THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: APRIL 2004

MANIFOLDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

In your answers you may use any theorem (or proposition or lemma) proved in the course, provided you state it clearly.

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1. (a) Let $f : M \to N$ be a smooth map, and let Z be a submanifold of N. Define what it means for f to be *transverse* to Z in N. Show that

$$f \pitchfork Z \quad \Leftrightarrow \quad \text{graph}(f) \pitchfork M \times Z$$

in $M \times N$, where $\text{graph}(f) = \{(x, y) \in M \times N : y = f(x)\}$ is the graph of f. **[6]**

Let $N^n \subset \mathbb{R}^p$ be a smooth manifold.

(b) Define the *tangent bundle* $TN \subset N \times \mathbb{R}^p$ and the *normal bundle* $\nu(N, \mathbb{R}^p) \subset N \times \mathbb{R}^p$. Prove that TN is a manifold of dimension $2n$. **[6]**

(c) Show that the natural projection $\pi : TN \to N$ is a submersion. **[6]**

(d) Show that if $M^m \subset N^n \subset \mathbb{R}^p$ then $\nu(M, \mathbb{R}^p) \cap TN = \nu(M, N)$, and show that this intersection is transverse in $N \times \mathbb{R}^p$. (Hint: (c) can help.) What can you conclude about $\nu(M, N)$? **[7]**

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2. (a) What is a *smooth atlas* on a manifold M? **[3]**

(b) Let $\sim$ be the equivalence relation on the sphere S^m obtained by declaring antipodal points to be equivalent, and let $\mathbb{RP}^n = S^n/\sim$. Construct an atlas on $\mathbb{RP}^n$ as follows. Each hyperplane P though 0 cuts the sphere into two open hemispheres. Let H be one of them. The map $q : S^n \to \mathbb{RP}^n$ is open, so $Q := q(H)$ is an open set in $\mathbb{RP}^n$ (and doesn't depend on which hemisphere we choose). As q is 1-1 on H, $q : H \to q(H)$ is a homeomorphism. Let B be the unit ball in P, centred at 0. Define the chart $\phi_P : Q \to B_H$ as the composite of $q^{-1} : Q \to H$ and $\pi : H \to B$, where π is orthogonal projection.

(i) Make a drawing or drawings for the case $n = 1$ (i.e. where S^n is the unit circle) showing

- two hyperplanes P_1 and P_2 (which you should choose);
- two hemispheres (= semicircles) H_1 and H_2,
- $\phi_{P_1}(Q_1 \cap Q_2)$ and $\phi_{P_2}(Q_1 \cap Q_2)$,

and showing also the crossover map

$$\phi_{P_2} \circ \phi_{P_1}^{-1} : \phi_{P_1}(Q_1 \cap Q_2) \to \phi_{P_2}(Q_1 \cap Q_2)$$

as a sequence of projections. **[5]**

(ii) Show that the atlas on $\mathbb{RP}^n$ just constructed is smooth. **[5]**

(c) An atlas $\{\phi_a : U_\alpha \to V_\alpha\}$ gives a manifold M a *Euclidean structure* if all of the crossover maps

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are Euclidean isometries.

(i) Show that the m-torus $S^1 \times S^1 \times \cdots \times S^1$ has a Euclidean structure. **[5]**

(ii) What is a *straight line* on the m-torus? Explain. **[4]**

(d) Does the atlas in (b) allow us to speak of straight lines in $\mathbb{RP}^n$? Justify your answer, possibly by means of a drawing. **[3]**

3. (a) Define the *degree* of a smooth map between oriented compact manifolds of the same dimension, explaining why it is well-defined. **[5]**

(b) State and prove Brouwer's fixed point theorem. **[6]**

(c) Define the *linking number* of disjoint oriented closed curves in $\mathbb{R}^3$. Calculate the linking number of the following pair of curves

taking care to explain your procedure. **[6]**

(d) In $\mathbb{R}^5$, consider the two 2-spheres

$$S_1 = \{(x_1, \ldots, x_5) : x_4 = x_5 = 0, \;\; x_1^2 + x_2^2 + (x_3 + \frac{1}{2})^2 = 1\}$$

and

$$S_2 = \{(x_1, \ldots, x_5) : x_1 = x_2 = 0, \;\; (x_3 - \frac{1}{2})^2 + x_4^2 + x_5^2 = 1\}$$

(whose centres are at $(0, 0, -1/2, 0, 0)$ and $(0, 0, 1/2, 0, 0)$ respectively). Show that it is not possible to move S_1 and S_2 in a homotopy, in such a way that they remain at all times disjoint, to positions on opposite sides of a hyperplane. **[8]**

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4. (a) Let ω be the 2-form $\cos x dy \wedge dz - \sin\, z dx \wedge dy$.

(i) Calculate the exterior derivative $d\omega$. **[2]**

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $f(u, v) = (u, v^2, uv)$. Find $f^*(\omega)$. **[2]**

(b) Let $\omega = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ be a nowhere-vanishing 1-form on $\mathbb{R}^3$.

- Show that if ω_1 is a 1-form such that $\omega \wedge \omega_1$ is identically zero then there is a smooth function c on $\mathbb{R}^3$ such that $\omega_1 = c\omega$. **[2]**

- Show that if ω_2 is a 2-form such that $\omega \wedge \omega_2$ is identically zero, then in some neighbourhood U of each point p there exists a 1-form ω_U such that $\omega_2 = \omega \wedge \omega_U$. **[4]**

(c) Let M^m be a manifold. Explain how to use a nowhere vanishing m-form to orient M. Prove that if M is orientable then there exists a nowhere vanishing m-form. **[6]**

(d) State Stokes's Theorem. Let ω be an r-form on $\mathbb{R}^n$ such that

$$\int_Z \omega$$

for every oriented submanifold $Z \subset \mathbb{R}^n$ which is diffeomorphic to the r-sphere S^r. Show that $d\omega = 0$. **[3], [6]**

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5. (a) What is an *orientation* of finite dimensional vector space V?
What is an *orientation* of a manifold? **[4]**

(b) Suppose that $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a short exact sequence of oriented finite dimensional vector spaces. What does it mean for the sequence to be *positive*? Explain how to use orientations of two of the spaces in a short exact sequence to orient the third (you do not need to *show* that your procedure is independent of choices, though it should be). **[5]**

Explain how to orient:

(i) the non-empty preimage of a regular value of a smooth map $f : M \to N$, where both M and N are oriented; **[4]**

(ii) the non-empty transverse intersection $M \cap Z$ of submanifolds M and Z of N, where all three are oriented. **[4]**

(c) (i) Show that the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$. **[4]**

(ii) Recall that $\mathbb{RP}^n$ is the quotient of the n-sphere S^n by the equivalence relation identifying antipodal points. Use (c)(i) to show that $\mathbb{RP}^n$ is orientable if and only if n is odd. **[4]**
