Mathematics and Math Education: how can they work together?[1]

Intake My old friend "Disgusted of Birmingham" has some things right. However, the practical question is one of diagnosis and possible 'treatments' that might happen. Because of the power of high-stakes tests over what is actually taught, sufficient explanation for the present state is the decision to base the National Curriculum tests (including GCSE) on a checklist of detailed elements of attainment (content topics) that *must be assessed separately*[2]. This led to the fragmentation of reasonable mathematical tasks into one-step subtasks.

The Mathematics community could change this by saying loudly and clearly at a political level that ***all worthwhile mathematics at every level involves (more or less) extended chains of reasoning***[3].

It is surprising that anyone of intelligence comes out of current 11-16 mathematics with any interest in the subject.

University teaching
Turning to the main subject of the meeting, university teaching, whose teaching do we want to make more effective:

1. Staff actively interested in improving their teaching?
2. The majority of staff?

These present very different problems.

Staff interested in their teaching are fairly well cared for through UMTC (see http://www.umtc.ac.uk/index.html; Bill Cox is on the organizing committee), and the ongoing networks that arise from the meetings. They would benefit from recognition of their design and development work: in terms of load (developing a new course element well takes serious time) and academically, providing it is (empirically) rigorous enough to be called research (in the engineering research tradition, see below[4]).

The majority of staff who, after all, teach the majority of students present a much greater challenge. (The original UMTC "trickle down" theory has, like its economic cousin, not worked) The following factors need to be born in mind:

- **What incentive is there for research-focused staff to devote the substantial time needed to change their courses so as to improve their students' learning?** (Excuse the periphrastics, but teaching is so often seen as a teacher-output issue – even Bill Cox has only "explanation" among his principles for teaching interventions!) Though there is much rhetoric on the importance of good teaching, are the 'teeth' in the promotion criteria no longer dominated by research/publications? (At science promotion meetings, *all* candidates were presented by their departments as teachers of near-Socratic distinction)

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[1] This note sets out to sketch a set of strategic options, with their various 'cost' implications. If wanted, I'd be happy to expand on any aspect. Hugh.Burkhardt@nottingham.ac.uk

[2] This fragmentation came to pass in the initial development of the National Curriculum, despite the best efforts of the Mathematics Working Group. It was an unintended consequence of senior civil servants' naïve view of performance in Mathematics as a bag of tricks at different levels which "pupils can or cannot do". They understood English, their working language, much better – hence those tests are not just spelling, grammar and syntax but involve extended reading and writing in various genres. The equivalent extended reasoning is entirely absent in the Mathematics tests.

[3] Ironically you may think, the current policy for "functional mathematics" provides a window of opportunity – the top people know that it cannot be assessed in short 'items'; however, there is a powerful "key skills" lobby that likes the fragmentary status quo, despite its manifest failure. (You hear the same thing in universities "They need a sound foundation in basics before they can solve problems" – though kids solve problems with their (only) counting skill when they come to school. The two need to be developed in parallel, each reinforcing the other – the one adding to the "toolkit", the other developing the essential multiple connections to existing knowledge, and to concrete embodiments of it)

[4] There are journals for reporting such work – *International Journal of Mathematical Education in Science and Technology*, to name but one.

Alternatively,

- **Would a department control the detailed content and pedagogy of its main courses,** so that selected well-engineered courses (see box opposite), developed by its own teaching-oriented staff or others elsewhere, are delivered byother assigned staff, after some specific training in handling the less-familiar learning activities involved?
- **Would a department consider changing the balance of course content so as to optimize the progress of (all) their students across a (changing) range of performance goals** that, partly at least, reflect their future needs? (This was the agenda of Adams, Atiyah, Griffiths et al at UMTC 1975 and the question remains; I mention some specifics in the Appendix to this note)

What is a well-engineered course?

University mathematicians (and indeed many in Education) are not much involved with research-based development of teaching so it may be worth sketching what needs to be involved if the products, and processes for their use, that are produced meet the standards of 'fitness for purpose' (to coin a phrase) expected of a well-engineered product in other fields. The Table opposite summarises the development process. There are basically three phases:

A. Drafting, trialling by the author, revision: This needs a knowledge of the field, the relevant research on teaching and learning, plus design skill (often neglected). The outcome is some teaching that pleases the author, working for him(her). This is as far as many innovations get. Some study of the student learning outcomes, absolutely and/or in comparison with 'standard' teaching, is needed to establish the gains to students from the change – and to make this 'alpha-version' defendable as research.

B. Alpha trials with a small number of colleagues are the first, and most important, step in developing materials that will work for other teachers. They test whether the materials communicate effectively with other teacher-users so that they can achieve learning activities and outcomes comparable with those the author managed. This phase needs detailed feedback from a handful of classes on what happens and (the distributions of) what the students can do at the end. This feedback often gives rise to substantial revision – and sometimes abandonment of part of the draft – in producing the 'beta-version'.

