THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2005

MANIFOLDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

In your answers you may use any theorem (or proposition or lemma) proved in the course, provided you state it clearly.

---

1. (i)

a) Let $f : M \to N$ be a smooth map, and let Z be a submanifold of N. Define what it means for f to be *transverse* to Z in N.

b) Here is a slight extension of the definition of transversality: given maps $f : M \to N$ and $g : Z \to N$, we say that $f \pitchfork g$ ("f is transverse to g") if for all points $x \in M, z \in Z$ such that $f(x) = g(z)$,

$$d_x f(T_x M) + d_z g(T_z Z) = T_{f(x)} N.$$

Show that $f \pitchfork g$ if and only if $f \times g : M \times Z \to N \times N$ is transverse to the diagonal $\Delta = \{(y_1, y_2) \in N \times N : y_1 = y_2\}$.

c) Let $W := \{(x, z) \in M \times Z : f(x) = g(z)\}$. What can you say about W, and about $T_{(x,z)}W$, if $f \pitchfork g$?

d) Assume that $f \pitchfork g$. Let π_M be the projection $\pi : W \to M$. What is the kernel of

$$d_{(x,z)}\pi_M : T_{(x,z)}W \to T_x M?$$

Show that π_M is an immersion at (x, z) if and only if g is an immersion at z, and a submersion at (x, z) if and only if g is a submersion at z.

(ii)

a) State Sard's theorem.

b) Let $M \subset \mathbb{R}^n$ be a manifold, and assume $0 \notin M$. Show that for almost every hyperplane $H \subset \mathbb{R}^n$, $M \pitchfork H$.
Hint: Consider the map $F : M \times S^{n-1} \to \mathbb{R}$, $F(u, x) = u \cdot x$, and show that 0 is a regular value.

---

2. a) What is a *smooth atlas* on a manifold M? **[3]**

b) Let $\mathrm{O}(n+1)$ be the group of orthogonal $(n+1) \times (n+1)$ real matrices, and let $\ell = [c_1, \ldots, c_{n+1}]$ be a point in $\mathbb{RP}^n$. Show that the map $f : \mathrm{O}(n+1) \to \mathbb{RP}^n$ given by $A \mapsto A(\ell)$ is smooth.

c) Construct an explicit embedding of $\mathbb{RP}^n$ into $\mathbb{R}^N$ for some N. Explain why your map is en embedding (that is, why its image is a manifold diffeomorphic to $\mathbb{RP}^n$).
Hint: use the fact that $\mathbb{RP}^n$ is homeomorphic to $S^n/\sim$, where $b \sim b$ if $a = \pm b$.

d) Show that the image Y of your embedding is not orientable if n is even. Hint: Consider the commutative triangle

$$\begin{tikzcd} T_x S^n \arrow[rr, "d_x a"] \arrow[dr] & & T_{-x} S^n \arrow[dl] \\ & T_{f(x)} Y & \end{tikzcd}$$

where a is the antipodal map.

e) Show that if n is even, it is *not* possible to find global regular equations for the image Y of $\mathbb{RP}^n$. That is, there is no collection of $N - n$ smooth functions $g_1, \ldots, g_{N-n}$ defined on an open set in $\mathbb{R}^N$ containing Y, such that
(i) $Y = \{x \in \mathbb{R}^N : g_1(x) = \cdots = g_{N-n}(x) = 0\}$ and
(ii) 0 is a regular value of the map $x \mapsto (g_1(x), \ldots, g_{N-n}(x))$.

---

3. a) Define the *degree* $\deg(f)$ of a smooth map f between oriented compact manifolds of the same dimension. **[5]**

b) Show that if $M \xrightarrow{f} N \xrightarrow{g} P$ are maps between oriented manifolds of the same dimension, then $\deg(g \circ f) = \deg(g)\deg(f)$.

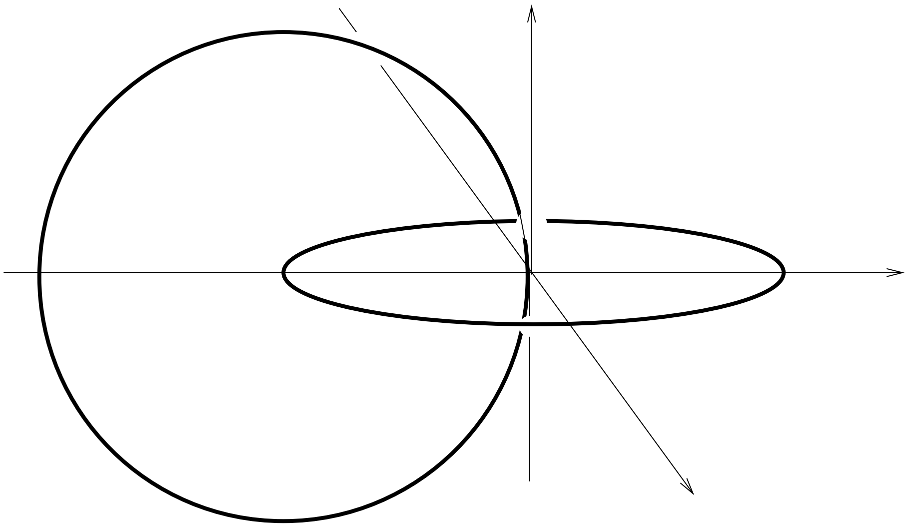
c) State the main theorem of oriented intersection theory (it concerns a map from a compact oriented manifold with boundary to another manifold).

