Introduction to Geometry

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1 Introduction

Elementary geometry is the most accessible part of mathematics. The objects can be seen, or at least imagined; they are idealised versions of the objects of the physical universe, and so the theorems of geometry are almost like theorems of physics.

Here are two theorems of Euclidean geometry:

Theorem 1.1 *Let ABC and DEF be triangles. If*

$$AB = DE, \quad BC = EF, \quad \text{and } CA = FD$$

then the two triangles are congruent.

Theorem 1.2 *In contrast, a quadrilateral is not determined up to congruence by the lengths of its sides.*

The two statements are born out by experience: if we make a triangle by pinning together three lengths of cardboard at their ends, then the structure we obtain is rigid: its only motions do not alter the relative positions of its parts to one another. On the other hand, a quadrilateral made in this way is not rigid. So carpenters have to find cleverer ways of joining wood to make rigid quadrilateral frames.

Thanks to our highly developed visual cortex, we can arrange and appreciate large quantities of information in diagrams, and one of the features of this course will be its frequent use of diagrams and of reasoning based on diagrams. I hope to convince you that drawing diagrams is worthwhile, and to provide you with the opportunity to develop your own diagram-drawing skills.

This course is meant also to emphasize that one can do interesting mathematics at a rather concrete level. Most of the time you will have no difficulty understanding the statements of the theorems, even though in some cases the proofs may be quite subtle; the same goes for the exercises.

Objectives of this course:

1. To review plane Euclidean geometry, focussing on proofs.
2. To practice drawing diagrams, and to use them as a means of constructing proofs.
3. To develop intuition and visualisation in 3 dimensions.

2 Theorems on Circles

Consider the following fact: through any three points in the plane there is a circle – which may have infinite radius (i.e. be a straight line), when the three points are collinear. This is easy to believe: consider the set of all circles. Each has equation

$$(x-p)^2+(y-q)^2=r^2.$$

To determine a circle, you have to fix the three numbers, p, q and r. To ask that the circle should pass through each of the three points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is to ask that

$$\begin{array}{rcl}(x_1-p)^2+(y_1-q)^2 &=& r^2\\ (x_2-p)^2+(y_2-q)^2 &=& r^2\\ (x_3-p)^2+(y_3-q)^2 &=& r^2.\end{array} \tag{1}$$

To determine the required circle, we have to find the values of p, q and r from these three equations. Three equations in three unknowns — it's easy to believe that because the number of equations equals the number of unknowns, there is a solution.

This heuristic principle is a good guide to what should happen in "favourable circumstances", but even when the three equations are linear, it's by no means certain that there will be a unique solution. Consider the three equations[1]

$$\begin{array}{rcl}\alpha_1^1 p+a_2^1 q+\alpha_3^1 r &=& A\\ \alpha_1^2 p+a_2^2 q+\alpha_3^2 r &=& B\\ \alpha_1^3 p+a_2^3 q+\alpha_3^3 r &=& C\end{array}$$

where the α_j^i and A, B, C are constants and p, q, r are the unknowns. A unique solution is only guaranteed if

$$\begin{vmatrix}\alpha_1^1 & \alpha_2^1 & \alpha_3^1\\ \alpha_1^1 & \alpha_2^1 & \alpha_3^1\\ \alpha_1^1 & \alpha_2^1 & \alpha_3^1\end{vmatrix} \neq 0.$$

The system of equations (1) we have above is not even linear, but quadratic, and clearly something peculiar must happen when the three points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear, as then r is infinite and p and q are not determined.

So the coordinate geometry approach seems beset with difficulties. Most likely it will run into the ground in a morass of equations in which it is impossible to see what is going on. Even more so in the apparently more difficult problem of finding a circle tangent to three given lines.

In this first section of the course we approach these two problems visually, find ruler-and-compass constructions which give the answers, and turn these back into coordinate geometry.

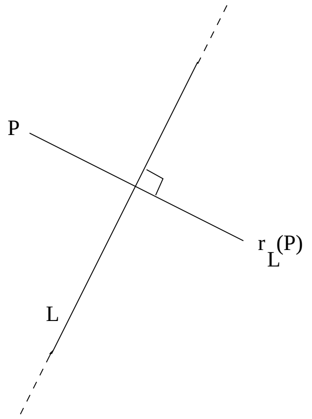
Some of you may have met this material at school. After all, it is "elementary", in the sense that it requires no mathematical sophistication to understand the subject matter. On the other hand, it is not necessarily easy.

[1]If you haven't come across 3×3 determinants, just think of the case of 2 simultaneous equations in 2 unknowns.

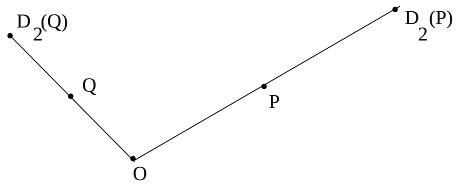
3 Congruence

A *map* from the plane to the plane is a rule which sends each point in the plane to another point in the plane, its *image* under the map.
Notation f : Plane $\to$ Plane. The image of a point P under f is $f(P)$.

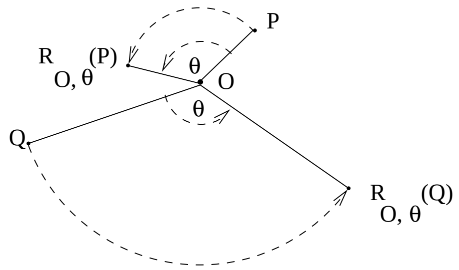
Example 3.1 1. Let L be a line in the plane. Then *reflection in* L is the map which sends a point P to the unique point P' such that L is the perpendicular bisector of the segment PP'. Reflection in L is often denoted r_L, so we write $r_L(P) = P'$. (Two lines L_1 and L_2 are *perpendicular* if all four angles where they meet are equal. The perpendicular bisector of a line-segment AB is the line perpendicular to AB passing through its midpoint.)



2. Fix a point O in the plane, and then for each $P \neq O$, map P to the point on OP (continued) whose distance from O is twice OP. This is "dilation with scale factor 2 and centre O". Dilation with scale factor α is defined similarly. If $\alpha = 1$, the map is just the identity map (leaving every point where it started).



3. Fix a point O in the plane, choose an angle θ, and rotate the plane about O through the angle θ in an anticlockwise sense. We denote this map $R_{O,\theta}$.



4. A map which moves every point the same distance in the same direction is called a *translation* and is denoted T_v, where v is the vector of movement. In coordinates, if $v = (v_1, v_2)$ then $T_v(x, y) = (x + v_1, y + v_2)$.

Definition 3.2 The map f of the plane is *an isometry* or *distance-preserving* if for every two points P, Q, we have

$$\text{distance}(f(P), f(Q)) = \text{distance}(P, Q).$$

It becomes tedious to write distance(P, Q) for the distance between P and Q; so we will usually just write $d(P, Q)$, or even PQ.

Exercise 3.3 (For those familiar with some Euclidean geometry): Assuming standard results of Euclidean geometry prove that the maps r_L, $R_{O,\theta}$ and T_v of the example are isometries

Dilation $D_{O,\alpha}$ with scale factor $\alpha \neq 1$ is not an isometry. What does $D_{O,\alpha}$ do to distance?

Proposition 3.4 *Isometries preserve angles. That is, if f is an isometry, A, B, C are three points, and we write $D = f(A), E = f(B), F = f(C)$ then $D\hat{E}F = A\hat{B}C$.*

Proof What is an angle? If you understand what is an angle, or, more precisely, what is the *size* (i.e. the numerical measure) of an angle, then you are in a strong position to prove the proposition. I postpone the proof a little, to give you time to think about the question. □

Two figures are congruent if there is an isometry of the plane which takes one to the other. However, when we say "the triangles ABC and DEF are congruent", we mean more: that there is an isometry sending A to D, B to E and C to F. Thus, unless a triangle ABC has some special symmetry, it is not congruent to the triangle ACB. The significance of this will become clear in the proof of 3.16(2) below.

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As one of our beginning points in Euclidean geometry, we make the following assumption:

Axiom 3.5 *Suppose A, B, C, D are four points in the plane. If $d(A, B) = d(C, D)$ then there is an isometry taking A to C and B to D.*

In other words, any two line-segments of equal length are congruent. You might object,

Objection 3.6 "but how on earth can you know that this is true?"

To which I reply,

Justification 3.7 How can I know that this is true of what? We are trying with our definitions and assumptions to construct a mathematical object which resembles the (more or less physical) plane than we experience. If we think that in the plane we know and sit on, there are isometries taking line- segments of equal length to one another, then we have every right to build such a property into our mathematical model of the plane. Our mathematical model won't have any properties at all unless we put some there. Of course, from the properties we endow it with at the outset, we hope to be able to deduce others — that's the point of the game. So we want to build in a relatively short set of properties at the beginning, and then build everything else from these.

You might make another objection:

Objection 3.8 "But surely the property you have just assumed as an axiom *could in fact be deduced from other more basic properties.*"

Justification 3.9 Fair enough; that's really a reproach based on your aesthetic sense that we should make as few assumptions as we can get away with. I agree that we should try to do so, and so does every mathematician who tries to build up some field of mathematics from an initial set of assumptions (axioms). Mathematicians spend a lot of time trying to see if all of their axioms are really needed, or if some can be dispensed with because they actually follow from the others. Besides loss of elegance, if I have too many axioms, lots of things can go wrong. For example, it might turn out that some of my axioms contradict one another, not overtly, because usually we notice that kind of thing straight away, but more subtly — in the sense that by putting some of the axioms together I can deduce something that is patently false, such as $1 = 0$, or "every straight line meets every other and also does not meet every other".

However, in this case we do need an axiom like this one.

Perhaps you return to the attack:

Objection 3.10 Your question in 3.7, "true of what?", does have an answer: the coordinate plane $\mathbb{R}^2$, the set $\{(x, y) : x, y \in \mathbb{R}\}$.

Justification 3.11 You mean, let's use this mathematical model for the plane, that has already been constructed. Well, we could. And indeed, our assumption (3.5) does hold for $\mathbb{R}^2$, with the distance between any two points (a, b) and (c, d) defined to be $\sqrt{(a-c)^2 + (b-d)^2}$. The interesting thing is, it also holds for lots of other already constructed objects - such as the sphere. If we insist on just thinking about $\mathbb{R}^2$, then we throw away the possibility of finding that some of our theorems are true for these other objects. This is a strong reason for keeping to our axiomatic approach, and building in to our system a hopefully fairly short list of desirable properties.

So this is how we will try to proceed. I say "try", because it is rather hard to be explicit about all of the assumptions (axioms) we will need. And rather than list them all at the start, I will introduce some others as we go along. In fact the first person to try to axiomatise geometry was of course Euclid, and he got it wrong. In developing his geometry from the axioms, he unwittingly used all sorts of additional assumptions. Fortunately he didn't arrive at any serious contradictions (that couldn't be fixed by a little tightening up of the definitions), perhaps because he was modelling something that undoubtedly does exist. Some PhD candidates in mathematics have not been so lucky, probably because they did not take the time to check that there were interesting examples of the mathamatical structures they thought they were writing about. To be honest, we will not write out lists of all of the assumptions we make. To do so would be too tedious. And it would spoil the fun of geometry, which is to a large extent based on the possibility of turning our intuition into mathematics.

Unfortunately this leaves you, the reader, in a rather uncomfortable position, at least at the start. What can you assume, and what must you prove from the axioms? In fact this discomfort will last only until we have proved a number of criteria for congruence of triangles: SAS, SSS, ASA and a variant of SAS called SS$\pi/2$. Almost all subsequent developments are based on these, so once the critera for congruence are proved, there need be no more doubts about what we can assume and what we must prove. You will find that we build rather a rich structure. It will certainly be clear enough and strong enough to prove lots of interesting and worthwhile theorems.

You might make one final objection:

Objection 3.12 *But this vagueness about the axioms means that geometry is less true and clear than, say, number theory or analysis.*

To which I reply, that is so if you look only at the mathematical structure we develop and not at the wonderful way that it reflects and enriches our intuition. In fact the doubts about axiomatics won't last very long, given the strong endorsement that your intuition should give to what we prove. In other words, it's worth it! I hope you end up by agreeing with me.

More on axiomatics in Chapter 5.

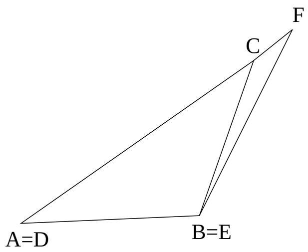
For now, I make one last assumption:

Axiom 3.13 Given a line L and a point P not on it, there is an isometry of the plane leaving every point on the line where it is, and moving the point P to the other side of the line.

Now, at last, we can prove something, using our explicit axioms, some logic, and (perhaps?) some not-so-explicit assumptions. It will be true of every mathematical object where our axioms (and un-noticed assumptions!) hold.

Proposition 3.14 *(The Side-Angle-Side criterion for congruence of triangles). Let ABC and DEF be two triangles in the plane. If $d(A,B) = d(D,E)$, $d(A,C) = d(D,F)$ and $B\hat{A}C = E\hat{D}F$ then the triangles are congruent.*

Proof By an isometry (Axiom 3.5) we can position the segment DE so that it coicides with AB. So now $D = A$, $E = B$. After another isometry, if necessary, we can ensure that F and C are on the same side of this line (by Axiom 3.13). As the angles BAC and BAF are the same, C and F lie on the same line through A. As $d(A,C) = d(A,F)$, C and F must coincide.



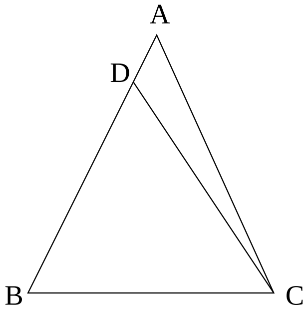
In other words, we have shown that by means of a suitable isometry, we can make A coincide with D, B with E and C with F. That, is, the triangles ABC and DEF are congruent. $\square$

Exercise 3.15 Show that reflection in a line is an isometry. Hint: SAS

Proposition 3.16 *(1) If $d(A,B) = d(A,C)$ then $A\hat{B}C = A\hat{C}B$.*
(2) If $A\hat{B}C = A\hat{C}B$ then $d(A,B) = d(A,C)$.

Proof (1) By SAS, the two triangles ABC and ACB are congruent. Therefore corresponding angles are equal, i.e. $A\hat{B}C = A\hat{C}B$.

(2) Proof by contradiction: suppose $d(A,B) > d(A,C)$. Draw D on AB so that $d(D,B) = d(A,C)$.



Then the triangles ABC and DBC are congruent; for we have

(1) $d(D,B) = d(A,C)$ by construction;

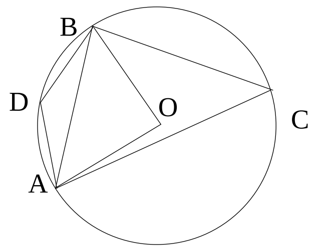
(2) BC is a common side, and

(3) the included angles are equal: $D\hat{B}C = A\hat{C}B$, by hypothesis,

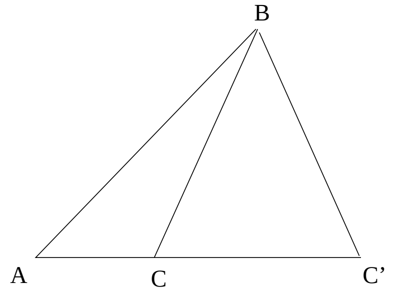
and congruence follows by SAS. But this is absurd: the triangle DBC is smaller than ABC, and therefore has smaller area, so the two cannot be congruent. $\square$

A triangle with two equal sides is called an *isosceles* triangle.

Exercise 3.17 In the diagram on the next page, A, B, C and D lie on the circumference and O is the centre of the circle. Prove that $A\hat{O}B = 2A\hat{C}B$. Which angle in the diagram is equal to $2A\hat{D}B$?



