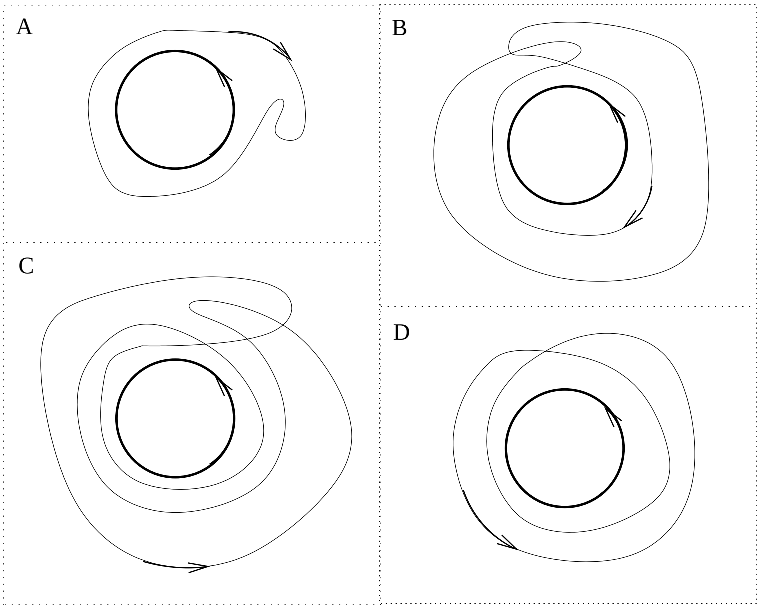
MA455 Manifolds Exercises IV Spring 2008

1. Suppose that $f : M^n \to N^n$ is a map of oriented manifolds, with M compact and N connected. Show that if f is not surjective then $\deg(f) = 0$.

2. Suppose that $f : M^n \to S^n$ is not surjective. Show that f is homotopic to a constant map.

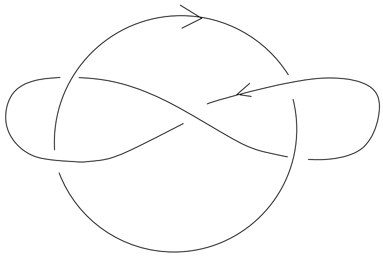
3. Each of the following pictures shows an oriented closed curve wrapping round the circle S^1. Each curve can be parametrised from S^1. Radial projection ρ maps each curve to S^1. By composing a parametrisation with the projection ρ, we get a map $f : S^1 \to S^1$.
(i) Mark the critical values of each of these maps.
(ii) For each map, check that $\deg(f; y)$ is independent of the choice of y.

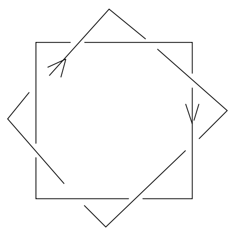


4. What is the degree of the map $S^1 \to S^1$ defined by $z \mapsto z^n$?

5. What happens to $\ell(C_1, C_2)$ if we interchange C_1 and C_2?

6. Find the linking numbers of the following pairs of curves.





7. Consider the map $S^1 \to S^1 \times S^1$ defined by $\theta \mapsto (p\theta, q\theta)$. Composing with the diffeomorphism $S^1 \times S^1 \to \mathbb{T}$ described in Section 1 of the Lecture Notes, we get a curve C on the torus $\mathbb{T}$. It is known as a (p, q) curve (because it winds around the torus p times in one direction and q times in the other).

(i) Draw the image on $\mathbb{T}$ of the $(2, 3)$ curve.

(ii) Let $C(\varepsilon)$ be the image on $\mathbb{T}^2$ of the curve $S^1 \to S^1 \times S^1$ given by θ to $(p\theta + \varepsilon, q\theta)$ for $\varepsilon > 0$.

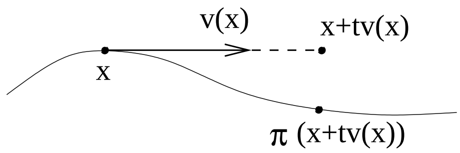
(a) Draw the curve $C(\varepsilon)$ for the case $(p,q)=(2,3)$.
(b) Show that if $\varepsilon>0$ is small then $C\cap C(\varepsilon)=\emptyset$.
(c) In the case $(p,q)=(2,3)$, find $\ell(C,C(\varepsilon))$ (for small $\varepsilon>0$).
(d) (Harder) For coprime p,q, what is $\ell(C,C(\varepsilon))$ (for small $\varepsilon>0$)?

