MA455 Manifolds Exercises VII Spring 2006

There are probably more exercises than you will want to do. Pick at least one on each theme of the last two chapters.

1. The (m-dimensional) *volume* of a compact oriented submanifold M^m of $\mathbb{R}^n$ is defined to be $\int_M \mathrm{vol}_M$, where vol_M is the volume form. Can you motivate this definition? How might one define the m-dimensional volume of a compact non-orientable submanifold (such as e.g. the Möbius band)?

2. Let $A_n(r)$ be the volume of the n-dimensional sphere of radius r, and let $B_n(r)$ be the volume of the n-dimensional ball of radius r. For example, $B_2(R) = \pi r^2, A_1(r) = 2\pi r, B_3(r) = \frac{4}{3}\pi r^3, A_2(r) = 4\pi r^2$. Notice that $A_1 = B_2', A_2 = B_3'$. Explain. Generalise. (hint: use Stokes's Theorem).

3. Show that the value of the right hand side of Equation (49) in the Lecture Notes is independent of the choice of partition of unity $\{\rho_i\}$.

4. Carry through the 3-dimensional analogue of the calculation in the pre-proof of Stokes's Theorem in the Lecture Notes.

5. (i) Show that if $\omega = a_1 dx_2 + a_2 dx_2 + a_3 dx_3$, with $a_1(x) \neq 0$ for all x, then there exists $\omega_2 \in \Omega^2(\mathbb{R}^3)$ such that $\omega \wedge \omega_1 = dx_1 \wedge dx_2 \wedge dx_3$.

(ii) Show that if $\omega = a_1 dx_2 + a_2 dx_2 + a_3 dx_3$, with at least one of a_1, a_2, a_3 non-zero at every point, then there exists $\omega_2 \in \Omega^2(\mathbb{R}^3)$ such that $\omega \wedge \omega_1 = dx_1 \wedge dx_2 \wedge dx_3$. Hint: use a partition of unity.

(iii) Generalise.

6. (i) Let $S(R)$ be a circle of radius $R > 0$ centred at 0. Use Stokes's Theorem to show that $\int_{S(R)} d\theta$ is independent of R.

(ii) Let C be the boundary of a compact manifold with boundary $W^2 \subset \mathbb{R}^3$, and suppose $0 \notin C$. Show that

$$\int_C d\theta = \begin{cases} 2\pi & \text{if } 0 \in W \\ 0 & \text{if } 0 \notin W \end{cases}$$

Hint: use Stokes's Theorem.

7.(a) Write down the equation of a vector field v on $\mathbb{R}^3 \setminus \{0\}$ such that:
(i) v is radial, and radially symmetric about 0.
(ii) if v were the velocity field of a fluid flow, then the flow across any sphere centred at 0 would be independent of its radius, and equal to 4π.

(b) Use this equation to find a 2-form $\omega \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$ such that if $S \subset \mathbb{R}^3 \setminus \{0\}$ is any sphere, then

$$\int_S \omega = \begin{cases} 1 & \text{if } 0 \text{ lies in the ball } B \text{ of which } S \text{ is boundary} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

(c) Suppose that $d\omega = 0$ (actually, is this true of the form you have found?). Let M^2 be the boundary of a compact oriented manifold- with-boundary $W^3 \subset \mathbb{R}^3$. Show that (1) holds also if we substitute M for S and W for B.

8. What is the relation between $d\theta \in \Omega^2(\mathbb{R}^2 \setminus \{0\})$ and the "complex-valued" 1-form dz/z? Hint: look at real and imaginary parts.

9. This exercise proves "Hadamard's Lemma": if f is a smooth function defined on some neighbourhood U of a in $\mathbb{R}^n$, then there exist smooth functions f_i on some possibly smaller neighbourhood U' of a such that for all $x \in U'$,

$$f(x) = f(a) + \sum_j (x_j - a_j) f_j(x). \tag{2}$$

To prove this, let U' be a ball with centre a, and let $x \in U'$. Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = f(a + t(x - a)).$$

Then evidently

$$f(x) = g(1) = g(0) + \int_0^1 g'(t)dt. \tag{3}$$

By applying the chain rule to the integrand in (3) and rearranging, obtain an expression like (2).