There are examples of the development of undergraduate course elements to this level, often in association with introducing technology (which justifies the greater cost). David Tall's work on calculus is a good example; he knows much more.

Research-based development: Development is a systematic process, typically:

The design process produces draft materials. The team has some evidence, albeit with atypical teachers (the authors), on the response of students, but none on how well the materials enable other teachers to create a comparable learning experience in their classrooms.

Systematic development turns fine drafts into robust and effective products. It involves successive rounds of trials, with rich and detailed feedback, in increasingly realistic circumstances.

Feedback at each stage guides the revision of the materials by the design team. Feedback can take many forms; the criterion for choosing what information to collect is its usefulness and cost-effectiveness for that purpose. This also depends on presenting information in a form that the designers can readily absorb – too much indigestible information is as useless as too little; equally, it depends on the designers' willingness to learn from feedback, and having the skills to infer appropriate changes from it.

Cost-effectiveness is a key issue, giving different balances of feedback at each stage. In favourable circumstances for, say, the development of teaching materials, these are:

Alpha trials in a handful of classrooms (~ 5-10), some with some robust teachers who can handle anything and others more typical of the target group. This small number is enough to allow observers to distinguish those things that are generic, found in most of the classrooms, from those that are idiosyncratic, limited to a particular teacher. The priority at this stage is rich detailed feedback from each classroom, including:

structured observation reports by a team of observers, covering in detail every lesson of each teacher (expensive but rich feedback)

samples of student work for analysis (inexpensive but limited feedback)

informal-but-structured interviews with teachers and students on their response to the lesson, and on the details of the lesson materials, line by line (expensive but rich feedback).

Feedback to the designers of what has been found is central to development process, but difficult to optimise. Written feedback and informal conversations are limited, and may lack structure. Meetings can be an effective way for the observers to provide feedback to the lead designer. In one model, observers present:

first, an analytic picture of each teacher in the trials, working without and with the new materials

then a step by step discussion of the materials, bringing out what happened in each of the classrooms, noting where the materials did not communicate effectively to teacher or students as well as the actual realisation of the intended activities.

The discussion in these sessions is primarily about clarifying the meaning of the data, but suggestions for revision also flow. The role of the lead designer here is that of an observer, absorbing the information and suggestions, and integrating them into decisions on revision.

Revision by the lead designer then produces the 'beta version'.

Beta trials The priorities are different now, focused on the realisation of the lessons in typical classrooms. A larger sample (20-50) is used. It should be roughly representative of the target groups. It is useful to have stratified, reasonably random samples (e.g. by invitation. "You have been chosen...." has good acceptance rates, particularly when the materials can be related to high-stakes assessment) With given resources, a larger sample means more limited feedback from each classroom, largely confined to written material from samples of students. Observation of the beta version in use in another small group of classrooms is an important complement to this.

Revision by the lead designer again follows, producing the final version for publication. This is not the end of the process.

Feedback 'from the field' will guide future developments. Both informal comments from users and more structured research will produce insights on which to build.

C. Beta trials

These are designed to discover how well the innovation works in the hands of a broader group of more typical teacher-users – and finding unexpected ‘side effects’. This would only be appropriate for courses that were going to be widely used. (The first year calculus/analysis courses, based on Maple, that are widely used by Ontario universities are an examples I know of)

Cost-benefit analysis and collaboration

It will be clear that this development process involves much more effort than Phase A alone. It is necessary if you want to have teaching outcomes that you can be reasonably sure of; however, it is hard to justify for ‘one’s own course’ – except as (engineering or design) research in mathematics education.

These kinds of research have problems of academic recognition in education schools but, with some encouragement from the RAE, progress is being made. Such research fits the talents and inclinations of many Math Ed staff much better than the traditional small neat academic studies which, because they are too small to have evidence of generalizability, do not contribute to design, development or practice in the field.

Thus this note on the rather-forbidding challenges in trying to improve undergraduate teaching can end on an opportunity for collaboration between mathematicians, who have ideas on design for improved course elements, and math educators who (should) have the expertise to handle the empirical development.

As always, it is worth finding out if a market exists for the product, and on what scale, or if one could be created. Given the attitude of lecturers, often combining independence and a wish to spend as little time as possible on preparation, this is a non-trivial challenge. (The most successful example I know was in the polytechnics, where CNAA controlled the balance of course content, and imposed innovations, including incidentally the teaching of modeling skills, on all departments under their aegis)

Appendix: Some thoughts on course content

I offer these tentatively, first confessing to two prejudices (with some empirical support in fact and policy):

1. Judgments of course success should be based on (*the distribution of*) student learning outcomes, not on what the lecturer 'covers'[5].
2. Most students would benefit more in their future lives and work from mathematical power[6] than precision of analysis[7].