8.(i) Show that if C_1 and C_2 are disjoint closed oriented curves in $\mathbb{R}^3$ then $\ell(C_1,C_2)$ is unchanged if we deform C_1 or C_2 in a homotopy during which the two curves remain at all times disjoint.

(ii) We say that we can *separate* C_1 and C_2 if it possible, by means of a homotopy during which the curves remain at all times disjoint, to place them on opposite sides of a plane. Show that if it is possible to separate C_1 and C_2 then $\ell(C_1,C_2)=0$. Hint: suppose they are separated by the plane H. Then it is possible to choose a vector $y\in S^2$ such that $f^{-1}(y)=\emptyset$ (where f is the map $C_1\times C_2\longrightarrow S^2$ whose degree we measure as our definition of $\ell(C_1,C_2)$).

9. Prove that if M is a compact oriented manifold and $\chi(M)\neq 0$ then any map $f:M\longrightarrow M$ which is homotopic to the identity map must have a fixed point.

10. Show that the map f_t defined in the proof of Theorem 4.16 in the Lecture Notes has no fixed point, provided the vector field [1] v has no zero on M. Hint: the *picture* should be clear enough:



How to make this into a precise argument? For which points $x'\in U_M$ is $\pi(x')$ equal to x?

11. Let v be the vector field on S^2 defined by $v(x)=\pi_x(0,0,1)$, where $\pi_x:\mathbb{R}^3\longrightarrow T_xS^2$ is orthogonal projection.
(i) Find the zeros of v.

(ii) Write down an expression for v in local coordinates around its zeros.

12. Compute $\chi(S^2)$ by considering the rotation $R_{\pi/2}$ through $\pi/2$ about the polar axis. This is homotopic to the identity, and has two fixed points, the north and south poles. The map

$$S^2\longrightarrow S^2\times S^2,\quad x\mapsto (R_{\pi/2}(x),x)$$

is homotopic to the diagonal embedding

$$x\mapsto (x,x)$$

and transverse to the image, Δ, of this embedding, so its intersection index with Δ is equal to $\chi(S^2)$.

13. There is a mapping $S^1\longrightarrow S^1$, homotopic to the identity and without fixed points. Generalise this to a map $S^{2k+1}\longrightarrow S^{2k+1}$. Deduce that $\chi(S^{2k+1})=0$.

14. Generalise the result of Exercise 8 to S^{2k}.

[1] There's a misprint in the proof of Theorem 4.16: the first line of the proof should say "Given a tangent vector field v on M", but omits the word "field".

15.(i) Let $F : W^{m+1} \to M^m$ be a smooth map, where W is a compact oriented manifold with boundary and M is an oriented manifold. Show that

$$\deg(\partial F) = 0.$$

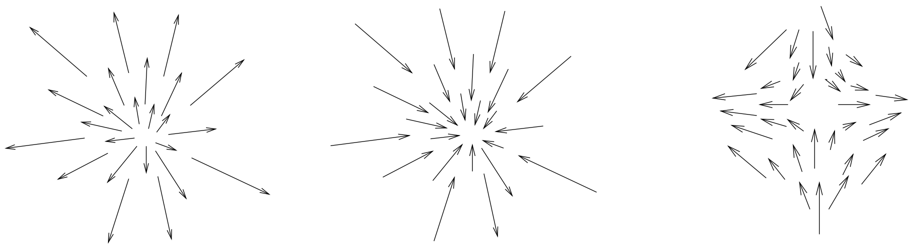
Hint: the proof of Proposition 4.1 in the Lecture Notes identifies the degree as an intersection number.

(ii) Let W be a compact oriented manifold with boundary. Show that there can be no smooth retraction $W \to \partial W$.

16. Suppose that M^m is a compact oriented manifold, and that $f_0, f_1 : M^m \to S^m$ are smooth maps. Show that if there is no $x \in M$ such that $f_0(x) = -f_1(x)$, then f_0 and f_1 are homotopic and hence have the same degree.