Note that SSA is not a valid criterion for congruence:

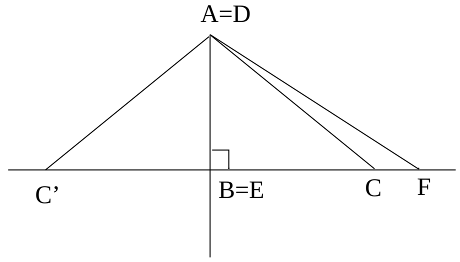


The triangles ABC and ABC' are not congruent, even though they have pairs of equal sides ($BC = BC'$, and AB is common to both triangles) and an equal angle ($B\hat{A}C = B\hat{A}C'$).

However, one particular case of SSA *is* valid: the case where the equal angles are right-angles.

Proposition 3.18 *If $AB = DE$, $AC = DF$, and $A\hat{B}C = D\hat{E}F = \pi/2$ then the triangles ABC and DEF are congruent.*

Proof After moving DEF by an isometry we can assume $A = D$ and $B = E$. Since the angles $A\hat{B}C$ and $A\hat{B}F$ are equal, C and F lie on the same line through B. Reflection in the line AB continued (if necessary) then places F on the same side of this line as C. We now know that reflection in a line is an isometry (by Exercise 3.15).



We would like to show that $C = F$, for then the two triangles will be equal. Suppose that C and F are *not* equal. Let C' be the reflection of C in the line AB. Note that $AC = AC'$, since reflection is an isometry (and this reflection leaves A fixed). As $AC = AF = AC'$, the three points C, F and C' all lie on the intersection of the line BC continued with the circle of centre A and radius AC. This is absurd: a line and a circle meet in no more than two points. $\square$

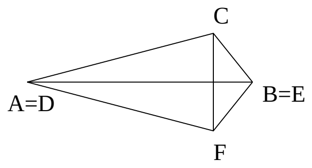
We will refer to this criterion for congruence as SS$\pi/2$.

Corollary 3.19 *If ABC is an isosceles triangle, with $AB = AC$, then the perpendicular bisector of BC and the bisector of the angle BAC coincide.*

Proof Exercise. □

Proposition 3.20 *The side-side-side criterion for congruence, or the rigidity of triangular frames: if $AB = DE$, $BC = EF$ and $CA = FD$ then the triangles ABC and DEF are congruent.*

Proof Use an isometry to bring D and E into coincidence with A and B respectively, and with F on the opposite side of AB from C. Then



(1) $A\hat{C}F = A\hat{F}C$, and $B\hat{C}F = B\hat{F}C$, by 3.16(1);
(2) $A\hat{C}B = A\hat{F}B$ by adding the two equalities of (1), so
(3) congruence of ABC and $DEF = ABF$ follows by SAS. □

We can now give a geometrical solution to the problem of finding a circle passing through three given points in the plane.

Lemma 3.21 *The perpendicular bisector of the line-segment AB is the set of all points equidistant from A and B.*

Proof Let F be the midpoint of AB. If P lies on the perpendicular bisector of AB then the triangles APF and BPF are congruent, by SAS, and so the sides AP and BP have the same length — i.e. P is equidistant from A and from B. Conversely, if P is equidisntant from A and from B then the triangles APF and BPF are congruent, by SSS, and so the angles AFP and BFP are equal. Hence FP is perpendicular to AB. □

Exercise Draw a diagram to accompany this argument.

Theorem 3.22 *If ABC is any triangle, the perpendicular bisectors of the three sides are concurrent (all three meet in a point). This point is the centre of the circle passing through A, B and C.*

Proof Let L_1, L_2 and L_3 be the perpendicular bisectors of AB, BC and CA respectively. Then by the lemma

$$\begin{aligned} L_1 \cap L_2 &= \{\text{points equidistant from } A \text{ and } B\} \cap \{\text{ points equidistant from } B \text{ and } C\} \\ &= \{\text{points equidistant from } A, B \text{ and } C\} \end{aligned}$$

Of course $L_1 \cap L_2$ consists of just one point. Since $L_1 \cap L_2$ is, in particular, equidistant from A and C, it must also lie on L_3, again by the lemma. $\square$

So if you know how to *construct* the perpendicular bisector of a line-segment, you now know how to draw the circle passing through A, B and C. In case you don't:

Construction 3.23 *To draw the perpendicular bisector of AB,*

1. *Draw a circle with centre A and radius $d(A, B)$.*
2. *Draw a circle with centre B and radius $d(A, B)$.*
3. *These two circles meet at two points, say P and Q. Then PQ is the perpendicular bisector of AB.*

Exercise Why does this construction work?

Now let's go back to coordinates.
Exercise To find the equation of the circle passing through $A = (x_1, y_1), B = (x_2, y_2)$ and $C = (x_3, y_3)$:

1. Find the gradient of the line-segment joining (x_1, y_1) and $x_2, y_2)$, and of the line segment joiing (x_2, y_2) and (x_3, y_3).
2. Find the midpoints of the line-segments.
3. Find the equations of the perpendicular bisector of AB and the perpendicular bisector of BC.
4. Find the point Q where these two lines meet. This is the centre of the required circle.
5. Find the radius of the circle - it is the distance $d(Q, A)$.

To check your calculations, note that since $d(Q, A) = d(Q, B) = d(Q, C)$, your expression for $d(Q, A)$ should be symmetric in the subindices. That is, its value should be unchanged if you replace x_1 and y_1 wherever you see them by x_2 and y_2, and vice versa — and the same for any other pair of indices. If this is not obvious from the way it's written, either you can make it obvious by judicious manipulation of the formula, or you've made a mistake.

Exercise Find an algebraic criterion for the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) to be collinear. In fact the answer to this exercise should be recognisable in your solution to the previous one.

Exercise How would you find the centre of a sphere passing through four given points in three-dimensional space?

Comment One of the points I'm trying to make here is that "looking" at a problem before setting to work on it with coordinate geometry may help you to break it up into bite-sized chunks. In fact looking at a problem is such an effective way of understanding it that most branches of mathematics try to recast themselves in geometrical terms, even if only in order to be able to use visual metaphors. A prime example of the success of this strategy is the recent solution by Andrew Wiles of one of the most outstanding problems of mathematics — Fermat's Last Theorem. This theorem asserts that if n is an integer greater than 2 then there are no positive integer solutions to the equation

$$a^n + b^n = c^n.$$

Fermat wrote in the margin of a book that he had a proof of this, but left no copy of his proof (if it existed, which most people now doubt). For 300 years no-one was able to give a correct proof, though innumerable cranks and some serious mathematicians produced false proofs. In the end, Wiles proved it using the ideas and techniques of (algebraic) geometry; a common characteristic of the cranky false proofs was their insistence on not stepping outside the realm of arithmetic.

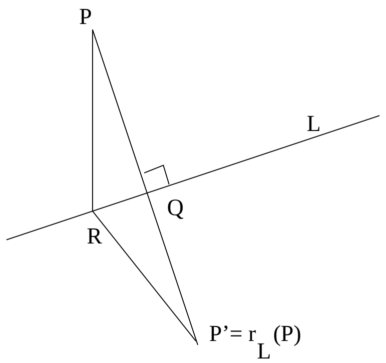
Now we begin work on a similar problem: that of finding a circle tangent to the three sides of a triangle.

Definition 3.24 A line L is *tangent* to a circle C at P if it meets C only at P.

We need some preparatory results:

Lemma 3.25 *Let L be a line and P a point not on L. The shortest line from P to L is perpendicular to L.*

Proof Let $P' = r_L(P)$, and let Q be the point where PP' crosses L. By definition of reflection, PP' is perpendicular to L. Let R be any other point on L. Then $d(P, P')$ is the length of the segment PP', and $d(P, Q)$ is half this length.



Since the route PRP' is *not* a straight line, its length is greater than the length of PP'. As $d(P, R)$ is half the length of this route, we deduce that $d(P, R) > d(P, Q)$. Thus, the shortest route from P to L is the perpendicular one, PQ. □

Lemma 3.26 *(1) Through any point P on a circle C there is a unique tangent line.*
(2) This line is perpendicular to the radius OP (where O is the centre of C).

Proof If the line L passes through P and is not perpendicular to OP, then by the preceding lemma there are points $Q \in L$ such that $d(O,Q) < d(O,P)$. As P lies *on* C, any such point Q must lie *inside* it. This means that L cannot be tangent to C, for a line meeting the region inside a circle must cut it in two points.

We have shown that a line not perpendicular to OP is not tangent to C. If the line L through P *is* perpendicular to OP, then every point $Q \neq P$ on L is further from O than is P, and thus lies outside C. So P is the only point of $L \cap C$, and L is indeed tangent to C. $\square$

Now we give another condition for the congruence of triangles:

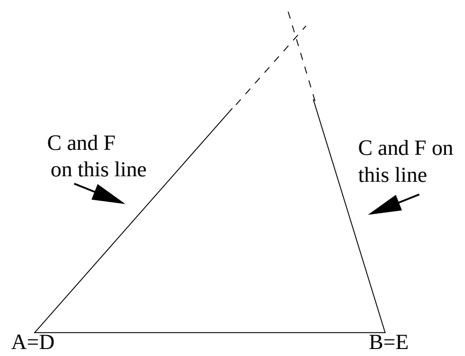
Proposition 3.27 ASA implies congruence. *Let ABC and DEF be two triangles. If $AB = DE$, and $B\hat{A}C = E\hat{D}F, C\hat{B}A = F\hat{E}D$ then ABC and DEF are congruent.*

Proof By an isometry, bring DE into coincidence with AB. After a reflection in the line AB, we can place F on the same side of this line as C.

Since $B\hat{A}C = B\hat{A}F$, C and F lie on the same line through A.

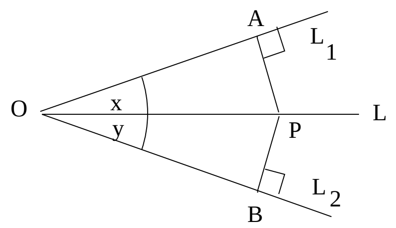
Similarly, as $A\hat{B}C = A\hat{B}F$, C and F lie on the same line through B.

As two distinct lines meet in at most one point, $C = F$ and the two triangles are therefore congruent.



Lemma 3.28 *The bisector of the angle between lines L_1 and L_2 is the set of points inside the given angle equidistant from the two lines.*

Proof



Let A and B be the points where the perpendiculars from P to L_1 and L_2 meet these lines. Then $d(P, L_1) = PA$ and $d(P, L_2) = PB$. Suppose first that $d(P, L_1) = d(P, L_2)$. We have to show that the angles x and y are equal. This follows from congruence of the triangles OAP and OBP; they are congruent because $PA = PB$, since $PA = d(P, L_1)$ and

$PB = d(P, L_2)$ and these are supposed equal, the side OP is common to the two triangles, and $P\hat{A}O = P\hat{B}O = \pi/2$, so congruence follows by SS$\pi/2$. Hence $x = y$.

To prove the converse, suppose that $x = y$. If we were to use the fact (not yet proved) that the angle sum in a triangle is $\pi/2$, then from the equality of x and y, and of $O\hat{A}P$ and $O\hat{B}P$, we would be able to deduce equality of $O\hat{P}A$ and $O\hat{P}B$, and then the triangles OAP OBP would be congruent, by ASA (for they have the common side OP). However, I do not want to use the angle sum in a triangle, as I want this proof to be valid even in circumstances where the angle sum in a triangle is not fixed — for example on the sphere. So instead, we argue as follows:

Suppose we reflect L_1 in L. We get a new line, L_2', on the other side of L from L_2 and making the same angle, y, with L as does L_2. Clearly this line must be equal to L_1, since we are assuming $x = y$. Hence, the point B' on L_2' closest to P must be equal to the point on L_1 closest to P, namely A. But as reflection in L is an isometry, and leaves P fixed, the closest point on L_2' to P must be the reflection of the closest point on L_2 to P, namely B. It follows that $PB' = PB$. As $B' = A$, we conclude that $PA = PB$, as required. □

Theorem 3.29 *(1) The three bisectors of the angles of a triangle are concurrent.*

(2) Their point O of intersection is the centre of a circle tangent to all thee sides. It is the only point in the interior of the triangle with this property.

Proof The proof of part (1) has exactly the same structure as the proof of 3.22, and I leave it as an **Exercise**.

(2) Let C be the circle with centre O and radius equal to the distance from O to any of the three sides. Then C meets each of the three sides at just one point — its point closest to O. Hence each side is tangent to C. □

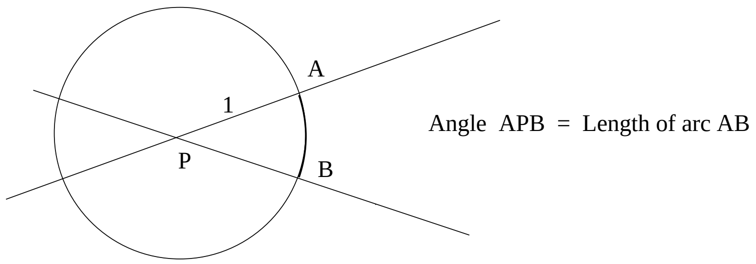
Exercise Is there always a circle tangent to three given lines? Is there always a circle passing through 3 given points?

Exercise How would you find the centre of a sphere tangent to four given planes in 3-space?

I end this section with a proof that isometries preserve angles.

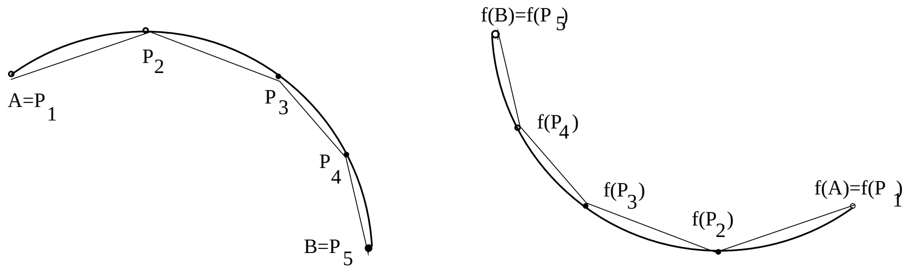
Proof of Proposition 3.4 The difficulty here is that we have not introduced a definition of angle. If we don't have a formal definition of what something is, it's very hard to give a formal proof of anything about it.

Definition 3.30 The *angle* between two lines at a point P where they meet is the length of the arc of the unit circle with centre at P, running between the points where the lines cut the circle.



Proposition 3.4 will be proved if we can show that an isometry must preserve the length of an arc. This seems plausible, even obvious, but a formal proof is hard to give because, once again, we don't have a formal definition of the length of an arc. For now, let me simply point you in the right direction with the following partial argument:

we can *approximate* the length of an arc of a circle by measuring the length of a polygonal approximation:



The length $P_1P_2 + P_2P_3 + \cdots + P_{n-1}P_n$ of any polygonal approximation to the arc is preserved by an isometry. Hence (and it really does follow, though here I do not make it clear why) an isometry preserves the length of the arc itself.

This (more or less) completes the proof that isometries preserve angles (Proposition 3.4). □

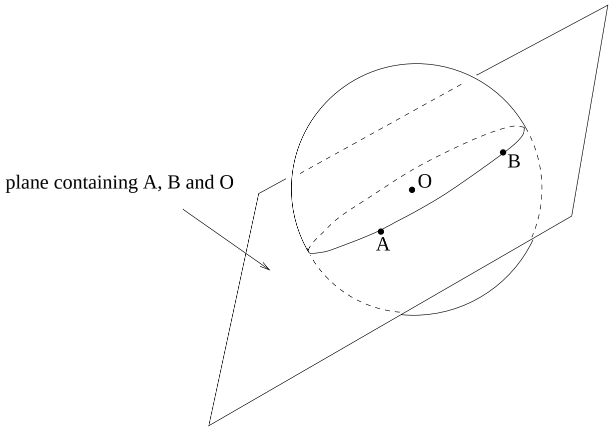
Exercise What is the precise relation between the length of the arc AB and the lengths of the polygonal approximations to AB? How would you complete the proof of 3.4?