These points are mainly relevant to 'core' courses; multiple options are a good thing, provided they can be afforded. (I have had enough arguments with colleagues over the years to know the profound polychotomies in mathematics schools that these issues can generate).

Analysis, and the new language of set theory, is a shock that should be left until the students are settled in to university life. (UMTC 1975, led by distinguished pure mathematicians, suggested it be left to the second year) At school, the mathematics has been conceptual (with luck) and procedural (always) but not formal or probing – physiology without pathology. The shift to more analytic thinking (and a 'foreign language') should be made gradually and with recognition of the symbolization load involved. The multiple shocks (in living, learning, resources management, relationships,...) of arriving at university are not a good time to add this intellectual one.

Cognitive conflict (e.g. the surprising counter-example that destroys an "inappropriate generalization")) is a powerful teaching tactic for learning in depth that could be used much more. Evidence suggests that it is crucial to get students to be brave enough to make mistakes and learn from them.

Non-routine problems, microworlds (and proofs), are at the heart of mathematical thinking. Their essence is *transfer* – the solver has to recognize connections to mathematics that may be useful in the problem, which may come from anything they have previously learnt[8]. (If it is a topic they have just been taught, the *transfer distance* is minimal – it is likely to be an exercise, where memory is almost enough on its own, and to be seen as such by students.

Non-routine problems thus carry a *strategic and tactical load* of deciding what to do; for a given total cognitive load (~difficulty), the technical load of doing it accurately must be lower. Without substantial teaching of problem solving, the "few year gap" between the mathematics that students use autonomously in solving problems and that they can do in the usual imitative exercises may be five years. (We call these false positives; life doesn't present challenges in standard form)

To develop some of these skills, there is great value in investigations of rich but unimportant mathematical 'microworlds', for example:

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[5] Brian Griffiths, a pioneer of improving undergraduate mathematics teaching, tells of the following dialogue with a colleague who was marking final year scripts:

> "It' so depressing. They know nothing?"
> "Have you thought of making the course easier?"
> "Oh no, I couldn't lower standards."

[6] Given the small proportion of graduates that will use their university level Pure Mathematics, should we be happy that physicists, engineers and economists learn more powerful mathematics (in some sense) than Honours Mathematicians?

[7] We theoretical physicists arrogantly say "If you can't be good, be careful". We know that you can find the convergence of a sequence by informal comparison to others – or by whether it blows up, oscillates or not when you compute it. We think Dirac was more useful than Laurent Schwartz. Vive la difference, of course. (I thought Kim soon got the essence of the 'upper bound' of the union of U and V with "the larger of u and v" – because 'least upper bounds' give you power, upper bounds being fairly useless unless they are close. I realize this is a peasant view but not unreasonable for those who regard mathematics primarily as a powerful toolkit for solving other problems.

[8] Professional mathematicians, pure and applied, can use mathematics they have *not* already learnt, but merely heard of. It is rare for teachers to try to develop this skill in students – but not out of the question.

CONSECUTIVE SUMS

9 = 4+5

9 = 2+3+4

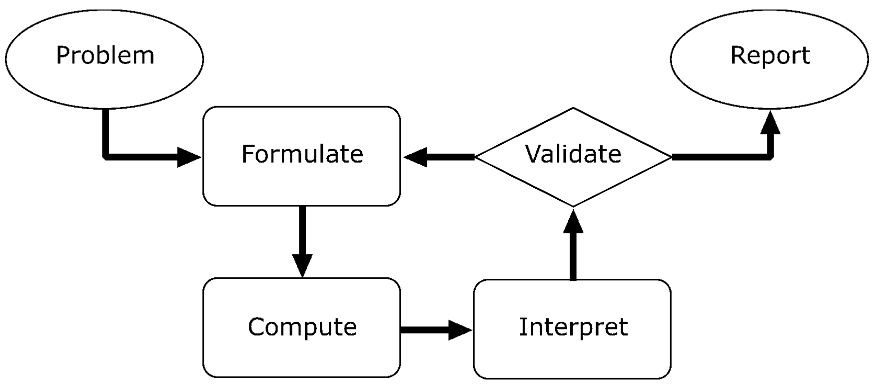
15= 1+2+3+4+5

Find out all you can about sums of consecutive natural numbers, giving justifications for your assertions where you can.

This investigation works well at sixth form level. It has many results and a nice ‘ramp’ of challenge from really easy (*sums of three are divisible by three*) through *Why not for four?* to quite challenging results (*Which numbers can’t be so represented?*, easy to explore and conjecture, and *Why*?, which is much harder.