17. Let the vector field v on $\mathbb{R}^m$ have an isolated zero at x_0. Let B be a closed ball centred at x_0, containing has no other zero of v. Define a map $v/\|v\| : \partial B \to S^{m-1}$ by $x \mapsto v(x)/\|v(x)\|$. Show that the degree of this map is unchanged if we replace B by a smaller ball with the same property. (Hint: if B' is a smaller ball centred at x_0, the formula $x \mapsto v(x)/\|v(x)\|$ defines a map on all of the compact manifold-with-boundary $B \setminus \text{interior}(B')$. The boundary of $B \setminus \text{interior}(B')$ is $\partial B \cup \partial B'$, but the orientation of $\partial B'$ as (part of) the boundary of $B \setminus \text{interior}(B')$ is opposite to its orientation as $\partial B'$.) This degree is known as the *index* of the zero of v, and denoted by $\text{ind}_{x_0}(v)$.

18. Compute the index of the following zeros of vector fields on $\mathbb{R}^2$.



Source Sink Saddle

19. Can you sketch a vector field with isolated zero of index 2? Of index -2? Hint: the next question suggests a method for finding vector fields in the plane, with zeroes of any desired index. But you still have to sketch them.

20. Let $p(z)$ be a polynomial in the complex variable z. The *multiplicity* of a zero z_0 of $p(z)$ is the highest integer m such that $(z - z_0)^m$ divides $p(z)$, or in other words, the (unique) integer m such that $p(z) = (z - z_0)^m q(z)$ for some polynomial $q(z)$ with $q(z_0) \neq 0$. It is denoted $\text{mult}_{z_0} p$.

Suppose that z_0 is a zero of p of multiplicity m. Show that if B is a ball around z_0 containing no other zero of p then

$$\deg(p/|p|) : \partial B \to S^1 = m.$$

Hint: prove it first for the map

$$z \mapsto \frac{(z - z_0)^m}{|z - z_0|^m},$$

e.g. by counting preimages. Then

$$\frac{p(z)}{|p(z)|} = \frac{(z - z_0)^m}{|z - z_0|^m} q(z)$$

for some smooth $q(z)$ with $q(z_0) \neq 0$. Find a neighbourhood of z_0 in which $q(z)$ is never a negative multiple of $q(z_0)$, and use Exercise 15. Then use the ideas of Exercise 16 to increase the radius of B.

21. If f is holomorphic on some punctured neighbourhood of z_0 and has a pole of order m at z_0, what is $\deg f/|f| : \partial B \to S^1$? Hint: first deal with $f(z) = 1/z$, and then use an argument involving composition to deal with $f(z) = 1/z^m$.

22. As t moves away from zero, what happens to the zero at 0 of the polynomial $p_t(z) = z^n + t$?

23. Suppose that

- p_t is a parametrised family of polynomials, and that z_0 is a zero of p_0, of multiplicity m;
- U is a neighbourhood of z_0 with smooth boundary ∂U, containing no other zero of p_0;
- if $|t| < \eta$, p_t has no zero on ∂U.

Show that if $|t| < \eta$ then $\sum_{z \in U} \mathrm{mult}_z(p_t) = m$. This equality is known as *conservation of multiplicity.*

24. Deduce from Brouwer's Fixed Point Theorem that if X is any topological space homeomorphic to the n-ball, then any continuous map $f : X \to X$ must have a fixed point.

25. Prove that an $n \times n$ real matrix M with all entries strictly positive must have an eigenvector with all components strictly positive, and associated eigenvalue also strictly positive. Hint: Exercise 24.

26.(i) Show that if we reverse the orientation of the compact oriented manifold M then this has no effect on the self-intersection number of the diagonal Δ in $M \times M$. So $\chi(M)$ is independent of orientation. (Be careful: reversing the orientation of M also may change the product orientation of $M \times M$, which is what we use to calculate $(\Delta \cdot \Delta)_{M \times M}$.)
(ii) Can you use this fact to define $\chi(M)$ when M is not orientable? Hint: M is *locally* orientable.

Section C: Harder

27. Show that if M is compact, oriented and of odd dimension then $\chi(M) = 0$.

28. In $\mathbb{R}^5$, consider the two 2-spheres

$$S_1 = \{(x_1, \ldots, x_5) : x_4 = x_5 = 0, \quad x_1^2 + x_2^2 + (x_3 + \frac{1}{2})^2 = 1\}$$

and

$$S_2 = \{(x_1, \ldots, x_5) : x_1 = x_2 = 0, \quad (x_3 - \frac{1}{2})^2 + x_4^2 + x_5^2 = 1\}.$$

Show that it is not possible to move S_1 and S_2 in a homotopy in such a way that they remain at all times disjoint, to positions on opposite sides of a hyperplane.