4 Geometry on the sphere

On the surface of a sphere there are no straight lines. Nevertheless, there are certain curves which share an important property with straight lines in the plane — they provide the shortest route between points. These are the so called "great circles" or "geodesics". By definition, a great circle is the intersection of the sphere with a plane passing through its centre. Any two great circles meet at two (antipodal) points, but other than that great circles behave in many ways like straight lines. For example,

Lemma 4.1 *Through any two non-antipodal points on a sphere there is a unique great circle.*

(Two points on the sphere are *antipodal* if they lie at opposite ends of a diameter of the sphere.)
Proof Let O be the centre of the sphere. If the two points A and B are not antipodal then O, A and B are not collinear, so



there is a unique plane containing all three. The intersection of this plane with the sphere is the unique great circle passing through A and B. $\square$

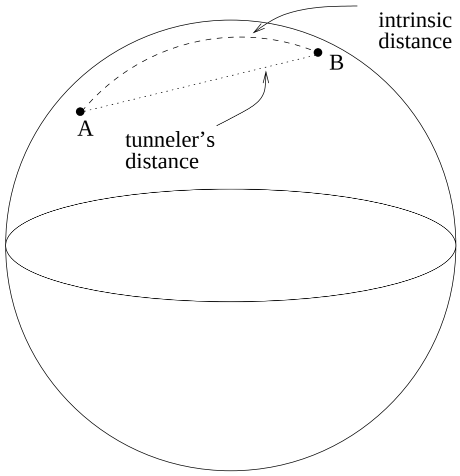
It is useful to be able to make (plane) pictures of great circles:

Drawing Hint The boundary of a drawing of the sphere is a circle. A great circle on the sphere then appears as an ellipse whose major axis is a diameter of the bounding circle.

Exercise Mark three points A, B and C on a drawing of sphere, with no two of them antipodal, and then draw the great circles through each pair of them. You may find it helpful to adopt the convention that curves "behind" the sphere should be drawn with a dotted line.

To do geometry on the sphere, we need a notion of distance. There are two natural choices:

1. the length of the straight line joining A and B (the tunneler's distance), and
2. the intrinsic distance, the distance on the sphere, which, by what we said above, is the length of the shorter of the two arcs of the great circle joining A and B.



This second notion of distance is more relevant to a sailor, and for this, and other reasons, is the one we choose to work with. An *isometry of the sphere* is a map which conserves distance. In fact it doesn't matter which of the two notions of distance is used: it is easy to see that if a map conserves one kind of distance then it conserves the other, as the two are related by a simple trigonometrical formula.

Example 4.2 Any isometry of 3-space which leaves fixed the centre of the sphere, must map the sphere to itself (as the sphere is the set of all points at distance r from the centre), so defines an isometry of the sphere. Thus, as isometries of the sphere we have

1. rotations of the sphere about an axis passing through the centre,

2. reflections of the sphere in planes passing through the centre,

3. but no translations.

Here I am of course assuming (correctly, but without proof) that rotations and reflections are isometries of 3-space.

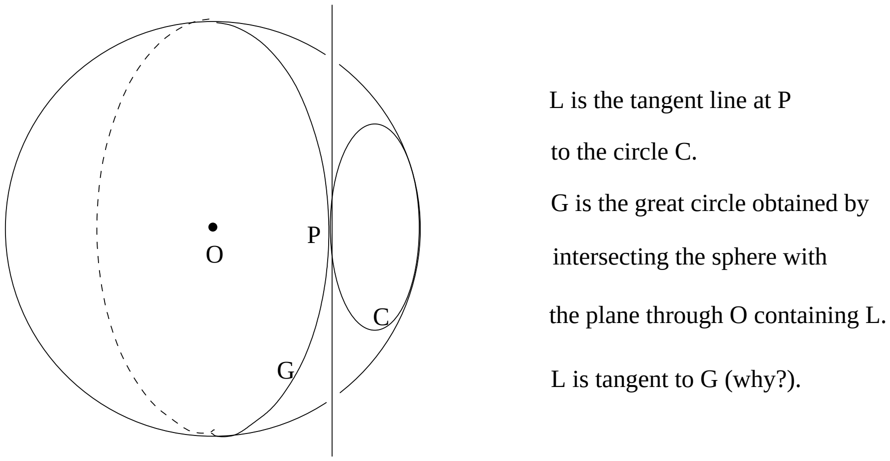
Exercise Are there any other isometries of the sphere?

Proposition 4.3 *If AB and CD are arcs of great circles, and have the same length, then there is an isometry of the sphere taking AB to CD.*

Proof **Exercise** □

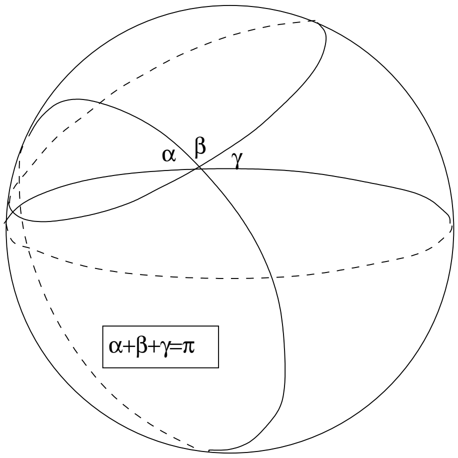
The biggest difference between spherical and plane geometry is that on the sphere *there are no parallel lines* - every pair of great circles meet. From this, many other differences follow. For example, the sum of the angles in a geodesic triangle (one whose edges are arcs of great circles) is *never* equal to π.

Before discussing this further, I had better say what I mean by "angles" on a sphere. One can show by calculus that any smooth curve C on the sphere has a well-defined *tangent line* at each point. If C is a circle (and thus the intersection of the sphere with a plane) then in fact we don't need calculus; since the circle lies in a plane, we define the tangent line to be the tangent line defined for a plane circle in 3.24 above. The *angle* between two circles at a point P on the sphere is then the angle between the tangent lines to the circles at P. If L is tangent at P to the circle C on the sphere, then L is necessarily orthogonal to the radius OP. So all tangent lines to circles at P lie in the plane through P orthgonal to OP; this is the *tangent plane* to the sphere at P.



Exercise Prove the last assertion.

In particular, the angles at a point P on a sphere add up to 2π, since they are in fact angles in the tangent plane to the sphere at P.



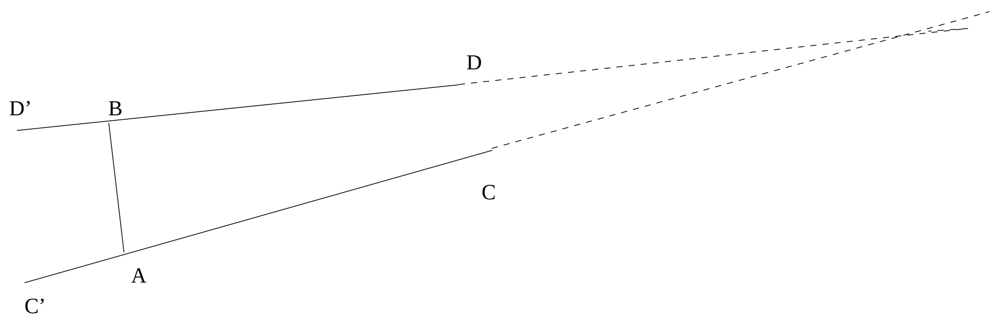
Exercise Draw some spherical triangles with one vertex at the north pole and the other two vertices on the equator. What can you say about their angle-sum?

We will take further looks at spherical geometry as the course goes on.

5 The parallel postulate

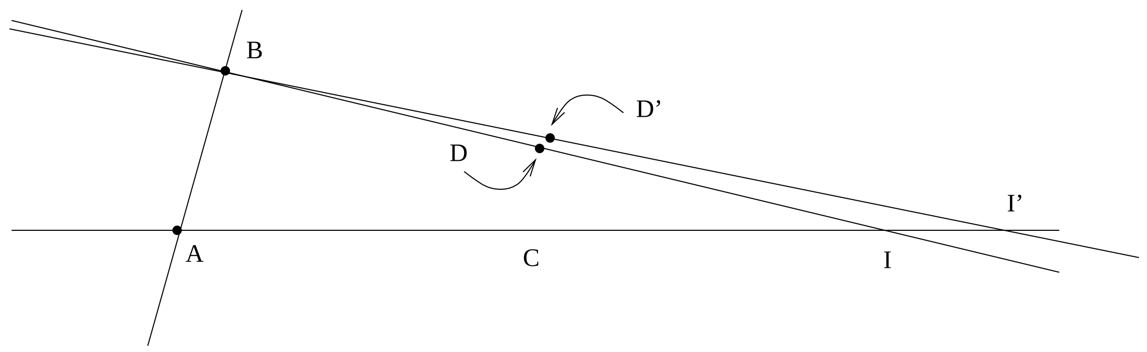
Euclid gave as axioms a set of definitions and propositions from which, he believed, all the other propositions of plane geometry would follow. The fith of these was the so-called parallel postulate:

Axiom 5.1 *If two lines AC and BD lie in the plane, with C and D on the same side of the line AB, and if the angles $C\hat{A}B$ and $A\hat{B}D$ add up to less than two right angles (=$180° = \pi$ radians)* *then the lines AC, BD, extended in the direction implied by the notation, will eventually meet.*



If the sum of the angles is greater than π then the postulate implies that the lines CA and DB extended will eventually meet (apply the postulate to the angles $C'\hat{A}B$ and $A\hat{B}D'$).

And if the sum of the angles is exactly π then the two lines will never meet — and so are *parallel*:



For suppose they meet at I, as shown in the diagram. Draw I' on AC continued, further from A than I. Then $I'\hat{B}A > I\hat{B}A = D\hat{B}A$, so

$$I'\hat{B}A + B\hat{A}C > \pi.$$

It follows from the postulate that the lines $I'B$ and CA extended must meet. But then thes two lines, extended in both directions, meet in two points - an absurdity.

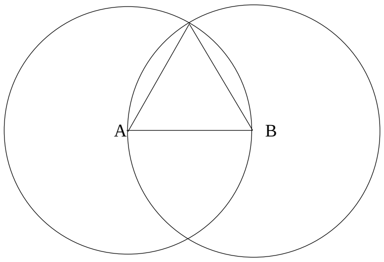
Thus, Euclid's postulate 5.1 implies

Axiom 5.2 (Playfair's Axiom) *Given any line L and a point P not on L there is a unique line through P parallel to L.*

In fact 5.1 can be deduced from 5.2, so the two are equivalent.

Euclid's parallel postulate is of a very different character from his other axioms — it seems more like a theorem. For many centuries, mathematicians tried to deduce it from the other axioms, fruitlessly as it turns out. The situation was finally clarified in the last century when Gauss, Bolyai and Lobachevsky indpendently constructed geometries in which the parallel postulate was false, but all of the other axioms held. This showed that the parallel postulate could not be a consequence of the others.

In fact, as I said earlier, Euclid's axiomatisation had a number of defects, and Euclid himself was not able to make all his deductions using *only* his axioms, even though he thought he had. For example, he constructs an equilateral triangle with given base AB by drawing two circles, one with centre A and radius AB and the other with the same radius and centre B, one with centre A and radius AB. He then asserts that the triangle whose vertices are A, B and one of the two points of intersection of the two circles is equilateral. This is correct, but there is nothing in his axiom system to guarantee that these two circles will meet at all.



We have made explicit use of a number of axioms, such as the following two which are in Euclid's list:

Axiom 5.3 *Through any two points in the plane there is a unique line.*

Axiom 5.4 *Any two distinct lines meet in at most one point.*

but our deductions sometimes make use of other assumptions which have not been made explicit. For example, in the proof of Lemma 3.26 I have assumed that a segment with one end inside a circle and one end outside it, must cut it at some point. This is equivalent to the assumption that Euclid inadvertently made in his construction of an equilateral triangle. I have also assumed

Axiom 5.5 A line divides the plane into two parts.

Axiom 5.6 A line and a circle meet in no more than two points.

Axiom 5.7 There are enough isometries: if AB and CD are line segments of equal length, then there is an isometry taking A to C and B to D.

Axiom 5.8 A line is the shortest route between any two of its points

.

And I have made a number of other unacknowledged assumptions.

There are two ways to deal with the problem of the "foundations" of geometry. One is to give a rigorous axiom system which is rich enough to prove the theorems of Euclidean geometry. A complete axiom system was first given by David Hilbert at the end of the last century. It runs for a number of pages, and so it would certainly not make a good beginning for a course. Nevertheless, some authors attempt such an approach. They (and their students) have to grapple with statements like:

- each line in the plane is in order-preserving bijection with the real line

- if two lines make the same angle with a third at a point P, then they coincide.

The other way is to say, for us "the plane" means $\mathbb{R}^2$, the set of ordered pairs of real numbers. The lines are all sets of the form $\{(x,y) \in \mathbb{R}^2 : ax+by+c=0\}$, and distance is the Pythagorean distance, $d(x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. Then it is possible to check that the assumptions one makes really do hold.

For example, we check Playfair's axiom: if L is the line $ax+by=c$ and $P \notin L$ has coordinates (P_1,P_2), then the line L' with equation $ax+by=aP_1+bP_2$ is parallel to L, and passes through P.

As another example, by explicit algebraic calculations involving the solution of a quadratic equation, we can find the coordinates of the points of intersection of two circles. Since $\mathbb{R}$ does contain the square roots of non-negative numbers, these coordinates will be real numbers provided the distance between the centres is no greater than the sum of the radii (try it) and therefore two circles whose centres are distant from one another by less than the sum of the radii really do meet in two points in $\mathbb{R}^2$. Although this approach seems simpler than giving a complete axiomatisation, it just pushes the problem of foundations into another course (not a bad idea in itself), in this case Analysis. Underlying the proof given above that Euclid's two circles (thought of as subsets of $\mathbb{R}^2$) do meet, is the fact that every positive real number has a positive square root in $\mathbb{R}$, and this itself requires proof. It is a consequence of the so-called completeness of $\mathbb{R}$. To postpone the study of geometry until all of these issues are dealt with is to throw away one of its principal attractions, its accessibility.

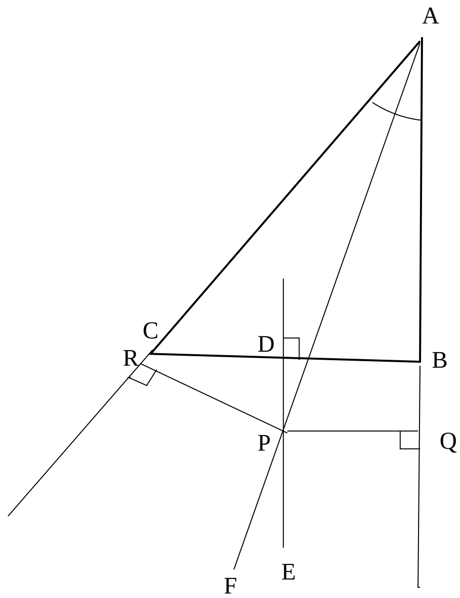
In Euclidean geometry diagrams are used as an aid to reasoning; they also motivate one to reason — to do maths — in the first place. Using what is "obvious", plus some reasoning, one can say things that are not obvious, about objects one is interested in for their own sake. At least for me, pictures play an important rôle in generating that interest. But one must be very careful with diagrams. Like any powerful tool, they can lead to disaster if not kept in check. Here is an example:

False Proposition 5.9 *Every triangle is isosceles.*

Proof We will prove that in the triangle ABC, $AB = AC$. To do this, we draw the bisector AF of the angle $B\hat{A}C$, and the perpendicular bisector DE of AB. These two lines meet at P, and PQ and PR are the perpendiculars from P to AB and AC continued.