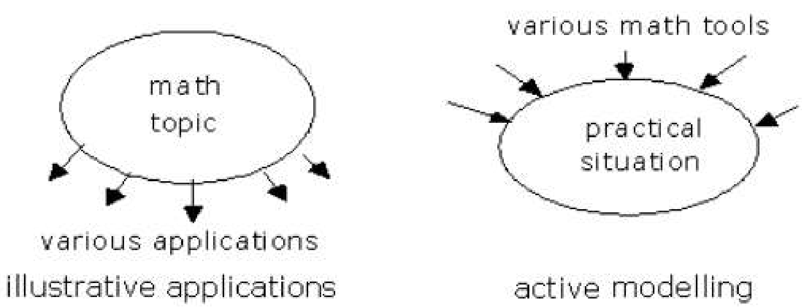
It does, of course, involve long chains of connected reasoning – but in a technically simple context.

Modelling ‘real world’ problems extends the range of strategic and tactical skills to those that link the mathematics to the practical situations, summarized in the diagram.



The phases of modelling

In school (and many university courses) only the compute phase is taken seriously. Taught applications are important but not enough, because of the distinction shown in the second diagram below.



Most curricula offer *illustrative applications*; there the focus is on a specific mathematical topic, showing the various practical domains where it can be useful and practising its use in those contexts. The student has no doubt as to the mathematics to be used – it is the topic just taught. In contrast, in *active modelling* the focus is on the practical situation and understanding it better. Usually, a variety of mathematical tools will be useful for different aspects of the analysis. (This is a good indication as to the real goal.) Choosing and using tools appropriately is a major part of the challenge to the student.

Word problems Many people feel they are "much more difficult than they should be" – after all it is "just" a matter of "interpreting the question". There is good research that shows "interpreting the question" into mathematics is a

comprehension +modeling

task[9]; since both are challenging processes, the difficulty is not surprising. First the student must visualise the situation, constructing a mental model. (This is actually made harder by the minimalist text that is often preferred). Then they have to formulate appropriate mathematics to describe it.

To take an example:

> *Two cyclists leave their homes, 42 Km apart, at 8am. They ride towards each other and meet at 11am. One cycles at 7.5 Km an hour[10]. How fast does the other go?*

What elements are involved?

• Picture the situation(a diagram helps) – 42 Km apart, 'facing each other'

• Recognise that the time *difference* (8am-11am = 3 hours) is the key

Solution route 1 (from conversations Friday):

• A goes at 7.5 Km/hr for 3 hours > 22.5 Km

• So, to meet, B must go 42-22.5 = 19.5 Km in 3 hours

• So B's (average) speed is 19.5/3 = **6.5 Kph**

Solution route 2: Recognize relative variables are all you need (more sophisticated?)

• Relative velocity = 42/3 = 14Kph

• So B's speed is 14-7.5 = **6.5 Kph**

Note every line above with numbers in is a modelling statement – obvious if you see it but, empirically, much more challenging "than it should be", particularly if you are not taught how to choose appropriate models for similar looking examples like the following:

a) Joe buys a six-pack of cola for £3 to share among his friends. How much should he charge for each bottle?
b) If it takes 30 minutes to bake 6 potatoes in the oven, how long will it take to bake one potato?
c) If King Henry 8th had 6 wives, how many wives had King Henry 4th?

In the last 40 years we have learnt how to teach modeling, using a mixture of rich non-routine examples and instruction on strategies and tactics that often pay off.

We have also learnt that a significant component of such work in maths courses transforms the attitude of many students to the subject, motivating them to go further with it.

Most of these modes of thinking are also essential to doing pure mathematics.

Proof Here I would only point to the relativity of rigour. (Peter Hilton once said to me "Rigour is not about proving everything, but avoiding false statements") Alan Bell's research, and others, indicates that few kids under 16 have any concept of a watertight argument. So, as with "interpreting the question", one should take students up the slope of rigour (reflecting history). Kaye Stacey[11] wrote a nice booklet called "Describing, Explaining, Convincing", when she worked with the Shell Centre some years ago, which does this very nicely. It was aimed at

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[9] Some believe that it is just a transcription task; unfortunately, the flexibility of natural languages make that approach highly unreliable (unless, as in some poor US text books, the problem is always presented in a standard linguistic form.

[10] Very slow cycling – surely they were walkers. (Stupid numbers don't help comprehension)

[11] Oxford D.Phil in Pure Mathematics, now Professor of Science and Mathematics Education at the University of Melbourne. A very able person

sixth forms but would benefit many undergraduates.

Status of mathematical statements In this as in so many other areas, students have no clear picture of the *status* of different mathematical statements – perhaps because this is rarely discussed. I used to begin a first year math course (Grade A students) by writing on the board, successively:

	ax^2+bx+c	What's that? (confidently) A quadratic!
then	$y=ax^2+bx+c$	What's that? (puzzled) A quadratic?
then	$ax^2+bx+c=0$	What's that? (no response)

Good luck with your enterprise.

I hope some of this may help.

Hugh