29. The *Jordan-Brouwer separation theorem* says that if $M^m \subset \mathbb{R}^{m+1}$ (with $m \geq 1$) is a compact connected oriented manifold (without boundary) then $\mathbb{R}^{m+1} \setminus M$ consists of two connected components, one of which is bounded, and that M is the boundary of each. In this exercise you will prove this.
(i) First step: show that if a manifold M is connected then it is path connected. To do this, for

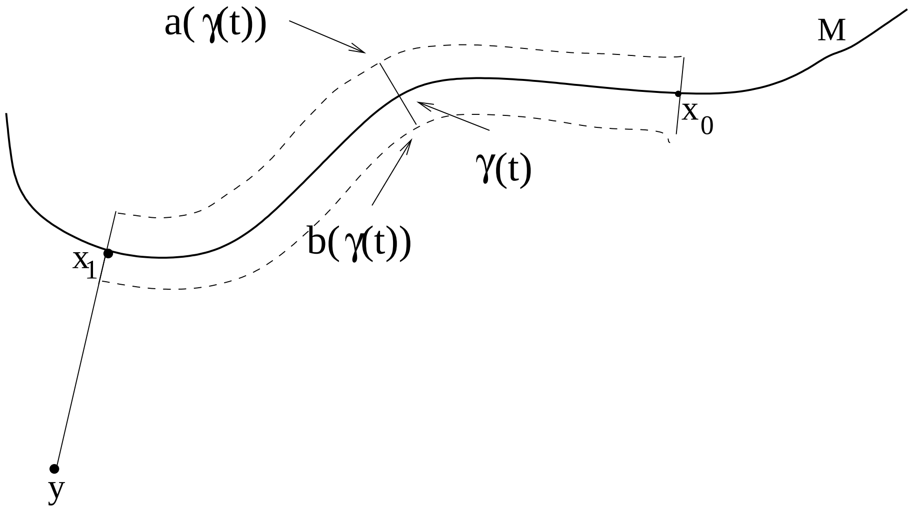
each point $x_0 \in M$ let $C(x_0)$ be the set of points that can be joined to x_0 by a path in M. Show that because M is a manifold, $C(x_0)$ is open. Next, observe that its complement is

$$\bigcup_{x \notin C(x_0} C(x),$$

so is also open.

(ii) Show that $\mathbb{R}^{m+1} \setminus M$ has at most two connected components. You can do this as follows: select a point $x_0 \in M$. Let U be a tubular neighbourhood of M in $\mathbb{R}^{m+1}$, of radius $\varepsilon > 0$. For each point $x \in M$, let L_x be a line segment orthogonal to $T_x M$, meeting M at x and projecting a distance $\varepsilon/2$ on each side of M. Let $a(x)$ and $b(x)$ be its two endpoints. We will show that every point $y \in \mathbb{R}^{m+1} \setminus M$ can be joined by a path in $\mathbb{R}^{m+1} \setminus M$ to either $a(x_0)$ or $b(x_0)$.

To show this, let $x_1 \in M$ be such that the line segment from y to x_1 meets M only at x_1 (e.g. x_1 minimises the distance from y to a point of M.) Let $\gamma : [0,1] \to M$ be a path with $\gamma(0) = x_0, \gamma(1) = x_1$. Then the two paths $a(\gamma(t)), b(\gamma(t))$ never meet M, and join $a(x_0)$ to $a(x_1)$ and $b(x_0)$ to $b(x_1)$. Show that y can therefore be joined either to $a(x_0)$ or $b(x_0)$ by a path that does not meet M.



(iii) Now we have to show that $\mathbb{R}^{m+1} \setminus M$ has at least two connected components. To do this, suppose $y \notin M$ and define a map $r_y : M \to S^m$ by

$$r_y(x) = \frac{x-y}{|x-y|}.$$

(a) Show that if y_1 and y_2 are in the same path-connected component of $\mathbb{R}^{m+1} \setminus M$, $\deg(r_{y_1}) = \deg(r_{y_2})$.

(b) Show that if the line L is transverse to M then for two points y_1, y_2 on $L \setminus L \cap M$ such that the line segment $[y_1, y_2]$ meets M exactly once, then $\deg(r_{y_1}) \neq \deg(r_{y_2})$.

(iv) Conclude that $\mathbb{R}^{m+1} \setminus M$ has two connected components, exactly one of which is bounded.

(v) Show that M is the boundary of each of the two connected components of $\mathbb{R}^{m+1} \setminus M$.