The proof proceeds through a number of steps.

1. **The triangles APQ and APR are congruent**; the angles $P\hat{A}Q$ and $P\hat{A}R$ are equal (AP bisects $B\hat{A}C$), the angles $P\hat{Q}A$ and $P\hat{R}A$ are equal (both are right-angles), and so the angles $A\hat{P}Q$ and $A\hat{P}R$ are equal (equality of two pairs of angles implies equality of the third pair, as the angles in a triangle add up to π — we have not proved this yet, but is is proved (correctly!) in the next section). As the side AP is common to both triangles, and $PR = PQ$ by Lemma 3.28, congruence follows by SAS.



2. From the congruence of APQ and APR it follows that $AR = AQ$.

3. **The triangles DPC and DPB are congruent**; they have the common side DP, $DB = DC$ as D is by definition the midpoint of BC, and $P\hat{D}B = P\hat{D}C = \pi/2$ as DP as by definition DP is perpendicular to BC, so congruence follows by SS$\pi/2$.

4. From congruence of DPC and DPB it follows that $PC = PB$.

5. **The triangles PRC and PQB are congruent**; $PC = PB$, $PR = PQ$ and $PQB = PRC = \pi/2$, so congruence follows by SS$\pi/2$.

6. From congruence of PQB and PRC it follows that $CR = PQ$.

7. As $AQ = AR$ and $BQ = CR$ it follows by subtraction that $AB = AC$. So ABC is an isosceles triangle.

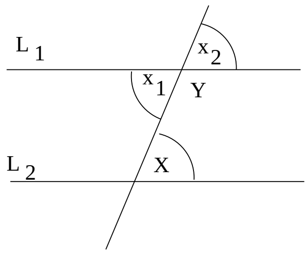
□

Exercise What is wrong with this proof?

6 Parallel lines

We now assume Euclid's fifth postulate. Thus, we know that

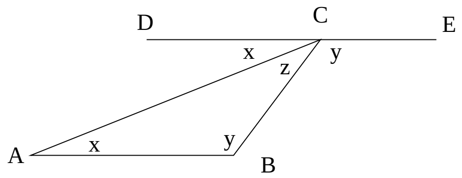
(i) the lines L_1 and L_2 are parallel if and only if the angles X and Y add up to π.



(ii) if they are parallel then the angle X is equal to the angle x_1 (they are "alternate angles") and to x_2 (X and x_2 are "corresponding angles"); these equalities hold because $x_1 + Y$ and $x_2 + Y$ are both equal to π, since each sum is the angle on a line.

Proposition 6.1 *The angles in a plane triangle add up to π.*

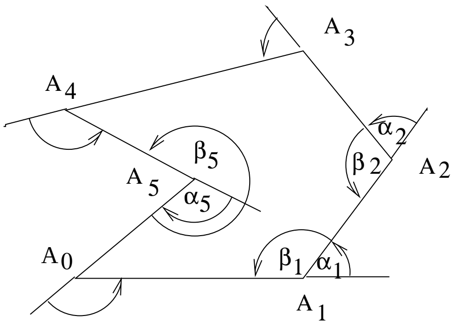
Proof Draw DE through C, parallel to AB. Then the equalities marked in the diagram hold: in each case they are alternate angles. Thus $x + y + z =$ angle in a line $= \pi$. □



Another argument is often used to prove 6.1; since it also proves a more general statement, here it is:

Proposition 6.2 *The angles in a plane $n - gon$ add up to $(n - 2)\pi$.*

Proof Order the edges of the polygon in an anticlockwise direction. Imagine a pencil placed along the first edge A_0A_1, pointing in the anticlockwise direction.



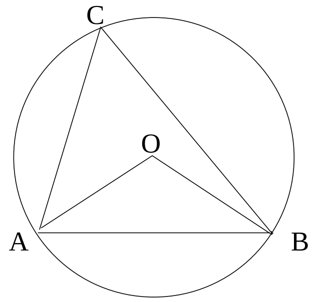
Then move it so it lies along A_1A_2, pointing once again in the anticlockwise direction. This movement involves rotating it through an angle α_1 in the clockwise direction. If β_1 is the interior angle at the vertex, then $\alpha_1 + \beta_1 = \pi$. Repeat this procedure at each vertex. Eventually the pencil returns to its original position. During its trip round the polygon, it has made one complete rotation in the anti-clockwise direction. Thus $\alpha_1 + \cdots + \alpha_n = 2\pi$. Since, for each i, $\beta_i + \alpha_i = \pi$, we have

$$\beta_1 + \cdots + \beta_n = (\pi - \alpha_1) + \cdots + (\pi - \alpha_n) = n\pi - (\alpha_1 + \cdots + \alpha_n) = (n-2)\pi.$$

Note that one should interpret this proof carefully: for example, in moving from the edge A_4A_5 to the edge A_5A_0 in the diagram, the pencil rotates in a *clockwise* direction. Thus, its rotation in the anticlockwise direction is *negative.* So the angle shown is counted negatively: in the diagram here, α_5 is negative. As before, $a_5 + \beta_5 = \pi$; β_5 is indeed greater than π. $\square$

Question Does this proof involve the notion of parallelism in any covert way?

Exercise 6.3 (i) Show that in the diagram below, $A\hat{O}B = 2 \cdot A\hat{C}B$, where O is the centre of the circle (Euclid III.20). Hint: use the fact that the angle sum in a triangle is π, and look for isosceles triangles. Drawing OC might help.
(ii) Draw D anywhere on the lower arc AB. Which angle in the diagram is then equal to $2 \cdot A\hat{D}B$?



Note that from 6.3 it follows that if C and C' lie on the same arc AB, then the angles $A\hat{C}B$ and $A\hat{C}'B$ are equal (Euclid III.21).

Exercise 6.4 *Suppose that C' is some point lying above the chord AB in the preceding diagram, and that $A\hat{C}'B = A\hat{C}B$. Show that C must lie on the circle. Hint: If not, let C'' be the point where AC (possibly continued) meets the circle, and compare the angles $A\hat{C}''B$ and $A\hat{C}'B$.*

7 Area

What we will understand by "area" is summarised by the following:

Definition 7.1

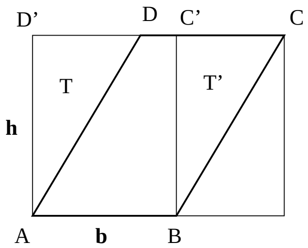
1. *The area of a rectangle is the product of the lengths of two adjacent sides.*
2. *If two figures are disjoint, or meet only along an edge or edges, then the area of their union is the sum of their areas.*
3. *Congruent figures have the same area.*

- Does every plane set have an area?

- Is this a silly question?

Proposition 7.2 *The area of a parallelogram is base times height.*

Proof The triangles T and T' are congruent. Hence they have the same area.



Therefore

$$\text{area}(ABCD) = \text{area}(ABC'D) + \text{area}(T) = \text{area}(ABC'D) + \text{area}(T') = \text{area}(ABC'D') = bh.$$

□

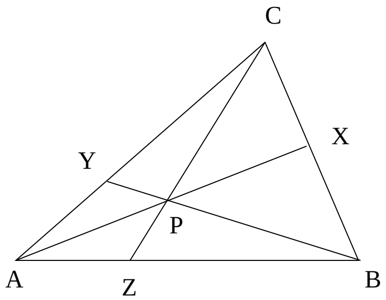
Proposition 7.3 *The area of a triangle is one half base times height.*

Proof **Exercise** Given any triangle T, there is a parallelogram with the same base and the same height, made up of T and a triangle T' congruent to T, with T and T' meeting only along an edge. □

Theorem 7.4 Ceva's Theorem *Let ABC be a triangle and X,Y and Z points on the edges AB, BC and CA respectively. If the segments AX, BY and CZ are concurrent then*

$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = 1.$$

Proof Suppose all three segments meet at P.



By the previous proposition,

$$\begin{aligned}
\frac{|AZ|}{|ZB|} &= \frac{\text{area}(AZC)}{\text{area}(BZC)} \\
&= \frac{\text{area}(AZP)}{\text{area}(BZP)} \\
&= \frac{\text{area}(AZC)-\text{area}(AZP)}{\text{area}(BZC)-\text{area}(BZP)} \\
&= \frac{\text{area}(APC)}{\text{area}(BPC)}.
\end{aligned}$$

For the third equality we are using the following elementary algebraic fact:

$$\text{if} \quad \frac{a}{b} = \frac{c}{d} \quad \text{then for all } \lambda_1, \lambda_2 \quad \frac{a}{b} = \frac{\lambda_1 a + \lambda_2 c}{\lambda_1 b + \lambda_2 d},$$

which I am sure you can prove for yourselves.

Similar arguments show

$$\frac{|BX|}{|XC|} = \frac{\text{area}(BPA)}{\text{area}(CPA)}$$

and

$$\frac{|CY|}{|YA|} = \frac{\text{area}(CPB)}{\text{area}(APB)}.$$

Hence

$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = \frac{\text{area}(APC)}{\text{area}(BPC)} \times \frac{\text{area}(BPA)}{\text{area}(CPA)} \times \frac{\text{area}(CPB)}{\text{area}(APB)} = 1.$$

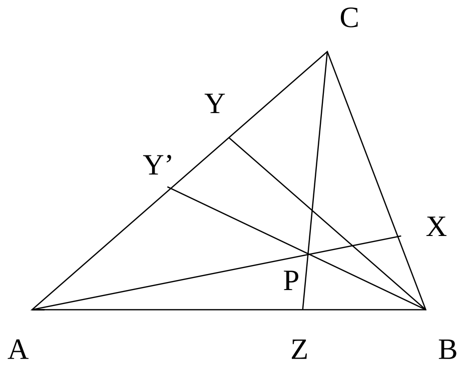
□

Corollary 7.5 Converse to Ceva's Theorem *If*

$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = 1$$

then the segments AX, BY and CZ are concurrent.

Proof Suppose that AX and CZ meet at P. We have to show that BY passes through P also. To do this, continue the segment BP until it meets AC, and call the point of intersection Y'. We have to show that $Y = Y'$.



We have

$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = 1$$

by hypothesis and also

$$\frac{|BX|}{|XC|}\frac{|CY'|}{|Y'A|}\frac{|AZ|}{|ZB|} = 1$$

by Ceva's theorem. It follows that

$$\frac{|CY|}{|YA|} = \frac{|CY'|}{|Y'A|}$$

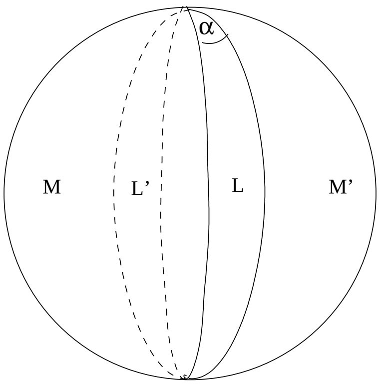
and as Y and Y' both lie on AC, this can only mean that $Y = Y'$, as required. $\square$

Exercise The *medians* of a triangle are the lines joining the vertices to the mipoints of the opposite sides. Show that the three medians of a triangle are concurrent.

8 Triangles on a sphere

We will use without proof the fact that the surface area of a sphere of radius R is $4\pi r^2$.

Two geodesics together divide the sphere into 4 *lunes*.



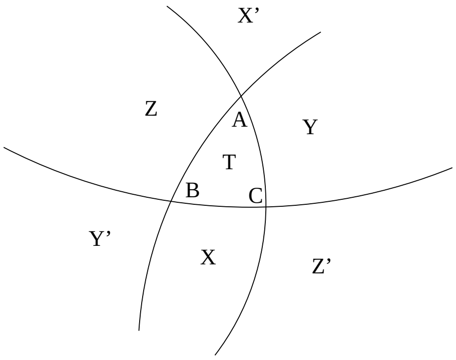
In this diagram I have marked two pairs of equal lunes L and L', and M and M'. It is evident that the area of L is proportional to the angle α: $\text{area}(L) = k\alpha$ for some constant of proportionality k. If $\alpha = \pi$ then $\text{area}(L) = (1/2)$area of sphere $= 2\pi r^2$. Thus $k\pi = 2\pi r^2$, $k = 2r^2$, and $\text{area}(L) = 2r^2\alpha$.

Theorem 8.1 *The sum of the angles in a geodesic triangle T on a sphere of radius r is equal to $\pi + area(T)/r^2$.*

This is one of the most beautiful elementary proofs that I know, and, when accompanied by a good diagram, is crystal-clear. I have drawn part of the diagram, making plain my notation.

Exercise Draw a complete diagram to accompany the following

Proof The three geodesics forming the edges of the triangle divide up the sphere into 8 triangles, each triangle having an antipodal twin with the same area. Call the triangle with we are interested in T and let T' be its twin. Let the vertices of T be A, B and C, and denote the interior angles in T at these vertices by $\hat{A}, \hat{B}$ and $\hat{C}$.



As shown here, we denote the triangles across the edges from T by X, Y and Z, and their antipodal twins by X', Y' and Z'. We use the symbols t, t', x, y etc. to denote the *areas* of the triangles T, T', X, Y, Z etc.

Together T and X make up a lune with angle $\hat{A}$, T and Y make up a lune with angle $\hat{B}$, and T and Z make up a lune with angle $\hat{C}$. Thus we have

$$\begin{array}{lcl} x+t & = & 2r^2\hat{A} \\ y+t & = & 2r^2\hat{B} \\ z+t & = & 2r^2\hat{C}, \quad \text{therefore} \\ 3t+x+y+z & = & 2r^2(\hat{A}+\hat{B}+\hat{C}). \end{array}$$

Since

$$2t+2x+2y+2z = t+t'+x+x'+y+y'+z+z' = \text{ area of the sphere } = 4\pi r^2$$

and thus $t+x+y+z = 2\pi r^2$, we deduce that

$$t = \hat{A}+\hat{B}+\hat{C}-\pi,$$

and the theorem follows immediately from this. □

Corollary 8.2 *There can be no isometric mapping from a region S of the sphere to a region T in the plane.*

Proof Let ABC be a triangle contained in S, and suppose that $f : S \to T$ is an isometry. Then $f(ABC)$ must be a straight-edged triangle in the plane. For the edges of ABC are arcs of geodesics, which are the shortest routes between points on the sphere, and their images under f must have the same property, as f preserves all distances. That is, their images must be straight lines. The isometry f must also preserve angles; but now we have a contradiction, for the sum of the angles of the plane triangle $f(ABC)$ must, like the sum of the angles of ABC, be greater than π. □

Exercise What goes wrong with the proof of Proposition 6.2 when we apply it to a spherical triangle?

Exercise 8.3 Show that in a geodesic n-gon on the unit sphere, the sum of the angles is $(n-2)\pi + A$, where A is its area.

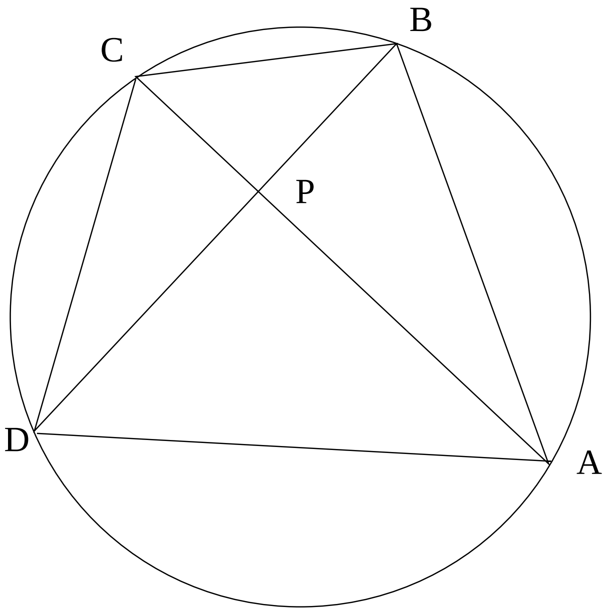
In Section 12 we will explore one of the ways in which one can map a sphere to the plane while preserving at least some of its geometry.

