There is a prize of £10 for the best solution to Exercise 11 received (in my (Jeremy Gray's) pigeon-hole in the staff/graduate common room) by Friday of week 7.

Introduction to Geometry Exercises 5
Section A

1. (i) List the isometries of a regular tetrahedron fixing a given vertex.
(ii) Ditto for a cube.

2. Show that in a regular polyhedron with p-gonal faces, in which q edges meet at each vertex, the number of edges is given by

$$E^{-1} = p^{-1} + q^{-1} - 1/2.$$

3. (i) What are the convex subsets of $\mathbb{R}$ called?

(ii) Show that the intersection of any collection (even an infinite number) of convex subsets of 3-space is itself convex. Deduce that for every subset X of 3-space, there is a (unique) smallest convex set containing X. It is called the *convex hull* of X. Hint: consider the collection of *all* convex subsets of $\mathbb{R}^3$ containing X.

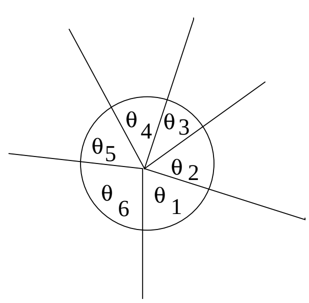
(iii) What is the convex hull of X if X consists of 2 points? of three non-collinear points? of 4 non-coplanar points?

(iv) Show that if X is a convex solid in $\mathbb{R}^3$ then any straight line meeting its interior must meet its surface at exactly two points. Hint: use (i).

4. Show that if P is a regular polyhedron (Definition 13.6 on page 43 of the Lecture Notes) and if E_1 and E_2 are any two edges of P, then there is an isometry of P taking E_1 to E_2. Ditto for any two vertices. Hint: use the definition.

Section B

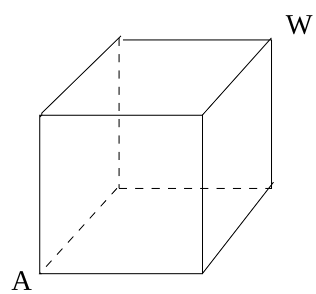
5. The inequality $(p-2)(q-2) < 4$ for a regular polyhedron with p-gonal faces and q edges meeting at each vertex, which we deduced from Euler's formula $F - E + V = 2$ in the lectures, can also be deduced from a consideration of the angle-sum at each vertex, as follows:



In the picture here, the sum of the angles at the vertex is 2π. Now imagine the vertex moving out of the plane of the page, leaving the other end of each edge resting on the page. Then each angle θ_i diminishes. This is exactly the situation at each vertex of a convex polyhedron. So the angle-sum at that vertex is less than 2π. Use this fact to deduce that in a regular polyhedron in which each face has p edges, and q edges meet at every vertex, then $(p-2)(q-2) < 4$.

6. (i) Draw a regular tetrahedron situated inside a cube so that its vertices are vertices of the cube.
In the rest of this exercise, you do not have to justify your answers, but you do have to state them clearly, e.g. by labelling the vertices of the cube.
(ii) Choose two faces of the tetrahedron, and now find an isometry of the cube which maps the tetrahedron to itself, and which maps one of the chosen faces to the other.
(iii) Choose two edges of one of the faces of the tetrahedron and find an isometry of the cube which maps the tetrahedron to itself, which maps the chosen face to itself, and which maps one edge to the other.
(iv) Choose an edge of the tetrahedron, and now find an isometry of the cube which maps the tetrahedron to itself, maps the chosen edge to itself, and interchanges its two ends.

7. (i) Imagine a plane passing middway between the opposite vertices A and W of a cube, and at right angles to the line AW. By marking points where this plane cuts the edges of the cube, or otherwise, make a careful drawing of the intersection of this plane with the surface of the cube. Now describe this intersection precisely, justifying your assertions.



(ii) What happens as this plane is moved towards A, remaining at all times parallel to its original position? Draw a sequence of pictures showing the different shapes of intersection of the plane with the cube.

8. Let $s_N : S^2 \setminus \{N\} \to \mathbb{R}^2$ and $s_S : S^2 \setminus \{S\} \to \mathbb{R}^2$ be stereographic projections, as described in Section 12 of the Lecture Notes. Careful! — this means, in which we project to the equatorial plane, as shown in the diagram on page 39. In my *lecture* on stereographic projection I used projection to the horizontal plane tangent to the sphere at the south pole. (In fact, as far as the proof that stereographic projection is conformal is concerned, it makes no difference which horizontal plane we project to.)

(i) Show that $s_S^{-1} \circ s_N$ takes $P = (x, y, z) \in S^2$ to $(x, y, -z)$. Hint: Make a drawing! And note that $s_S(P)$ and $s_S^{-1}(s_N(P))$ both lie in the vertical plane containing P and the centre of the sphere.

(ii) Show that $s_N \circ s_S^{-1}$ is inversion i_C in the unit circle C (the intersection of the sphere and the plane). (cf. Example 3.1 (5) in the Lecture Notes). Bear in mind the hint for (i), and draw a clear diagram.
(iii) Deduce that i_C is a conformal map.

