THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2007

MANIFOLDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

In your answers you may use any theorem (or proposition or lemma) proved in the course, provided you state it clearly.

---

1. (i) Define the *tangent bundle* of a manifold M, and show that it is a smooth manifold of dimension $2 \dim M$. **[5]**

(ii) What is a *submersion*? Show that the natural projection $TN \to N$ is a submersion. **[5]**

(iii) Let $M^k \subset N^\ell \subset \mathbb{R}^n$. Define the *normal bundles* $\nu(M, N)$ and $\nu(M, \mathbb{R}^n)$. What is the dimension of $\nu(M, \mathbb{R}^n)$? **[3]**

(iv) What does it mean for submanifolds P and Q of a manifold R to be *transverse*? Show that $\nu(M, \mathbb{R}^n)$ is transverse to TN in $N \times \mathbb{R}^n$. Hint: (ii) will be useful. **[8]**

(v) Prove that $\nu(M, N)$ is a smooth manifold, carefully stating any theorem you make use of. What is the dimension of $\nu(M, N)$? **[4]**

---

2. (i) Let $M \subset \mathbb{R}^N$ be a smooth manifold, with regular defining equations $g : U \to \mathbb{R}^c$ on some open set U. Suppose that $F : U \to \mathbb{R}^p$ is a smooth map, and let $f : M \to \mathbb{R}^p$ be the restriction of F to M. Show that f is an immersion at $x \in M$ if and only if the map $(F, g) : U \to \mathbb{R}^p \times \mathbb{R}^c$ is an immersion at x. **[3]**

(ii) Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Y be the image of the map $f : S^2 \to \mathbb{R}^6$ defined by

$$f(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2, x_2x_3, x_1x_3, x_1x_2).$$

(a) Show that Y lies in a 5-dimensional affine subspace of $\mathbb{R}^6$. **[3]**

(b) Show that it is possible to choose $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$ so that the map $h : S^2 \to \mathbb{R}^5$ defined by

$$h(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2, a_1x_2x_3 + a_2x_1x_3 + a_3x_1x_2, b_1x_2x_3 + b_2x_1x_3 + b_3x_1x_2)$$

is an immersion. **[8]**

(c) Show that in this case $h(S^2)$ is a manifold, homeomorphic to $S^2/\sim$, where $\sim$ is the equivalence relation identifying antipodal points. You may assume without proof that h identifies only antipodal points (this is true, but please don't prove it here). **[7]**

(d) Continuing with the assumption of (c), find a smooth manifold in $\mathbb{R}^4$ homeomorphic to $S^2/\sim$. **[4]**

---

3. (i) Let $f : M^k \to N^n$ be a smooth map of smooth oriented manifolds, with M compact, and let Z^{n-k} be a closed oriented submanifold of N. Explain how to define the oriented intersection number $(f \cdot Z)_N$. **[4]**

(ii) How can oriented intersection numbers be used to define the Euler characteristic of a compact oriented manifold M? **[3]**

(iii) Show that if dim M is even then $\chi(M)$ is independent of the orientation of M. **[4]**

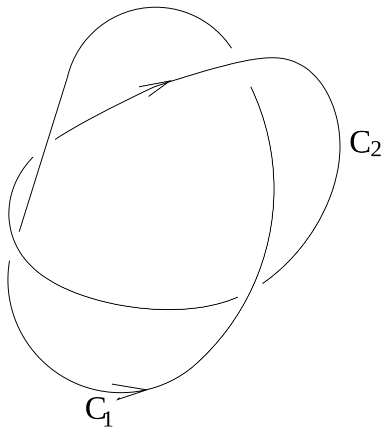
(iv) What if dim M is odd? Does $\chi(M)$ depend on the orientation of M? **[2]**

(v) Let $\Delta \subset S^2 \times S^2$ be the diagonal $\Delta = \{(x, y) \in S^2 \times S^2 : x = y\}$, and let $r_\theta : S^2 \to S^2$ be rotation about the z-axis through an angle of θ. Using r_θ we define a map $i_\theta : S^2 \to S^2 \times S^2$ by $i_\theta(x) = (x, r_\theta(x))$. Calculate the intersection number $(i_{\pi/2} \cdot \Delta)_{S^2 \times S^2}$ from first principles. **[9]**

(vi) What is $\chi(S^2)$? Justify your answer. **[3]**

4. (i) Explain how to define the linking number of two disjoint closed curves in $\mathbb{R}^3$. **[4]**

(ii) Find the linking number of C_1 and C_2 in the following picture, taking care to explain your procedure.



[6]

(iii) For which values of k and ℓ can you define a linking number for disjoint compact oriented manifolds M_1^k, M_2^ℓ in $\mathbb{R}^n$? Explain. **[6]**

(iv) Give an example of a circle and a 2-sphere in $\mathbb{R}^4$ with non-zero linking number, using your definition of linking number to prove that this is the case. **[9]**

5. The method of *Lagrange Multipliers* is used to find the maxima and minima of functions on submanifolds of $\mathbb{R}^n$. It says the following:

Let U be open in $\mathbb{R}^n$, suppose that the smooth map $g : U \to \mathbb{R}^k$ has 0 as a regular value, and let $M = \{x \in U : g(x) = 0\}$. Let $F : U \to \mathbb{R}$ be a smooth function and let $f : M \to \mathbb{R}$ be the restriction of F to M. Then the critical points of f are located at points $x \in M$ where the gradient vector $\nabla F(x)$ is a linear combination of the gradient vectors $\nabla g_1(x), \ldots, \nabla g_k(x)$ of the component functions $g_1, \ldots, g_k$ of g.

To do:

(i) Explain why this works.

[5]

(ii) Let M be the ellipsoid with equation

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 0$$

(where $a, b, c > 0$) and let P be a point of M.

(a) Find the equation of the tangent space T_PM, and the equation of the affine translate of T_PM passing through P. **[5]**

(b) Denote this translate by A_PM. Suppose that P is in the octant where $x_1, x_2, x_3 > 0$. Let R be the solid bounded by A_PM and the three coordinate planes. Show that

$$\mathrm{vol}(R) = \frac{a^2b^2c^2}{6P_1P_2P_3}.$$

Hint: use the fact that R is a pyramid, with volume $\frac{1}{3}$ base $\times$ height. **[8]**

(c) What is the minimum value of $\mathrm{Vol}(R)$ for P in the octant where $x_1, x_2, x_3 > 0$?

[7]