d) Use it to show that if W^{m+1} is a compact oriented manifold with boundary, and N^m is an oriented manifold, then deg$(\partial F) : \partial W \to M$ is zero.

e) What is the definition of the *linking number* $\ell(C_1, C_2)$ of disjoint oriented closed curves C_1 and C_2 in $\mathbb{R}^3$? Show that if C_1 and C_2 are disjoint oriented closed curves in $\mathbb{R}^3$ and there exists a compact oriented 2-manifold with boundary $S \subset \mathbb{R}^3$ such that
(i) $\partial S = C_1$ and
(ii) $S \cap C_2 = \emptyset$
then $\ell(C_1, C_2) = 0$.

f) For which pairs of submanifolds M, N of $\mathbb{R}^5$ can one define a linking number $\ell(M, N)$? Explain.

g) Give an example of a pair of submanifolds $M, N \subset \mathbb{R}^5$ such that $\ell(M, N) \neq 0$. You do not need to give a rigorous calculation of the linking number. Hint: raise the dimensions in the diagram



---

4. a) What is an *orientation* of finite dimensional vector space V? What is an *orientation* of a manifold? **[4]**

b) Suppose that $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a short exact sequence of oriented finite dimensional vector spaces. What does it mean for the sequence to be *positive*? Explain how to use orientations of two of the spaces in a short exact sequence to orient the third (you do not need to *show* that your procedure is independent of choices, though it should be). **[5]**

c) Given that each copy of $\mathbb{R}^n$ has the standard orientation (in which $(1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ is a positive basis), determine the sign of the short exact sequence

$$0 \to \mathbb{R} \xrightarrow{i} \mathbb{R}^3 \xrightarrow{j} \mathbb{R}^2 \to 0,$$

where the matrices of i and j with respect to the standard bases are

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & -1 & 0 \\ 0 & -3 & 2 \end{pmatrix}.$$

d) Explain how to orient:
(i) the non-empty preimage of a regular value of a smooth map $f : M \to N$, where both M and N are oriented; **[4]**
(ii) the non-empty transverse preimage $\mathrm{f}^{-1}(Z)$ of a submanifold $Z \subset N$, where M, N and Z are oriented. **[4]**

e) Define the *Euler characteristic* $\chi(M)$ of a compact oriented manifold M.

f) Sketch a proof that if $\chi(M) \neq 0$ then every smooth tangent vector field on M must have a zero.

g) Show that $\chi(S^{2m-1}) = 0$. **[4]**

5. a) Let G be a Lie group and let e be its neutral element. Find the derivative at (e, e) of the multiplication map $p : G \times G \to G$ $p(g_1, g_2) = g_1 g_2$, and the derivative at $e \in G$ of the inversion map $g \mapsto g^{-1}$.

b) Find the tangent space at the identity matrix I_n to the group $\mathrm{O}(n)$ of $n \times n$ orthogonal matrices. You may asume that an auxiliary map you use in your calculation is a submersion.

c) Find $T_A\mathrm{O}(n)$ for a general $A \in O(n)$.

d) Let $\mathrm{Sym}_n(\mathbb{R})$ be the vector space of $n \times n$ symmetric matrices with real entries, and let

$$S_n^r = \{A \in \mathrm{Sym}_n(\mathbb{R}) : \text{rank}\, A = r\}.$$

Show that S_n^r is a submanifold of $\mathrm{Sym}_n(\mathbb{R})$. What is its codimension?
Hint: first deal with the intersection of S_n^r with the open set of matrices of the form

$$\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

where A is of size $r \times r$ and has rank r. Find invertible matrices E, F such that

$$E \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} F = \begin{pmatrix} A & 0 \\ 0 & C' \end{pmatrix}$$

for some C'. Use the expressions you obtain to define a smooth map $f : U \to \mathrm{Sym}_{n-k}(\mathbb{R})$ such that $S_n^r \cap U = f^{-1}(0)$.