9 Similarity in the plane

Definition 9.1 *The plane triangles ABC and DEF are* similar *if*

$$B\hat{A}C = E\hat{D}F, \quad A\hat{B}C = D\hat{E}F \quad \textit{and} \quad B\hat{C}A = E\hat{F}D.$$

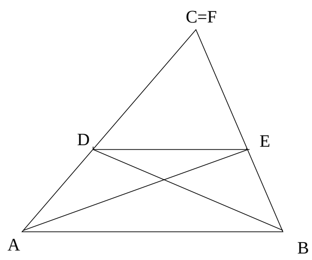
Exercise 9.2 *Show that the triangles APB and DPC are similar.*



Proposition 9.3 *(Euclid VI.4) If ABC and DEF are similar then*

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

Proof After an isometry we can place DEF so that F coincides wtih C and the angle $B\hat{C}A$ coincides with the angle $E\hat{F}D$ – i.e. A and D are on the same half-line beginning at C, and B and E are on the same half-line beginning at C, as shown in the diagram.

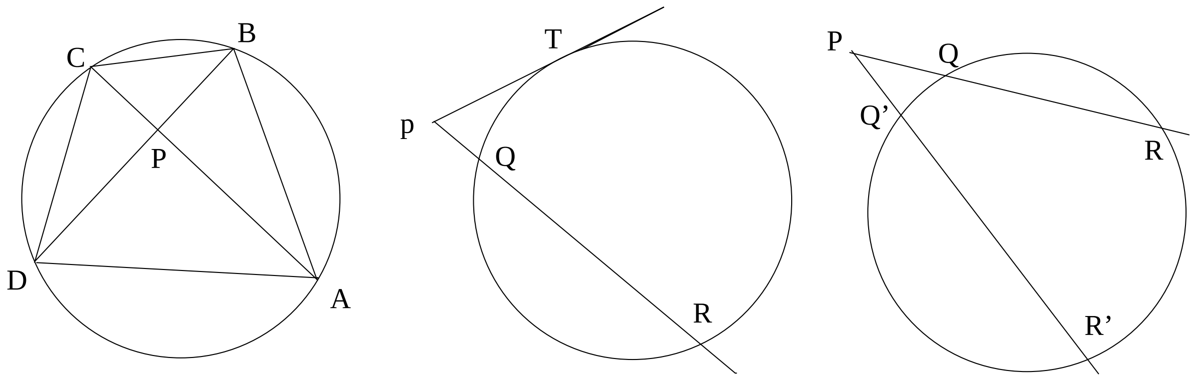


Then

1. AB and DE are parallel (by equality of $C\hat{A}B$ and $F\hat{D}E$)
2. area(BDE) = area(ADE), as these two triangles have the same base (DE) and the same height.
3. $$\frac{BC}{EC} = \frac{\text{area}(BDC)}{\text{area}(EDC)}$$
   — as the triangles BDC and EDC have the same height, the ratio of their areas is the same as the ratio of the lengths of their bases, BC and EC.
4. $$\frac{AC}{DC} = \frac{\text{area}(AEC)}{\text{area}(DEC)}$$
   by a similar argument.
5. area(BDC) = area(BDE)+area(EDC) = area(ADE)+area(EDC) (by (2)) = area(AEC).
6. $$\frac{BC}{EC} = \frac{\text{area}(BDC)}{\text{area}(EDC)} \quad \text{(by (3))} \quad = \frac{\text{area}(AEC)}{\text{area}(DEC)} \quad \text{(by (5))} \quad = \frac{AC}{DC} \quad \text{(by (4))}.$$

As $C = F$, we've shown that $AC/DF = BC/EF$, one of the two equalities we had to prove. A similar argument (superimposing E on B instead of F on C) shows that $AB/DE = CB/FE$, and this completes the proof. □

Exercise 9.4 (i) Show that in Figure (i) below, $AP \cdot PC = BP \cdot PD$ (Euclid III.35).



(i) (ii) (iii)

(ii) Show that in Figure (ii), $PT^2 = PQ \cdot PR$ (Euclid III.36).
(iii) Show that in Figure (iii), $PQ \cdot PR = PQ' \cdot PR'$.

Corollary 9.5 *(*Converse to 9.3*) If*

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

then ABC and DEF are similar.

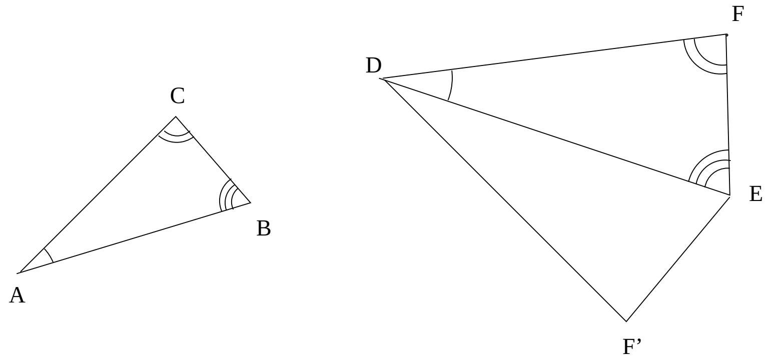
Proof Draw F' so that $D\hat{E}F' = A\hat{B}C$ and $E\hat{D}F' = B\hat{A}C$. Then the triangles DEF' and ABC are similar. Hence

$$\frac{EF'}{DE} = \frac{BC}{AB}$$

by 9.3. As also

$$\frac{EF}{DE} = \frac{BC}{AB}$$

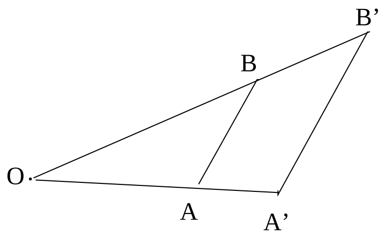
by hypothesis, we conclude that $EF = EF'$.



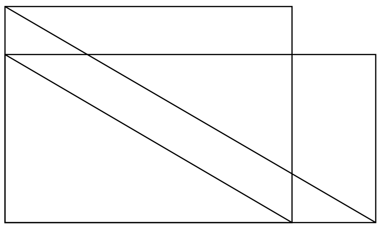
A similar sequence of steps shows that $DF = DF'$. It follows by SSS that the triangles DEF and DEF' are congruent: we have just shown that two pairs of sides are equal, and the third side DE is common to both triangles.

Now congruence implies similarity (see 3.4); thus DEF has the same angles as DEF' which has the same angles as ABC (by construction), and so DEF and ABC have the same angles. □

Exercise Show that if $OA'/OA = OB'/OB$ then the triangles OAB and $OA'B'$ are similar.

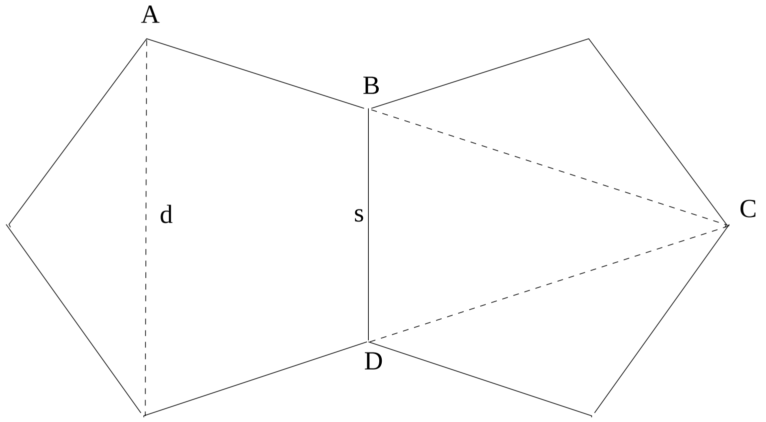


Exercise 9.6 In the diagram below, show that the two rectangles have the same area if and only if the two sloping lines are parallel.



Exercise 9.7 Let A', B' and C' be the midpoints of the three sides BC, CA, AB (respectively) of the triangle ABC. Show that $A'B', B'C'$ and $C'A'$ are parallel to AB, BC and CA respectively. Deduce that $A'B'C'$ is similar to ABC.

Exercise 9.8 The diagram shows two regular pentagons side by side.

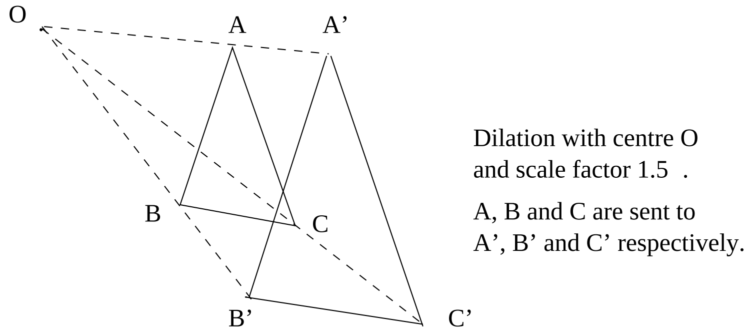


(i) Find the angle $B\hat{C}D$. (Hint: mark the centre of the right-hand pentagon and use Exercise 3.17.)
(ii) Show that ABC is a straight line.
(iii) Find the ratio d/s.

In the next proposition we denote dilation with centre O and scale factor a (cf. Example 3.1(2)) by $D_{O,a}$.

Proposition 9.9 *For any two points P and Q, we have*

$$D_{O,a}(P)D_{O,a}(Q) = aPQ.$$



Proof **Exercise** . $\square$

Definition 9.10 *A* similarity *is any map f from the plane to the plane with the property that there exists a real number a (the "scale factor") such that for all points P, Q, we have $f(P)f(Q) = aPQ$.*

A dilation (Example 3.1(2)) is a similarity.
An isometry is a similarity with scale factor 1.

Proposition 9.11 *A similarity preserves angles.*

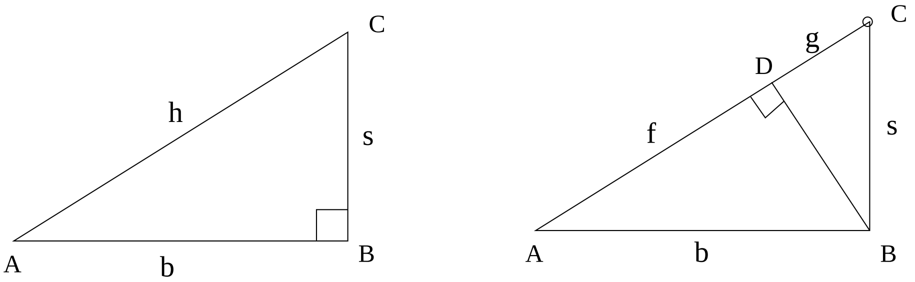
Proof If f is a similarity of the plane and ABC is a triangle then $f(A)f(B)/AB = f(B)f(C)/BC = f(C)f(A)/CA$ (all are equal to the scale factor), so by 9.5 ABC and $f(A)f(B)f(C)$ are similar.

Exercise Show that a similarity maps circles to circles.

Exercise Show that if ABC and EFG are similar triangles then there is a similarity taking ABC to EFG.

Theorem 9.12 Pythagoras's Theorem *In a right-angled triangle the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.*

Proof The triangles ABC and ADB are similar (equality of two out of three pairs of angles is enough for similarity), so $f/b = b/h$, and therefore $fh = b^2$.

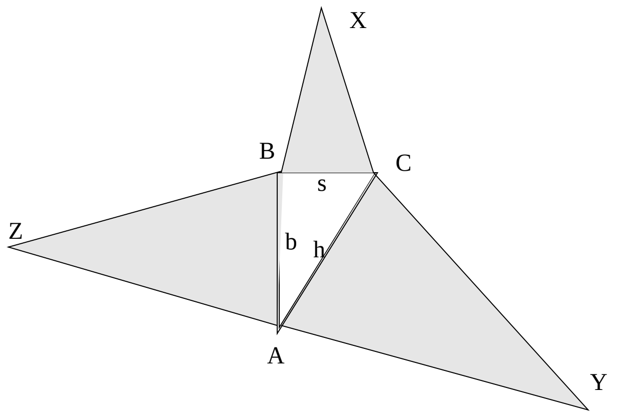


In the same way, the triangles ABC and BDC are similar, so $g/s = s/h$, and therefore $gh = s^2$.

Adding together the equalities obtained, we get $h^2 = (f+g)h = fh + gh = b^2 + s^2$. $\square$

Exercise Prove that Pythagoras's theorem remains true if we replace "square" by "equilateral triangle".

Exercise Prove that it remains true if we replace the squares on the three sides by triangles ABZ, BCX and CAY all similar to one another.



□

Exercise Does Pythagoras's Theorem remain true if we replace the squares by three shapes of any kind, provided they are similar to one another? Why?

10 Similarity implies congruence on the sphere

Theorem 10.1 *On the sphere, similar triangles are congruent.*

Here we do *not* allow degenerate triangles with one or more angles of π: the statement does not hold for these.

To prove 10.1, it is useful to introduce the notion of the *equator with respect to a point* X on the sphere: it is where the equator would be if X were the north pole. More precisely,

Definition 10.2 *Let X be a point on the sphere. The* equator with respect to X *is the great circle obtained by intersecting the sphere with the plane passing through the centre O and orthogonal to OX.*

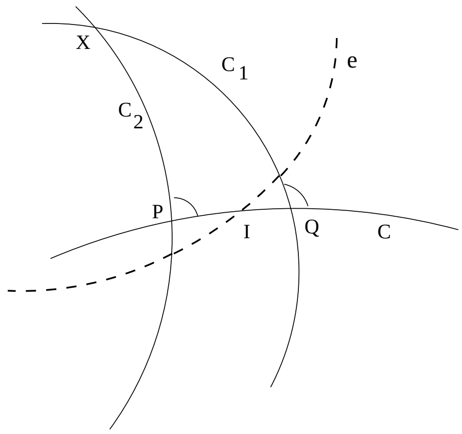
Evidently if X' is the antipode of X then the equator with respect to X' is the same as the equator with respect to X.

Lemma 10.3 *Suppose that two great circles C_1 and C_2 meet a third, C, at the same angle, at points P and Q. Suppose that C_1 and C_2 meet at X and X'. Then*

1. *the equator with respect to X and X' passes through the midpoint of the arc PQ, and*

2. *the great circle bisecting the angle $P\hat{X}Q$ passes through the midpoint of the arc PQ.*

Proof



Let I be the midpoint of the arc PQ and I' its antipode (which is the midpoint of the other arc PQ). Rotation through π about the axis II' interchanges P and Q, and maps C to itself; since the circles C_1 and C_2 meet C at the same angles at P and Q, this rotation also must interchange C_1 and C_2. As the rotation interchanges C_1 and C_2, it also interchanges X and X'; the points it leaves fixed are precisely those lying on the equator with respect to X and X' (E in the diagram). As I and I' are left fixed, they must lie on E.

As the rotation interchanges C_1 and C_2, I must be equidistant from C_1 and C_2; so I lies on the bisector of the angle $P\hat{X}Q$.