(iv) Show also that the map on the complex plane $f(z) = 1/z$ is (very nearly) the same thing as inversion in the unit circle; in fact, find the precise relation between f and i_C, and use it to deduce from (iii) that f is conformal.

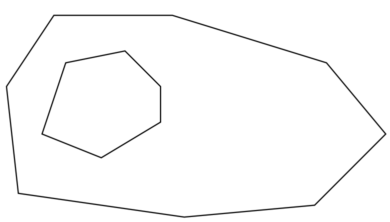
(v) Show that $i_C \circ s_N = s_N \circ u$, where u is the map $S^2 \to S^2$ given by $u(x, y, z) = (x, y, -z)$. (The equality $i_C \circ s_N = s_N \circ u$ is often expressed by saying that the diagram

$$\begin{array}{ccc} S^2 \setminus \{N, S\} & \xrightarrow{s_N} & \mathbb{R}^2 \setminus \{O\} \\ u \downarrow & & \downarrow i_C \\ S^2 \setminus \{N, S\} & \xrightarrow{s_N} & \mathbb{R}^2 \setminus \{O\} \end{array}$$

is "commutative" — both possible routes ($s_N \circ u$ and $i_C \circ s_N$) from the top left corner to the bottom right corner give the same result.)

Section C

9.



The diagram shows two convex polygons, one contained in the interior of the region bounded by the other. Show that the perimeter of the inner polygon is shorter than the perimeter of the outer polygon.

(However, it is possible for a tetrahedron to be contained inside another, and yet have longer total edge length. See http://www1.ics.uci.edu/ eppstein/junkyard/triangulation.html)

(ii) Let C be a convex plane curve. Show that C is rectifiable.

10. Describe the section of a regular tetrahedron by a plane middway between two opposite edges (i.e. edges which have no common end-point) and parallel to both of them, justifying your assertions. Make a clear drawing, clearly showing your answer.

11. Question 7 one dimension up: what is the intersection of the four-dimensional "hypercube" $\{(w, x, y, z) \in \mathbb{R}^4 : -1 \leq w, x, y, z \leq 1\}$ with a 3-dimensional hyperplane midway between its opposite vertices $(-1, -1, -1, -1)$ and $(1, 1, 1, 1)$?

12. (i) Let Q_2 be the intersection of the plane $\{x_1 + x_2 + x_3 = 1\}$ in $\mathbb{R}^3$ with the positive octant $\{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. Show that Q_2 is a regular 3-gon (i.e. an equilateral triangle).

(ii) Let Q_3 be the intersection of the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ in $\mathbb{R}^4$ with the positive octant $\{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\ x_4 \geq 0\}$. Show that Q_3 is a regular tetrahedron.
Hint: (i) can easily be done by looking at lengths of edges but a more sophisticated method, and one which is more useful in (ii), is to consider the symmetries (isometries) of Q_2. The isometries of $\mathbb{R}^3$

$$\begin{aligned} (x_1, x_2, x_3) &\mapsto (x_2, x_1, x_3) \\ (x_1, x_2, x_3) &\mapsto (x_1, x_3, x_2) \\ (x_1, x_2, x_3) &\mapsto (x_3, x_2, x_1) \end{aligned}$$

map Q_2 to itself, since they map the plane $\{1_1 + x_2 + x_3 = 0\}$ to itself and also map the positive octant to itself. Hence, so do their composites. So Q_2 has a lot of symmetries ....

13. Although a polyhedron cannot be said to be curved in the usual sense of the word, there is an interesting definition of curvature for polyhedra, closely connected with the observation in Exercise 4 above. This curvature is concentrated at the vertices; at each vertex, the curvature is defined to be

$$2\pi - \quad \text{the sum of the angles at the vertex.}$$

For example, at a vertex of a cube, three faces meet, each having an angle of $\pi/2$ there. Thus, the curvature at the vertex of the cube is $2\pi - 3 \times \pi/2 = \pi/2$.
To do
(i) What is the sum of the curvatures at the vertices of

1. a cube?
2. a tetrahedron?
3. a dodecahedron?
4. a cube-with-a-hole?

(ii) Show that in any polyhedron (not necessarily convex), the sum of the curvatures at all the vertices is equal to $2\pi(F - E + V)$. (Hint: first prove this when all the faces are triangles, using the fact that if there are F triangular faces then the total of all of the angles at all of the vertices is $F\pi$. Then do the general case by subdividing any non-triangular faces into triangles.)

The result proved here is a polyhedral version of the *Gauss-Bonnet Theorem* of differential geometry.

14. Let P be a regular polyhedron. Here is a procedure for constructing a new polyhedron.

From each vertex v of P we obtain a plane as follows: on each of the edges ending at v, mark the midpoint. Because P is regular, all these midpoints are coplanar (i.e. there is a plane passing through all of them). Call this plane H_v. Then the new polyhedron P^* is the polyhedron enclosed by all of the planes H_v obtained in this way (one for each vertex). The polyhedron P^* is called the *reciprocal*, or *dual*, of P.

