MA455 Manifolds Exercises VI Spring 2006

There are probably more exercises than you will want to do. Pick at least one on each theme of the last two chapters.

1. This exercise ses the notation introduced on page 83 of the Lecture Notes.
(a) Write down precise definitions of

(i) the parallelipiped $P(v_1, \ldots, v_k)$ — it should look something like

$$P(v_1, \ldots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k : \quad ? \leq \lambda_1 \leq ?, \ldots, ? \leq \lambda_k \leq ?\}$$

(ii) the region $P(v_1, \ldots, v_k) \setminus P(v_1, \ldots, v_i + \lambda v_j, \ldots, v_k)$

(iii) the region $P(v_1, \ldots, v_i + \lambda v_j, \ldots, v_k) \setminus P(v_1, \ldots, v_k)$.

(b) Show that the regions in (ii) and (iii) are congruent (Hint: they are related by a translation) and deduce that $|P(v_1, \ldots, v_k)| = |P(v_1, \ldots, v_i + \lambda v_j, \ldots, v_k)|$.

2. Let $\omega_1 = x_1 dx_2 \wedge dx_3 - e^{x_2} dx_1 \wedge dx_2$ and $\omega_2 = dx_1$, and let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\phi(u_1, u_2) = (u_1, u_2^3, u_1 u_2 + u_2^5)$. Calculate

(i) $\omega_1((2, 1, 3), (1, 3, -3))$

(ii) $\omega_1 \wedge \omega_2$ and $\omega_1 \wedge \omega_2\big((1, 0, 1), (2, 3, 1), (0, 1, 2)\big)$

(iii) $d\omega_1$

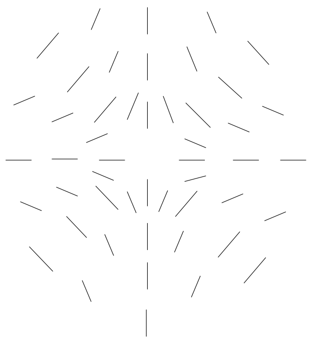
(iv) $\phi^*(\omega_1), \phi^*(\omega_2), \phi^*(d\omega_1)$ and $d\phi^*(\omega_1)$, checking that the last two are equal.

3. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\phi(x_1, x_2) = (x_1, x_1 x_2 + x_2^3).$$

Calculate $\phi^*(dx_1), \phi^*(dx_2)$ and $\phi^*(dx_1 \wedge dx_2)$. Wkat is the geometrical significance of the coefficient of $dx_1 \wedge dx_2$ in $\phi^*(dx_1 \wedge dx_2)$?

4. Let $\omega \in \Omega^1(\mathbb{R}^2)$. For each $x \in \mathbb{R}^2$, ω determines a line $L_x \subset T_x\mathbb{R}^2$, namely the line $\{v \in T_x\mathbb{R}^2 : \omega(v) = 0\}$. The assignment of a line in $T_x\mathbb{R}^2$ for each x is called a *line field.* A line field can be represented graphically by drawing a small piece of the affine translate $x + L_x$ of each line L_x, passing through x. For example, the following picture shows the line field for $\omega = x_2 dx_1 - x_1 dx_2$.

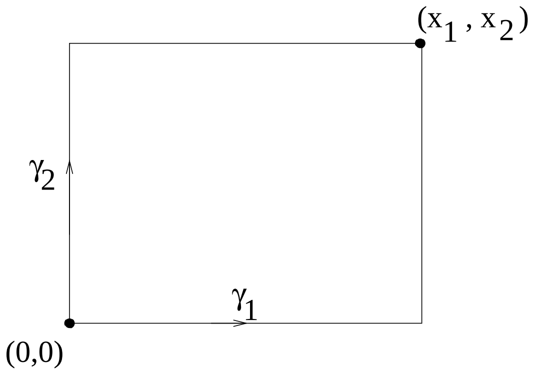


Sketch the line fields determined by the 1-forms (i) $x_1 dx_1 + x_2 dx_2$; (ii) $x_1 dx_2 + x_2 dx_1$ (iii) $dx_2 - x_2 dx_1$.