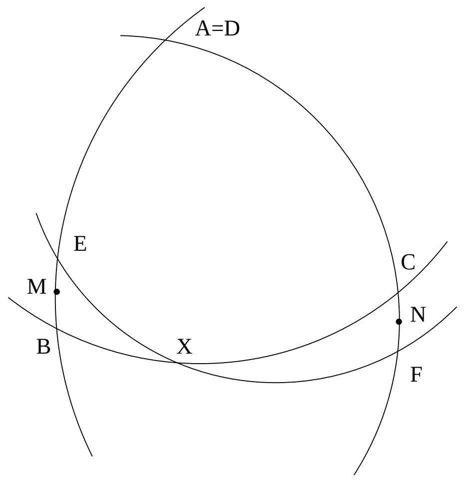
□

Proof of 10.1 If the triangles ABC and DEF have the same angles then they have the same area, by theorem 8.1.

By an isometry we can place D so that it coincides with A, and so that the edge DE is superimposed on AB and DF is superimposed on AC, though we do not assume that the superimposed edges have the same length as the edges they lie on.

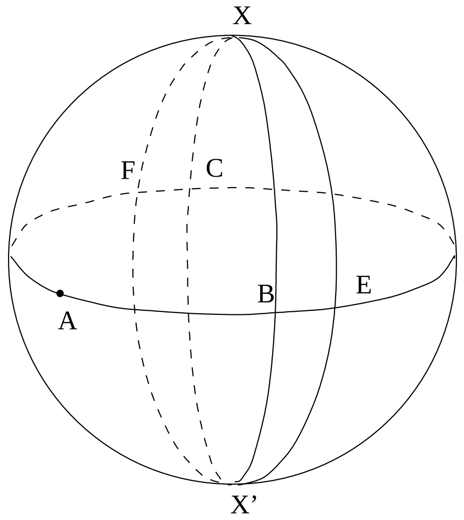
If E coincides with B then the great circles BC continued and EF continued must coincide, as the angles $\hat{B}$ and $\hat{E}$ are equal. It follows that C and F coincide, so the two triangles are coincident.

If E and B do not coincide, suppose that E lies between A and B on the edge AB ("E comes after B along the great circle, starting from A"). Then C must come before F (i.e. lie between A and F); for if C comes after F then the triangle ABC strictly contains the triangle DEF, and as they have the same area this cannot happen. The situation is thus as shown in the following diagram.



By the lemma, the midpoint M of BE lies on the equator e with respect to X, and on the bisector b of $B\hat{X}E$. The same is true of the midpoint N of CF (the great circle bisecting $C\hat{X}F$ also bisects $B\hat{X}E$). Thus M and N must be antipodal (for $e \cap b$ consists of a pair of antipodal points).

As M and N are antipodal, the great circle containing A and M must also contain N, and the great circle containing A and N must also contain M. That is, the edges AB and AC lie on the same great circle. So the angle $B\hat{A}C$ is equal to π. Our triangles ABC and $DEF = AEF$ have degenerated to lunes, as shown in the following diagram.



As we have explicitly excluded this possibility from our hypothesis, this completes the proof. $\square$

Remark 10.4 If we allow lunes as (degenerate) triangles, our theorem is false: in the diagram above, if we slide the vertex A along the horizontal great circle to a new position anywhere between B and C, we leave all of the angles of the triangle ABC unchanged; but evidently by doing this we can obtain triangles ABC no longer congruent to our original ABC *in the sense* *that there is now no isometry taking vertices of one to vertices of the other.*

This is a somewhat curious situation: our theorem is false in the degenerate case. Sometimes theorems remain true in the degenerate case; for example, if we allow a straight line as a circle with infinite radius, then the theorem that through every three points in the plane there passes a circle remains true in the degenerate case (where the points are collinear).

On the other hand, even by bending the meaning of "circle" we cannot arrange for a circle to be tangent to three parallel lines, so the theorem about the circle tangent to three given lines is false in the degenerate case (where the lines are parallel).

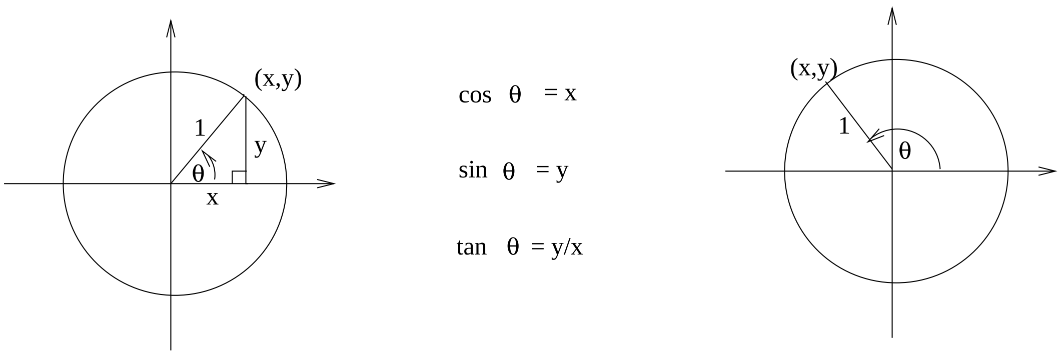
Exercise Look back at theorem 9.3 and see if in fact it remains true in degenerate cases.

11 Plane Trigonometry

If ABC and DEF are plane right-angled triangles, with right angles at B and E, and if the angles $\hat{A}$ and $\hat{D}$ are equal, then the triangles are similar, so

$$\frac{AB}{AC} = \frac{DE}{DF}, \quad \frac{BC}{AC} = \frac{EF}{DF}, \quad \frac{BC}{AB} = \frac{EF}{AF}$$

by 9.3. Thus, each of these ratios depends only on the angle $\hat{A}$. The first is $\cos\hat{A}$, the second $\sin\hat{A}$ and the third $\tan\hat{A}$. The following diagram shows how to extend these definitions to angles not found in a right-angled triangle:



It is clear from this definition that we have

$$\begin{array}{ll} \cos(\theta+\pi) = -\cos\theta & \sin(\theta+\pi) = -\sin\ \theta \\ \cos(\pi-\theta) = -\cos\ \theta & \sin(\pi-\theta) = \sin\ \theta \\ \cos(\theta-\pi/2) = \sin\theta & \sin(\theta-\pi/2) = -\cos\theta \end{array}$$

From Pythagoras's theorem we immediately deduce the equality

$$\cos^2\theta + \sin^2\theta = 1.$$

The other trigonometric functions sec, cosec and cot are defined by

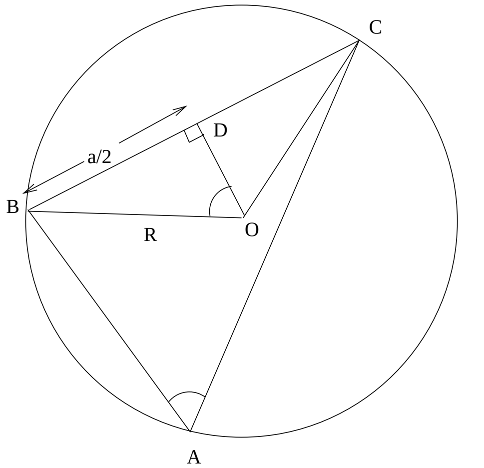
$$\sec\ \theta = 1/\cos\theta, \mathrm{cosec}\ \theta = 1/\sin\theta \quad \text{and} \quad \cot\ \theta = 1/\tan\theta.$$

Proposition 11.1 The sine rule *If ABC is a plane triangle with edges BC, CA and AB of length a, b and c respectively, then*

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} = 2R$$

where R is the radius of the circumcircle.

Proof In the diagram below, $B\hat{O}C = 2\hat{A}$, by Exercise 3.17.



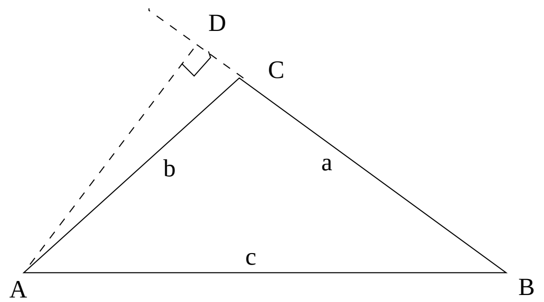
If D is the midpoint of BC then $B\hat{O}D = (1/2)B\hat{O}C = \hat{A}$, and $B\hat{D}O = \pi/2$. Hence $a/2 = BD = R\sin\ B\hat{O}D = R\sin\ \hat{A}$, and thus $a = 2R\sin\ \hat{A}$. Similarly $b = 2R\sin\ \hat{B}$ and $c = 2R\sin\ \hat{C}$, and the proposition follows. □

I leave you to draw the appropriate diagram if the centre O of the circumcircle lies outside the triangle, and to check that essentially the same argument works.

Proposition 11.2 The cosine rule *If ABC is a plane triangle then*

$$c^2 = a^2 + b^2 - 2ab\cos\ \hat{C}.$$

Proof Drop a perpendicular from A to the edge BC (continued if necessary, as in the diagram here), and let it meet it at D.



Then by Pythagoras

$$c^2 = (a + b\cos\ D\hat{C}A)^2 + (b\sin\ D\hat{C}A)^2.$$

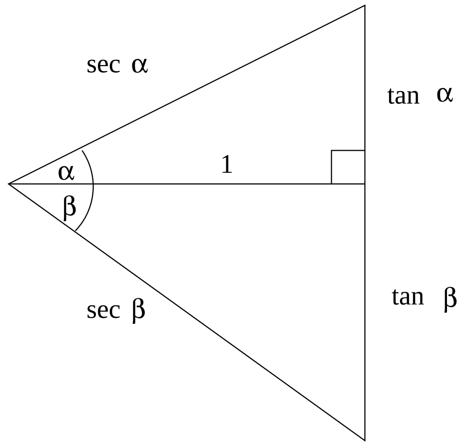
Since $\sin D\hat{C}A = \sin \hat{C}$ and $\cos D\hat{C}A = -\cos \hat{C}$, the proposition follows. □

Again, a different diagram is required in case $\hat{C} < \pi/2$.

Proposition 11.3 Addition Formulae

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta.$$

Proof For angles $\alpha, \beta \in [0, \pi/2)$, we apply the cosine rule to the following diagram,



obtaining

$$(\tan\alpha + \tan\beta)^2 = \sec^2\alpha + \sec^2\beta - 2\sec\alpha\sec\beta\cos(\alpha + \beta).$$

Multiplying thoughout by $(\cos\alpha\cos\beta)^2$ we get

$$\sin^2\alpha\cos^2\beta + 2\sin\alpha\sin\beta\cos\alpha\cos\beta + \sin^2\beta\cos^2\alpha = \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha + \beta).$$

Grouping together the terms with factor $\cos^2\alpha$ and the terms divisble by $\cos^2\beta$ and using $\cos^2 + \sin^2 = 1$, this becomes

$$2\sin\alpha\sin\beta\cos\alpha\cos\beta = 2\cos^2\alpha\cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha + \beta)$$

from which the formula for $\cos(\alpha + \beta)$ follows by division by $\cos\alpha\cos\beta$.

The formula for $\sin(\alpha + \beta)$ can be deduced, somewhat more easily than the formula for $\cos(\alpha + \beta)$, by applying the sine rule to the same diagram.

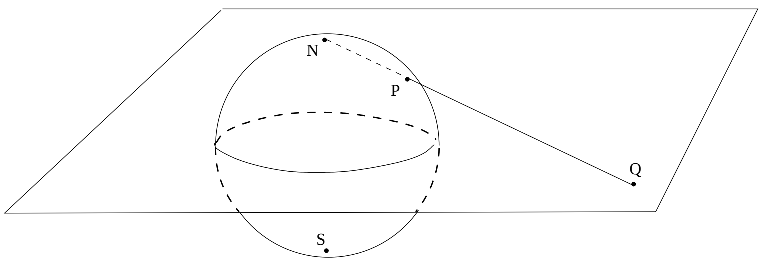
To extend these formulae to values of α and β outside the range $[0, \pi/2)$, use the formulae for $\cos(\theta - \pi/2), \sin(\theta - \pi/2)$ which followed the definition of cos and sin. For example, if $\alpha \in [\pi/2, \pi)$ and $\beta \in [0, \pi/2)$ then

$$\cos(\alpha+\beta) = -\sin(\alpha-\pi/2+\beta) = -\sin(\alpha-\pi/2)\cos\beta - \cos(\alpha-\pi/2)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

□

12 Stereographic Projection

Because it is not possible to draw an accurate map of any portion of the earth's surface on a plane sheet of paper, all sorts of compromises have been used by different cartographers. For political purposes, it may be convenient to have a map which accurately represents the areas of different countries, and in fact this can be achieved by *cylindrical projection*, shown at the end of this section. For the purposes of navigation, a map which preserves angles is also useful, and we now investigate one of several maps which do this.



Stereographic projection maps the sphere S^2 minus the North pole N onto the plane of the equator, as shown in the diagram. Each point $P \in S^2 \setminus N$ is mapped to the point Q in the plane where the straight line NP continued meets it. We will denote this projection by $s_N : S^2 \setminus N \to \text{Plane}$.

Exercise Sketch the image under s_N of each of the following sets in S^2:

1. a line of longitude (i.e. a great circle passing through both North and South poles);
2. a line of latitude (a circle made by intersecting the sphere with a horizontal plane);
3. the southern hemisphere;
4. the northern hemisphere;
5. a circle which is neither a line of longitude nor a line of latitude (here you may have to guess).

Exercise 12.1 Find a formula for $s_N(x, y, z)$, assuming that the radius of the sphere is 1, and a formula for its inverse $s_N^{-1} : \mathbb{R}^2 \to S^2 \setminus N$.

The map s is a bijection from $S^2 \setminus N$ to the plane, and it is easy to see the inverse map s^{-1} in the same diagram: if Q is a point in the plane, then $s^{-1}(Q)$ is the point P where the line joining Q and N meets $S^2 \setminus N$.

Exercise Find a formula for $s^{-1}(u, v)$, again assuming that the radius of the sphere is 1.

Definition 12.2 A map f from a surface S_1 to another surface S_2 is *conformal* if it preserves angles. That is, f is conformal if whenever C_1 and C_2 are curves in S_1, meeting at the point $P \in S_1$ at an angle of θ, then $f(C_1)$ and $F(C_2)$ meet at an angle of θ at $f(P)$.

In fact the angle between two curves at a point means the angle between their tangents at that point, and for a reasonable mapping the tangent to $f(C_1)$ at $f(P)$ depends only on the tangent to C_1 at P (and not on the choice of curve C_1 having this given tangent). By "reasonable" I mean a map with the following two properties:

1. If C_1 is a curve in S with a well-defined tangent at P then $f(C)$ is a curve with a well-defined tangent at $f(P)$, and

2. If C_1 and C_2 have the same tangent at P then $f(C_1)$ and $f(C_2)$ have the same tangent at $f(P)$.

In courses in calculus in several variables you will come across quite simple conditions which guarantee these two properties.

We have already met some examples of conformal maps: 3.4 says that an isometry of the plane is conformal; 9.11 says that a similarity of the plane is conformal.

Theorem 12.3 *Stereographic projection is a conformal map.*

To prove this we will need a couple of preliminary steps.

