THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2006

MANIFOLDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

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1. a) (i) Let M be a manifold. What is *tangent bundle* TM of M? **[2]**

(ii) Let $\phi : U \to V \subset \mathbb{R}^k$ be a chart on M. Show that there is a diffeomorphism $\Psi : V \times \mathbb{R}^k \to p^{-1}(U)$ such that $p \circ \Psi = \phi^{-1}$. Go on to show that TM is a smooth manifold, and that the projection $p : TM \to M$ is a submersion. **[5]**

b) (i) Denote by $\text{Mat}_{3\times 2}(\mathbb{R})$ the space of all 3×2 real matrices, and by $\sum^1$ its subset consisting of all 3×2 matrices of rank 1. Show that in the open set of matrices

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}$$

with $(x_{11}, x_{12}) \neq (0, 0)$, $\sum^1$ has, as regular equations, the two 2×2 minor determinants

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}$$ **[9]**

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x_1, x_2) = (x_1, x_2^2, x_1x_2)$, and let $jf : \mathbb{R}^2 \to \text{Mat}_{3\times 2}(\mathbb{R})$ be the map sending (x_1, x_2) to the Jacobian matrix of f at (x_1, x_2). Show that jf is transverse to $\sum^1$. **[9]**

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2. a) State the main theorem of oriented intersection theory, and use it to prove that if W is a compact oriented manifold with boundary ∂W, then there is no retraction $W \to \partial W$. **[7]**

b) State Brouwer's Fixed Point Theorem for continuous maps from the unit ball to itself. **[2]**

c) Prove Brouwer's Fixed Point Theorem for smooth maps. **[5]**

d) Show that if D is homeomorphic to the closed unit ball in Euclidean space $\mathbb{R}^n$ and $f : D \to D$ is continuous then f must have a fixed point. **[3]**

e) Let A be an $n \times n$ matrix with real entries, all of which are strictly positive. Prove that A must have a strictly positive eigenvalue with associated eigenvector in the positive orthant of $\mathbb{R}^n$. **[8]**

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3. (i) Show that if $f : M \to N$ is a smooth map of smooth manifolds, Z is a submanifold of N, and $f(x) = z \in Z$, then f is transverse to Z at x if and only if

$$\dim\,(d_x f)^{-1}(T_z Z) = \dim M - \dim N + \dim Z.$$

[4]

(ii) Let $F : M \times P \to N$ be a smooth map of smooth manifolds, suppose that Z is a smooth submanifold of N, and suppose that $F \pitchfork Z$, with $F^{-1}(Z) \neq \emptyset$. Let $\pi : F^{-1}(Z) \to P$ be the restriction of the usual projection $M \times P \to P$. For each $y \in P$, let $f_y : M \to N$ be the map $f_y(x) = F(x, y)$.

a) Show that if $F(x, y) = z \in Z$ then

$$\ker d_{(x,y)}\pi = (d_x f_y)^{-1}(T_z Z).$$

[5]

b) Prove that

$$f_y \pitchfork Z \quad \Leftrightarrow \quad y \text{ is a regular value of } \pi.$$

[5]

c) Show that if $F \pitchfork Z$, then for almost all $y \in Y$, $f_y \pitchfork Z$. **[4]**

(iii) Stating clearly each theorem that you use, prove that if $f_0 : M \to N$ is a smooth map of smooth manifolds with $N \subset \mathbb{R}^N$ and compact, and Z is a smooth submanifold of N, then there exists a smooth map $f_1 : M \to N$, homotopic to f_0, such that $f_1 \pitchfork Z$. **[7]**

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4. a) What is an *orientation* of a finite dimensional vector space V? **[2]**

b) Suppose that $g : \mathbb{R}^n \to \mathbb{R}^c$ is a smooth map with 0 as a regular value, and that $g^{-1}(0) \neq \emptyset$. Show that $g^{-1}(0)$ is orientable, taking care to show that the orientation you define satisfies the local compatibility condition. **[5]**

c) Suppose that $f : M^m \to N^m$ is a smooth map, with M compact, and M and N oriented. For a fixed $y \in N$, let $i_y : M \to M \times N$ be the map $i_y(x) = (x, y)$. What is the precise relation between $\deg(f)$ and $(i_y \cdot \text{graph}(f))_{M\times N}$? Justify your answer. **[7]**

d) What is the *Euler characteristic* of a smooth compact oriented manifold? **[3]**

e) Suppose that M and N are both compact oriented manifolds. What is the relation between $\chi(M), \chi(N)$ and $\chi(M \times N)$? Justify your answer. `Hint:` Combine a homotopy of the diagonal embedding of M in $M \times M$ and a homotopy of the diagonal embedding of N in $N \times N$ to get a homotopy of the diagonal embedding of $M \times N$ in $(M \times N) \times (M \times N)$. **[8]**

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5. a) Write down the equation of a vector field v on $\mathbb{R}^3 \setminus \{0\}$ such that

- v is radial, and radially symmetric about 0.
- the flux of v outward across any sphere centred at 0 is equal to 1 (and in particular is independent of the radius).

[5]

b) Find a 2-form ω such that for every compact surface S, $\int_S \omega$ is equal to the flux of v across S. **[5]**

c) Show that ω is closed. **[5]**

d) State Stokes's Theorem. **[3]**

e) Suppose that ω is a closed 2-form on $\mathbb{R}^3 \setminus \{0\}$ such that for every closed 3-ball B centred at 0, $\int_{\partial B} \omega = 1$. Show that if $B \subset \mathbb{R}^3$ is a closed 3-ball with boundary $\partial B \subset \mathbb{R}^3 \setminus \{0\}$ then

$$\int_{\partial B} \omega = \begin{cases} 1 & \text{if } 0 \in B \\ 0 & \text{otherwise.} \end{cases}$$

[7]
