MA455 Manifolds Exercises V Spring 2008

1.(a) Show that the function

$$f_1(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is smooth (the question is really to show that it is smooth at 0) - and has all its derivatives at 0 equal to zero. (Unless it comes easy, don't spend too long working out a formal proof. Perhaps this question should read "convince yourself that" instead of "show that".)

(b) Show that the function

$$f_2(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is smooth (again, the question is really to show that it is smooth at 0).

(c) Draw graphs of the smooth functions

(i) $f_3(x) = f_2(1 - x^2)$;

(ii) $f_4(x) = \frac{1}{I} \int_{-\infty}^{x} f_3(t)dt$, where $I = \int_{-\infty}^{\infty} f_3(t)dt$;

(iii) $f_5(x) = f_4(a - x^2)$, where $a > 1$, showing what happens as a increases.

(d) Construct a smooth function $\rho : \mathbb{R} \to [0, 1]$ taking the value 1 in the interval $[a, b]$ and the value 0 outside the interval $(a - \delta, b + \delta)$.

2. Suppose that M^m is a smooth orientable submanifold of $\mathbb{R}^{m+1}$. Show that it is possible to construct a smooth unit normal vector field on all of M. Hint: in a neighbourhood of each point there are two smooth unit normal vector fields. Use an orientation to choose one consistently on all of M.

3. Let ℓ_1 be the x_1-axis in $\mathbb{R}^{n+1}$. Define a map $f : \mathrm{Gl}(n + 1, \mathbb{R}) \to \mathbb{RP}^n$ by $f(A) = A \cdot \ell_1$. Show that f is smooth.

4.(i) Suppose that $v_1, \ldots, v_k$ are smooth maps $\mathbb{R}^\ell \to \mathbb{R}^n$, and their values $v_1(x), \ldots, v_k(x)$ are linearly independent for every $x \in \mathbb{R}^\ell$. Together they define a map $V : \mathbb{R}^\ell \to G_{k,n}$, sending x to $\mathrm{Sp}_{\mathbb{R}}\{v_1(x), \ldots, v_k(x)\}$.
(i) Let W be an $(n - k)$-plane whose intersection with $V(x_0)$ is 0, for some $x_0 \in \mathbb{R}^n$. Show that there is an open neighbourhood A of x_0 in $\mathbb{R}^\ell$ such that for all $x \in A$, $V(x) \in U_W$.
(ii) Compute the composite $\phi_W \circ V$. Hint: choose the basis E of W and its extension E^e to be orthonormal. This makes it easy to write down the expression of any vector with repect to these bases.
(ii) Show that V is a smooth map.

5. Let M^k be a smooth submanifold of $\mathbb{R}^n$. For each $x \in m$, T_xM is a k-dimensional vector subspace of $\mathbb{R}^n$, and therefore can be viewed as a member of $G_{k,n}$. Show that the map $x \mapsto T_xM \in G_{k,n}$ is smooth.

6. Fix a k-plane $P_0 \in G_{k,n}$. Define a map $\mathrm{SO}(n, \mathbb{R}) \to G_{k,n}$ by $f(A) = AP_0$. Show that f is smooth. Write down a description of $G_{k,n}$ as a quotient of $\mathrm{SO}(n, \mathbb{R})$ by a subgroup, and deduce that $G_{k,n}$ is compact.