To do:

(i) Make a drawing of this construction for a cube. Start by drawing the planes H_v corresponding to the vertices around the top face of the cube.

(ii) What is the dual of each of the regular polyhedra? A good drawing would provide a satisfactory justification for your answer, but might be hard to do in the case of the icosahedron or the dodecahedron. For these, some argument is needed. For example, how many faces does the dual polyhedron have? From your drawing in the case of the cube you can probably guess the relation between the faces, edges and vertices of the original polyhedron P and the faces, edges and vertices of P^*.

(iii) Show that each isometry of P is also an isometry of P^*.

15. If a tetrahedron is drawn inside a cube as described in Exercise 5, where are the vertices of the dual polyhedron (cf. Exercise 9) situated?

16. The intersection of a regular tetrahedron with a plane which meets its interior is evidently a convex polygon. How many edges can it have?

17. The n-dimensional hypercube $[-1,1]^n$ is the set

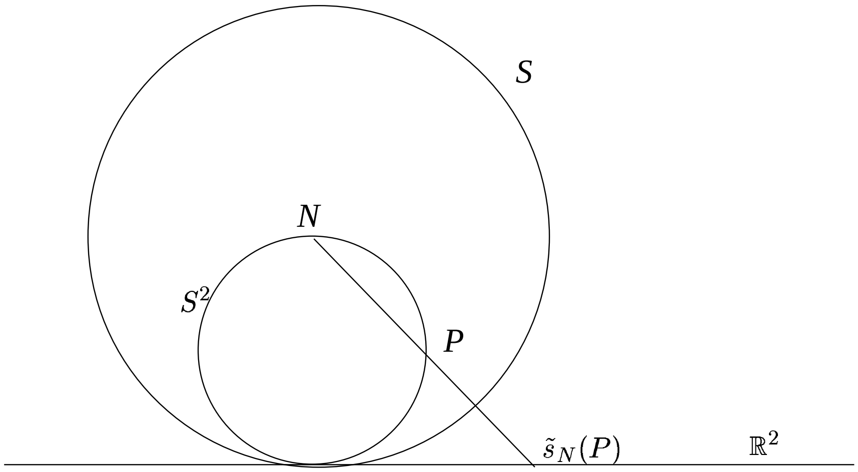
$$\{(x_1,\ldots,x_n) \in \mathbb{R}^n : -1 \le x_i \le 1 \text{ for } i = 1,\ldots,n\}.$$

Prove that if L is a 2-dimensional plane passing through two opposite vertices of the hypercube (e.g. $(-1,\ldots,-1)$ and $(1,\ldots,1)$), then the intersection of P with the hypercube is a parallelogram (i.e. has 4 edges, with opposite edges parallel).

18. In this exercise we consider the version of stereographic projection, in which we project to the tangent plane to the sphere at the south pole, rather than to the equatorial plane, as in exercise 8. We uncover another interesting relation between stereographic projection and inversion. Denote this variant of stereographic projection by $\tilde{s}_N : S^2 \setminus \{N\} \to \mathbb{R}^2$. For any sphere S in $\mathbb{R}^3$, with centre A and radius r, we define inversion in S, which we denote by i_S, in the same way that we defined inversion in a circle in the plane: we map each point P in $\mathbb{R}^3 \setminus \{A\}$ to the unique point P' on the half line AP continued, such that $AP' \cdot AP = r^2$.
To do: show that for every point $P \in S^2 \setminus \{N\}$,

$$\tilde{s}_N(P) = i_S(P),$$

where S is the sphere with centre N and radius 2.



19. Imagine N equally charged particles which are constrained to lie on the unit sphere. Each pair of particles repel one another with a force inversely proportional to the square of the distance between them. Thus the force affecting particle P_j is

$$C\sum_{j\neq i}\frac{P_j - P_i}{\|P_j - P_i\|^{3/2}}$$

where C is some constant depending on the charge. When this force points radially in or out, then particle P_j does not move, since it can only move on the surface of the sphere. So if the force acting on each particle is radial, then the configuration is in equilibrium. Show that if the P_j lie at the vertices of a regular tetrahedron, then the configuration is in equilibrium. Ditto for a cube. Can you find an argument which proves the corresponding statement for every regular solid? For this last you will have to make use of the *definition* of regularity, of course.

20. The proof of Theorem 12.3, that stereographic projection $S^2 \setminus \{N\} \to \mathbb{R}^2$ is conformal, can be adapted to prove that stereograpic projection $S^n \setminus \{N\} \to \mathbb{R}^n$ is conformal. The only difficulty is establishing that all of the steps, which are obvious in 2 dimensions because you can visualize them, are still obvious even though in higher dimensions you can't visualize them. For example, in order to emulate the step described in the first paragraph of the proof of 12.3, you have to show that every line tangent to the sphere S^n at any point $P \neq N$ is in fact tangent at P to a (unique) circle, contained in S^n, which also passes through N. Generalising a geometrical theorem out of the range of visibility is a good exercise in transforming geometrical intuition into formal argument.