Lemma 12.4 *If two circles in $\mathbb{R}^3$ meet at points P_1 and P_2, then the angle between their tangents at P_1 is the same as the angle between their tangents at P_2.*

Proof Let M be the plane passing through the centres of the two circles and the midpoint of P_1P_2. Then reflection in M maps each circle to itself (for the intersection of M with the plane of either circle is a diameter of that circle) and interchanges P_1 and P_2. It thus interchanges the two angles in question. As reflection is an isometry, the two angles must be equal. $\square$

Lemma 12.5 *Let H and H' be two parallel planes in $\mathbb{R}^3$, and let M_1 and M_2 be two further planes, parallel neither to each-other nor to H. Then the angle between the lines $H \cap M_1$ and $H \cap M_2$ is equal to the angle between the lines $H' \cap M_1$ and $H' \cap M_2$.*

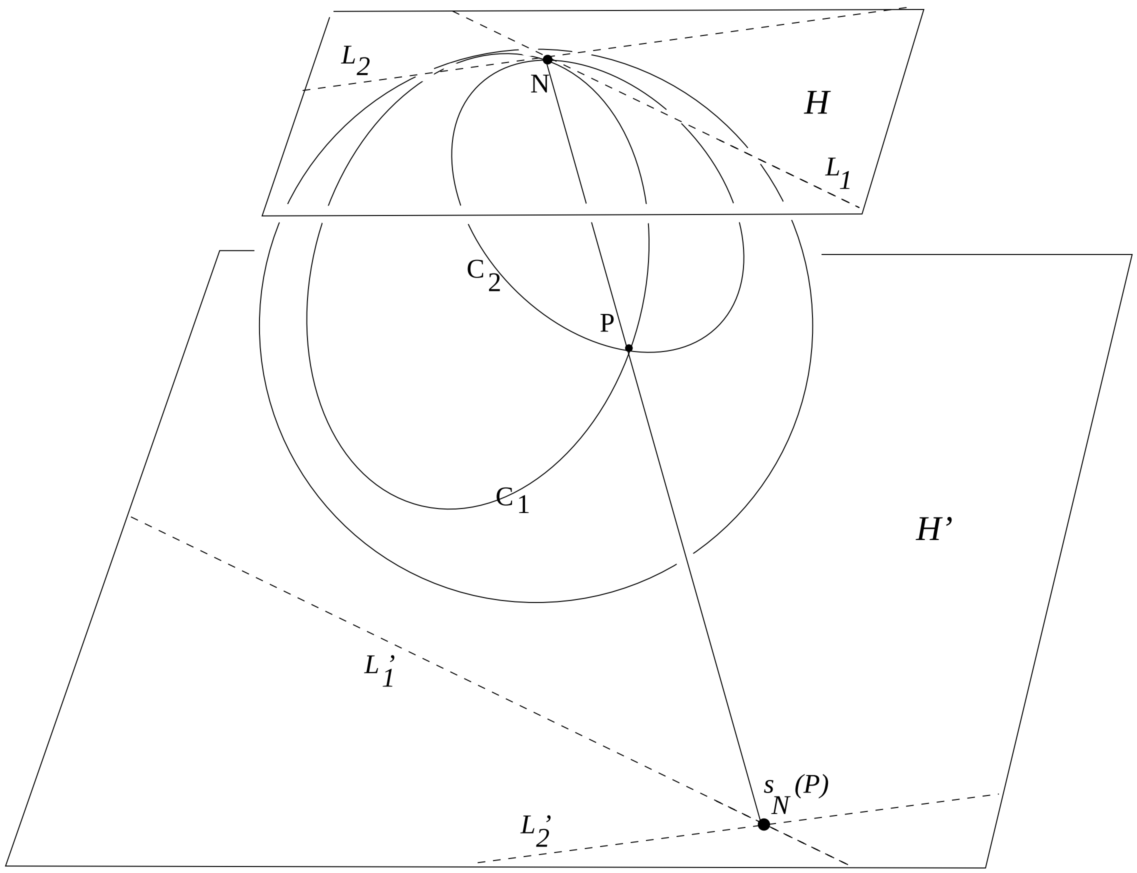
Proof Let $P = H \cap M_1 \cap M_2$, and $P' = H' \cap M_1 \cap M_2$. Let w be the vector PP'. Translation T_w by w is an isometry of $\mathbb{R}^3$, and therefore preserves angles. Now T_w takes H to H', and maps M_1 and M_2 to themselves. Hence it maps $H \cap M_1$ to $H' \cap M_1$ and $H \cap M_2$ to $H' \cap M_2$. Equality of the angles now follows. $\square$

Proof of 12.3 We have to show that if C_1 and C_2 are two curves in $S^2 \setminus N$ meeting at P, then the angle between $s(C_1)$ and $s(C_2)$ at $s_N(P)$ is the same as the angle between C_1 and C_2 at P. Stereographic projection is a reasonable map in the sense I described after the definition of the term "conformal", so to check that it is conformal, we can choose the curves C_1 and C_2 as we wish. Let $P \in S^2 \setminus N$, and let C_1 and C_2 be *circles* on the sphere, passing through P and N. We can indeed choose C_1 and C_2 passing through P and N, and having any tangent lines that we please at P: given lines ℓ_1 and ℓ_2 tangent to the sphere at P, let M_1 be the plane containing ℓ_1 and passing through N, and let $C_1 = M_1 \cap S^2$; similarly C_2 is the intersection with S^2 of a plane containing ℓ_2 and passing through N.

The circles C_1 and C_2 meet at P and also at N. Let H be the horizontal plane passing through N; it is tangent to the sphere there. Then the lines $L_1 = M_1 \cap H$ and $L_2 = M_2 \cap H$ are tangent to C_2 and C_2 at N, and thus the angle between C_1 and C_2 at N is the angle between L_1 and L_2.

Let H' be the equatorial plane, onto which s_N maps the sphere. H' is of course parallel to H.

The following diagram shows the planes H and H', and two circles involved in our construction, except that instead of drawing H' as the equatorial plane I have drawn it lower down. The reason for this is that my drawing with H' in the correct position became impossibly messy. For the same reason I have not drawn the planes M_1 and M_2. However, in fact it makes no difference to the truth of the theorem, or its proof, which horizontal plane we project onto.



Let $L_1' = M_1 \cap H'$ and $L_2' = M_2 \cap H'$. Then L_1' and L_2' are in fact *equal* to $s_N(C_1)$ and $s_N(C_2)$ respectively (why?).

By Lemma 12.5, the angle between L_1' and L_2' at $s_N(P)$ is equal to the angle between L_1

and L_2 at N, which is the angle between C_1 and C_2 at N.

By Lemma 12.4 the latter angle is equal to the angle between C_1 and C_2 at P. This proves that the angle between $s_N(C_1)$ and $s_N(C_2)$ at $s_N(P)$ is equal to the angle between C_1 and C_2 at P. $\square$

Exercise Why are L_1' and L_2' equal to $s_N(C_1)$ and $s_N(C_2)$ in the proof we've just finished?

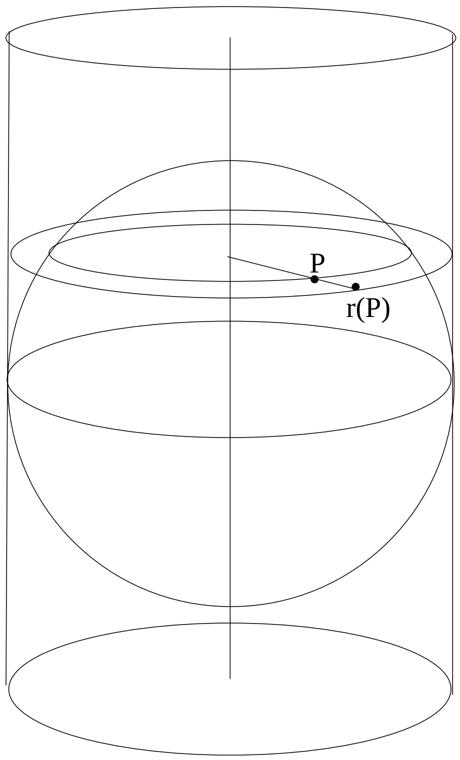
Exercise Stereographic projection $s_S : S \backslash \text{South pole} \to \mathbb{R}^2 = \text{equatorial plane}$ can be defined by the same technique as used for s_N. What is the domain of $s_N \circ s_S^{-1}$? Show that for $(x, y) \in \mathbb{R}^2$,

$$s_N \circ s_S^{-1}(x, y) = \frac{(x, y)}{x^2 + y^2}.$$

Hint: there is a simple proof involving similar triangles and no calculation. **Exercise** Describe the map $s_N^{-1} \circ s_S$.

Exercise The image under stereographic projection of any circle on the sphere is a circle or a line in the plane. Prove this by using the co-ordinate formulae for s_N and its inverse that you obtained in Exercise 12.1. Can you find a geometric proof, in the spirit of the proof of 12.3 — one, that is, that avoids the use of co-ordinates? (I should admit that I don't know of one).

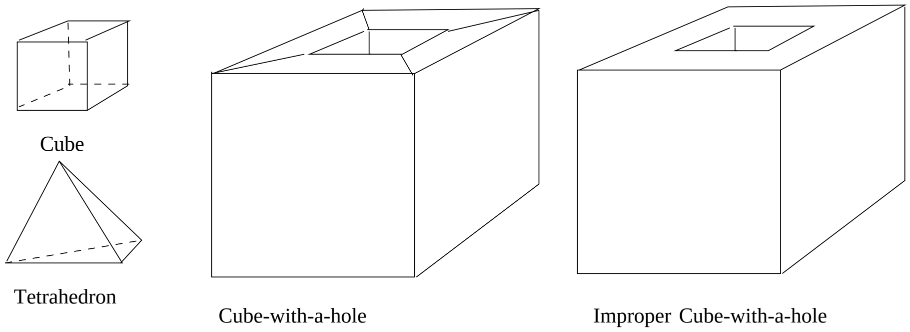
Stereographic projection of course distorts area considerably; for example, regions close to the north pole have images with greatly exaggerated area. The projection shown below maps each region of the sphere to a region of equal area. Proof of this fact can be found, for example, in *Modern Geometry with Applications*, by George Jennings, which is also the source of the proof of theorem 12.3 given above.



13 Polyhedra

A subset R of $\mathbb{R}^2$ or $\mathbb{R}^3$ is *convex* if whenever points P and Q lie in R then the line segment PQ joining them also lies in R. A disc, a square and a (solid) ball are convex; an annulus and a (hollow) sphere are not.

A *polyhedron* is a surface made up of convex plane polygonal regions joined two-by-two at their edges.



The top of the cube-with-a-hole (and the bottom, though you cannot see it) in the picture here are divided up by diagonal lines into convex regions; the same figure without these lines is not a polyhedron, at least according to our definition.

A polyhedron is *convex* if the region R of space it encloses is convex. In the picture above, the cube and tetrahedron are convex while the cube-with-a-hole is not.

A *face* of a polyhedron is one of the convex plane polygons making it up. An *edge* is a straight-line segment along which two faces meet. A *vertex* is a point where two or more edges meet. In the picture above, the number of edges, faces and vertices are as follows:

polyhedron	F	E	V
cube	6	12	8
tetrahedron	4	6	4
cube-with-hole	16	32	16.

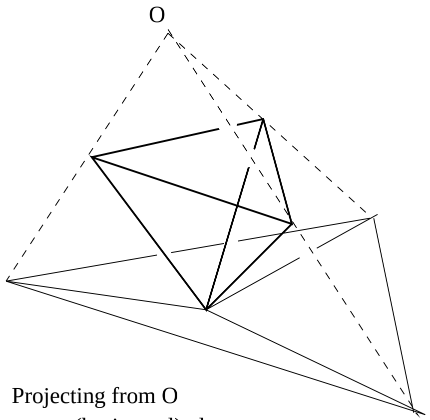
The alternating sum $F - E + V$ is equal to 2 for both the cube and the tetrahedron. This is in fact a property of every convex polyhedron, a fact first noticed, apparently, by the Swiss mathematician Leonhard Euler.

Theorem 13.1 *For every convex polyhedron, $F - E + V = 2$.*

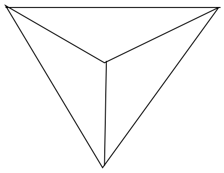
We will give two proofs. The first is similar in spirit to Euler's original proof, given in 1751; the second is due to Legendre, and was published in 1794. **Proof 1** We need

Plane diagrams of convex polyhedra

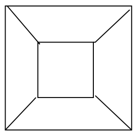
If a convex polyhedron is viewed in perspective frm a point O just outside the centre of one of its faces, then the polygon bounding this face appears as a large plane polygon, and all the remaining edges and vertices appear inside it. Technically, this view is obtained by projecting the polygon from O to a plane beyond the polygon.



Projecting from O
onto a (horizontal) plane



plane diagram of tetrahedron



plane diagram of cube

Let the face nearest to O be F_0. There are bijections

faces of the polyhedron other than F_0	$\leftrightarrow$	bounded regions of the plane diagram
edges of the polyhedron	$\leftrightarrow$	edges of the plane diagram
vertices of the polyhedron	$\leftrightarrow$	vertices of the plane diagram.

(F_0 itself corresponds to the whole of the plane diagram.)

Denote the number of bounded regions, edges and vertices in the plane diagram by f, e and v. In view of the bijections above, we have to show $f - e + v = 1$.

Lemma 13.2 *Let Y be a connected plane graph (that is, a finite collection of vertices, some of which are joined by edges, with the property that from any vertex one can get to any other by travelling along edges. Both ends of any edge are at vertices. Edges do not cross each-other). Let f be the number of bounded regions in the complement, e the number of edges, and v the number of vertices. Then*

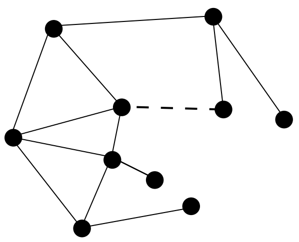
$$f - e + v = 1.$$

The following argument is more or less Euler's. Although convincing, by modern standards it leaves a lot to be desired, and we will look at some of its weaknesses in a moment.

Proof We use induction.

Begin with a single vertex and no edges. Then $f = 0, e = 0$ and $v = 1$, so $f - e + v = 1$, and the equation holds in this case.

The graph Y is obtained from this initial graph by a sequence of steps of the following two types:

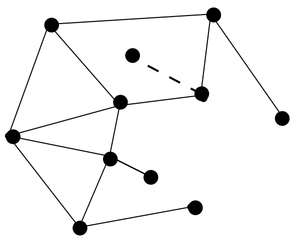


Type 1

Draw a new edge (shown here with a dotted line) joining two old vertices.

f increases by 1
e increases by 1
v is unchanged

So f-e+v is unchanged



Type 2

Draw a new edge (shown here with a dotted line) joining an old vertex and a new one.

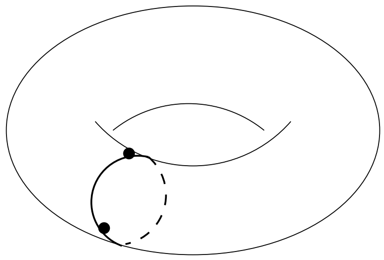
f is unchanged
e increases by 1
v increases by 1

So f-e+v is unchanged

Since $f - e + v = 1$ holds at the start, it holds at the finish. $\square$

Euler's theorem follows immediately From the Lemma. $\square$

Remark 13.3 The problem with this proof is that it gives no formal argument why f increases by 1 in Step 1, and is unchanged in Step 2. It may seem obvious that it does change in this way, but consider for a moment the analogous assertions on the surface of a torus (doughnut): as the diagram shows, the assertion in Step 1 is false.



Here, on adding the dotted edge, the number of regions into which the torus is divided is unchanged. So the fact that in Step 1 f *does* increase by 1 for *plane* graphs, is somehow a property of the plane, which it does not share with other surfaces. But how to prove it? In fact the assertion of Step 1 is equivalent to the following statement:

Theorem 13.4 Polygonal Jordan curve theorem *Any simple closed plane polygon separates the plane into two regions.* □

Here "simple" means one that does not cross itself. The fact that this statement (with "curve" in place of "polygon") needed a proof was first realised by the French mathematician Camille Jordan, and he gave what turned out to be an incorrect proof. A correct proof was later given by Osvald Veblen in 190?. The "obviousness" of the theorem comes from the fact that it is so strongly confirmed by our experience; again we see the difficulties involved in studying geometry: the fact that it makes statements about things that we see and care about means that we bring to it a host of often unnoticed assumptions. Modern mathematics, and modern school mathematics, have made radically different responses to this difficulty. Mathematicians have created the subject of topology, which provided the precise definitions and techniques needed to prove a statement like the Jordan curve theorem, and which has enormously enriched and influenced the sciences as well as mathematics; school mathematics syllabuses in Britain have simply given up doing geometry.