5. If the 1-form ω on $\mathbb{R}^2$ is equal to df for some smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, then $d\omega = 0$. This exercise shows that the converse is true also. Suppose that $d\omega = 0$.
(i) Let

$$\gamma_1(t) = \begin{cases} (t, 0) & \text{if } 0 \le t \le x_1 \\ (x_1, t - x_1) & \text{if } x_1 \le t \le x_1 + x_2 \end{cases} \qquad \gamma_2(t) = \begin{cases} (0, t) & \text{if } 0 \le t \le x_2 \\ (t - x_2, x_2) & \text{if } x_2 \le t \le x_1 + x_2 \end{cases}$$

Let $f_i(x_1, x_2) = \int_0^{x_1+x_2} \omega(\gamma_i'(t))dt$ for $i = 1, 2$. Show that $f_1 = f_2$. Hint: use Green's Theorem together with the fact that $d\omega = 0$.



(ii) Show that $df_1 = \omega$. Hint: to compute $\partial f_1/\partial x_1$, use the path γ_2, and to compute $\partial f_1/\partial x_2$ use the path γ_1.

6.(i) Show that if $\omega = a dx_1 + b dx_2 \in \Omega^1(\mathbb{R}^2)$ and $a(x) \neq 0$, then in some neighbourhood of x there exists a smooth function of x_1 alone[1], $f(x_1)$, such that $f\omega$ is the derivative of a smooth function $g(x_1, x_2)$. Hint: use the previous exercise and the theorem on the existence of solutions to ODE's.

(ii) Is anything similar true if $\omega \in \Omega^1(\mathbb{R}^3)$?

7. (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ have 0 as regular value, and let $X = f^{-1}(0)$. Show that if $v \in T_xX$ then

$$\frac{dx_1(v)}{\partial f/\partial x_2} = -\frac{dx_2(v)}{\partial f/\partial x_1}$$

provided the two derivatives in the denominator do not vanish at x.

(ii) Suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ has 0 as regular value, and let $X = f^{-1}(0)$. Show that if $v_1, \ldots, v_n \in T_xX$, then the value of

$$(-1)^{j-1} \frac{dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1}(v_1, \ldots, v_n)}{\partial f/\partial x_j}$$

is independent of j (provided the derivative in the denominator does not vanish at x).

8. Calculate $\int_{S^1} d\theta$.

9. Suppose that $A : (\mathbb{R}^m)^k \to \mathbb{R}$ is k-linear and alternating. Show that A shares the following properties with det:

(i) If $v_i = v_j$ for some $i \neq j$ then $A(v_1, \ldots, v_k) = 0$. Ditto if $v_i = \lambda v_j$.

[1] known as an *integrating factor*

(ii) $A(v_1, \ldots, v_i + \lambda v_j, \ldots, v_k) = A(v_1, \ldots, v_i, \ldots, v_k)$. (Hint: use linearity in the i-th argument to write the left hand side as a sum of two terms.)

(iii) For any permutation σ of k objects, $A(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sign}(\sigma) A(v_1, \ldots, v_k)$ (Hint: write σ as the composite of a sequence of transpositions.)

10. Suppose that $A : (\mathbb{R}^m)^m \to \mathbb{R}$ is m-linear and alternating. Show that A is a scalar multiple of det. Hint: Use the strategy of the proof of Theorem 7.8: the properties listed in the previous exercise enable you to reduce any m-tuple of linearly independent vectors $v_1, \ldots, v_m$ of $\mathbb{R}^m$ to an m-tuple of vectors lying along the coordinate axes, without altering the value of $A(v_1, \ldots, v_m)$.

11.(i) Suppose that $\omega_1, \ldots, \omega_k$ are 1-forms, and $v_1, \ldots, v_k$ are vectors. Show that

$$\omega_1 \wedge (\omega_2 \wedge (\cdots \wedge \omega_k) \cdots)\,(v_1, \ldots, v_k) = \begin{vmatrix} \omega_1(v_1) & \cdots & \omega_1(v_k) \\ \cdots & \cdots & \cdots \\ \omega_k(v_1) & \cdots & \omega_k(v_k) \end{vmatrix} \tag{1}$$

Hint: induction on k. Apply the definition of the wedge product to the product of the 1-form ω_1 and the $(k-1)$-form $\omega_2 \wedge (\cdots \wedge \omega_k)$. Compare with the expansion of a $k \times k$ determinant by its first row.

(ii) Conclude that the parentheses on the left hand side of (1) are unnecessary.