Second proof of Euler's formula Imagine the convex polyhedron placed inside a sphere, and imagine a light shining from inside the polyhedron. Thw shadows of its edges form a polygonal subdivision of the sphere. Replacing each edge of this subdivision of the sphere with a segment of great circle, we obtain a subdivision of the sphere into F geodesic polygons. The area of each polygon is the sum of its angles, minus $(n-2)\pi$ (Exercise 8.3), where n is the number of its edges. The sum of all these areas can be separated out into

1. the sum of all of the angles in all of the polygons; this is equal to $2\pi V$, since the angles around each vertex of the subdivision add up to 2π.

2. the sum of all the negative contributions of $(n-2)\pi$. Each edge of the subdivision belongs to two of the spherical polygons, so the total number of edges of all of the

spherical polygons together is $2E$. That is, all the n's add up to $2E$. Since we have a -2π for each face, the sum of all of the $(n-2)\pi$ is therefore equal to $(2E-2F)\pi$.

The area of the sphere is 4π; it follows that

$$4\pi = \sum \text{areas of polygons} = 2\pi V - \sum(n-2)\pi = 2\pi V - 2\pi E + 2\pi F,$$

and from this Euler's formula $F - E + V = 2$ is immediate. □

Consequences of Euler's formula

Proposition 13.5 *1. There is no convex polyhedron with 7 edges.*

2. In every convex polyhedron there is at least one face with $3, 4$ *or* 5 *edges.*

3. In every convex polyhedron there is at least one vertex from which emanate $3, 4$ *or* 5 *edges.*

To prove these, we make some elementary observations:

1. Every polygon has at least 3 edges. If we count the edges around each face of a polyhedron, and then add together the F numbers we get (one for each face), then we get $2E$, since each edge is counted twice. Thus,

$$2E = \text{sum of } F \text{ numbers, each at least } 3 \ \geq 3F.$$

2. At each vertex of any polyhedron, at least 3 edges must meet. Again, if we count the number of edges at each vertex and add the totals, we get $2E$, since each edge is counted twice. Thus, $2E = $ sum of V numbers, each at least $3 \ \geq 3V$.

I leave the remainder of the proof of 13.5(1),(2) and (3) as an
Exercise. Hint for (2): let F_3 be the number of 3-sided faces, F_4 the number of 4-sided faces, etc. Then $F = F_3 + F_4 + \cdots$. How is E related to $F_3, F_4, \cdots$?

As a second application of Euler's formula, we classify and describe the regular polyhedra, or *Platonic solids.*

Regular Polyhedra

What is a regular polygon? Everyone knows the answer: one in which all the interior angles are equal, and all the edges are of the same length. What about a regular polyhedron? By analogy with the definition of regular polygon, we might ask that

- every face be the same as every other,

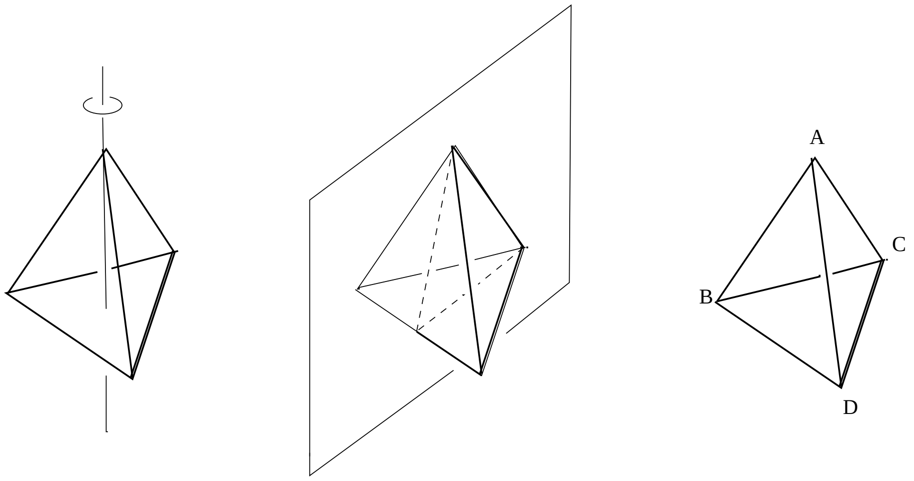
and then impose some requirement on the way that the faces meet. Or we might add to the requirement that the faces all be the same,

- all the edges must have the same length, and perhaps that

- the same number of edges meet at each vertex (so "every vertex is the same as every other").

But personally I am not convinced that these requirements will guarantee what I intuitively understand by "regular". My intuition seems to be based more on the idea that the polyhedron should "look the same" from a number of different viewpoints, though *a priori* it is not clear quite how many, or which. One way of formalising the notion of "looking the same", is to introduce the idea of the *isometries of ambient (3-dimensional) space which map the polyhedron to itself.* If a polyhedron is regular, there are a lot of these. They are known, for short, as *isometries of the polyhedron.*

Example 13.6 Consider the "regular tetrahedron", shown below.



Rotation about the axis shown, through an angle of $2\pi/3$ and $4\pi/3$, and reflection in the plane shown, each map the tetrahedron to itself. Of course, there are other axes of rotation and planes of reflection.

Exercise Find isometries of the labelled tetrahedron interchanging

1. vertices A and B
2. edges AB and CD
3. faces ABC and BCD

How many possible answers are there in each case?

Definition 13.7 A polyhedron P is *regular* if

1. for each pair F_1, F_2 of faces of P there is an isometry of P taking F_1 to F_2;
2. for each pair E_1 and E_2 of edges of the same face F of P there is an isometry of P sending F to itself and taking E_1 to E_2.

3. for each edge E of P, there is an isometry of P mapping E to itself and interchanging the two ends of E.

Exercise Show that if a polyhedron is regular then for any two edges E_1 and E_2 of P there is an isometry of P taking E_1 to E_2. Ditto with vertices.

Exercise Show that if a polygon is regular then each of its faces is a regular polygon.

Exercise Can a regular tetrahedron be situated inside a cube so that its vertices are vertices of the cube?

Exercise Must a regular polyhedron be convex?

Classification of regular polyhedra

There are infinitely many different regular polygons, one for each $n \geq 3$: the equilateral triangle, the square, the regular pentagon, ... For polyhedra the situation is very different: there are only five, the tetrahedron, cube, octahedron, dodecahedron and icosahedron. We will now prove a theorem showing that these are the only five possibilities, based on an analysis of the number of faces, edges and vertices.

If a polyhedron is regular then all its faces are regular polygons, each one with, say, p edges. Moreover, at each of its vertices the same number of edges, say q, meet. Suppose that our polyhedron has F faces, E edges and V vertices. Collecting all the edges together, one might think that there are qV of them, since q edges meet at each vertex. However, each edge is counted twice in this enumeration, for each belongs to 2 vertices. Thus, we have

$$E = qV/2.$$

Similarly, as each edge belongs to 2 faces, there are not pF but $pF/2$ edges:

$$E = pF/2.$$

Thus,

$$\frac{V}{1/q} = \frac{E}{1/2} = \frac{F}{1/p}$$

and each of these ratios is equal to

$$\frac{V - E + F}{1/q - 1/2 + 1/p}$$

(see the algebraic remark in the proof of Ceva's theorem, 7.4).

Proposition 13.8 *A regular polyhedron is convex (and thus $F - E + V = 2$).*

Thus, the expression we had before reduces to

$$\frac{2}{1/q - 1/2 + 1/p}$$

which, clearing denominators, becomes

$$\frac{4pq}{2p + 2q - pq}.$$

So

$$V = \frac{4p}{2p + 2q - pq} \quad E = \frac{2pq}{2p + 2q - pq} \quad F = \frac{4q}{2p + 2q - pq}.$$

As p, q, V, E and F must all be positive integers, the possible values of p and q are rather limited. We have $2p + 2q - pq > 0$, or equivalently $(p-2)(q-2) < 4$. But p and q themselves must be greater than 2, so $p - 2$ and $q - 2$ are positive integers whose product is less than 4. The table below lists all the possibilities; in the table d is the number $2p + 2q - pq$.

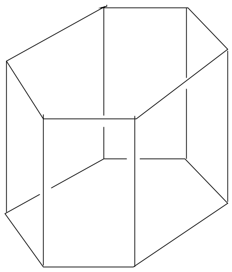
$\mathbf{p-2}$	$\mathbf{q-2}$	**p**	**q**	**face**	**d**	**F**	**E**	**V**	**name**
1	1	3	3	triangle	3	4	6	4	tetrahedron
1	2	3	4	triangle	2	8	12	6	octahedron
2	1	4	3	square	2	6	12	8	cube
3	1	5	3	pentagon	1	12	30	20	dodecahedron
1	3	3	5	triangle	1	20	30	12	icosahedron

It is slightly premature to give names to these regular polyhedra - we have not yet shown that there really do exist regular polyhedra with the requisite values of p and q. All we have done is find out when the combinatorial data allows the possibility. In principle there might be other constraints which we have not yet thought of. So we must find a way of constructing them. Following Coxeter, *Introduction to Geometry*, we now give a construction for each one, but *we do not show that the polyhedra we construct really do satisfy the criteria of our definition.* In some cases, such as the cube and the tetrahedron, this is easy. In others, notably the dodecahedron and the icosahedron, it is harder. What is evident from the construction is that in every case the polyhedron we obtain has all its faces congruent to one another, and the same number of edges meeting at each vertex. It is not evident, though it is true, that the polyhedron constructed meets the more stringent demands of the definition.

Constructing the regular polyhedra

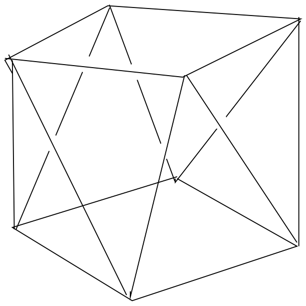
Let P be a regular n-gon lying in the plane. Let A be a point vertically above its centre, and join each vertex of P by a straight line to A. We obtain a pyramid with apex A and base P, and whose other faces are isosceles triangles. By adjusting the height of A above the plane containing P, the triangles can be made equilateral. When $n = 3$, so P is an equilateral triangle, we obtain the **regular tetrahedron**.

Again consider a regular n-gon P lying in a horizontal plane. Situate a copy P' of P in another horizontal plane in 3-space, with each vertex lying vertically above a corresponding vertex of P. By joining corresponding vertices, we obtain a prism whose vertical faces are rectangles.



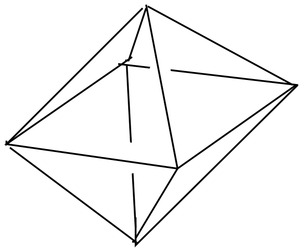
When P is a square, by adjusting the height of P' above P we can make each of the vertical faces a square also. We have obtained the **cube**.

Now rotate P' through π/n – one half of the angle subtended at its centre by one of the edges. The vertices of P' now do not lie vertically over the vertices of P, but by joining each vertex of P' to the two nearest vertices of P, we obtain a "twisted prism" — a polyhedron with P' and P as top and bottom faces, and whose remaining faces are isosceles triangles.



In this diagram, P and P' are squares.

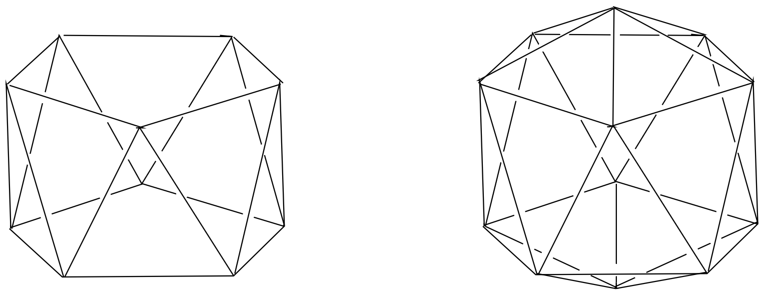
By adjusting the height of P' above P, these triangles can be made equilateral. When P itself is an equilateral triangle, we obtain in this way the **octahedron**.



The diagram of course suggests another construction for the octahedron, as the union of two pyramids with square bases.

Exercise Draw the octahedron following the method of construction described above.

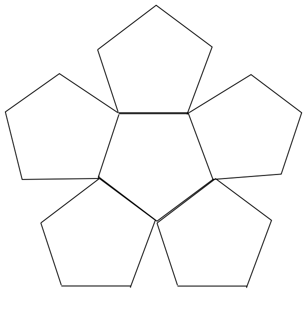
Suppose we have constructed a twisted prism, as in the previous paragraph, with base P a regular pentagon. Now construct a pyramid as described in the first paragraph, on top of the pentagon P', with its other faces equilateral triangles. Do the same with the pentagon P, only now with the apex of the pyramid below P. We have constructed a polyhedron with 20 faces, each one an equilateral triangle, the **icosahedron**. The diagram shows the two stages in the construction: on the left the twisted prism, on the right the icosahedron.



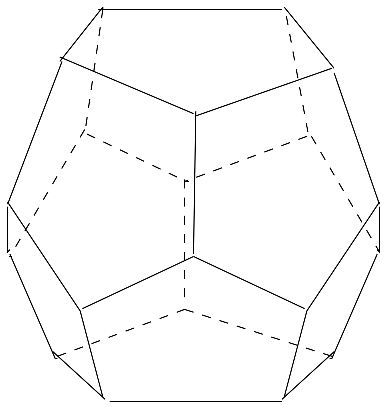
Exercise Show, without using the classification of regular polyhedra, that if P is a regular n-gon with $n \neq 5$ then the polyhedron we obtain by following this recipe is not regular.

Exercise Make a freehand drawing of the icosahedron, following the method of construction described here, and without copying the drawing here.

It remains only to construct the dodecahedron. Make two pentagonal bowls, each consisting of a regular pentagon surrounded by five other regular pentagons. With one upside down, the two bowls fit perfectly together to form a regular dodecahedron. It is possible to make a cardboard model by constructing two plane "nets" each consisting of a regular pentagon with five others joined to it:



Place one net on top of the other so that the vertices of the outer pentagons alternate, and run a string or elastic band round the edge so that it passes alternately over and under the protruding vertices. When the string is pulled tight, the model rises up out of the plane as a perfect dodecahedron.



To make an accurate model it is necessary to construct a regular pentagon with precision. Here is one construction. I leave you as an exercise to prove that it works. I used this construction on the computer screen, in the drawing package *xfig*, for the pentagon at the centre of the net shown above.

Draw a line-segment AB of length s. AB will be one of the edges of the regular pentagon. Construct a perpendicular segment BP of length $s/2$. The side of AB on which P lies will be referred to as "below AB". Draw the line AP and extend beyond P. Draw an arc, centre P, radius $s/2$. This crosses the line AP continued at two points. Denote by Q the further from A of these two, and let $AQ = d$. raw arcs centred at A and B of radiDi s and d. The point below AB where the two arcs of radius d cross, and the two points below AB where the arcs of radius d cross the arcs of radius s, are the remaining three vertices of the pentagon.

Exercise Prove that this construction does indeed give a regular pentagon. Hint: use Exercise 9.8.

References

These notes owe a considerable debt to the following three books:

1. H.S.M. Coxeter, *Introduction to Geometry*, second edition, Wiley Classics Library, John Wiley and Sons, 1989.

   This is a remarkable compendium, a classic of the subject. Unfortunately Coxeter places brevity and elegance at the top of his scale of mathematical virtues, so his book is not always easy to follow.

2. George A. Jennings, *Modern Geometry with Applications*, Springer-Verlag 1994.

   This is a great book, more fun to read than Coxeter, though considerably less encyclopaedic.

3. Elmer Rees, *Notes on Geometry*, Springer-Verlag 1983.

   Another excellent book, though aimed at readers with a lot more mathematical background than the other two books.