Introduction to Geometry Exercises 4
Section A

1. Prove Proposition 9.9 in the Lecture Notes.

2. Prove that $\cos \pi/3 = 1/2, \sin \pi/3 = \sqrt{3}/2$.

3. Use the angle-sum formulae to prove

$$\sin A + \sin B = 2 \sin 1/2(A + B) \cos 1/2(A - B).$$

Section B

4. State and prove a formula like that of Exercise 3 for $\cos A + \cos B$.

5. (i) Show that every isometry of the plane has an inverse, which is also an isometry (Hint: Exercises I number 8.)

(ii) Show that if two isometries of the plane agree on three non-collinear points, then they agree everywhere (Hint: Exercises 1 number 6, and part (i) of this question).

6. (i) Show that the composite of two similarities is a similarity.

(ii) Show that by composing a similarity with a suitable dilation, you get an isometry.

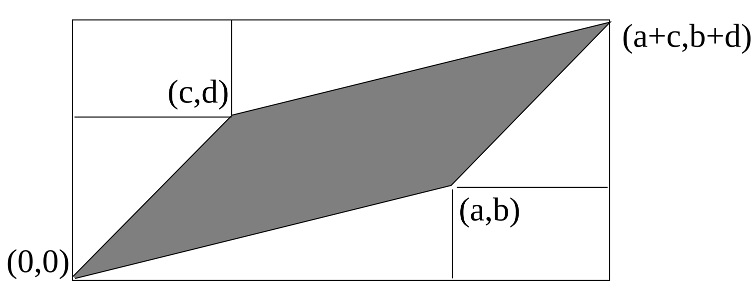
(iii) Show that every similarity is the composite of a dilation and an isometry.

(iv) Suppose two similarities agree on three non-collinear points. Show that they agree everywhere. Hint: (iii) of this question and (ii) of question 5.

7. Use the following diagram to prove that, ignoring its sign, the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is the area of the shaded parallelogram.



8. Write down a proof of the cosine rule (11.2) for a triangle each of whose angles is less than $\pi/2$. The proof given in the text, of course, is for a triangle with one angle of at least $\pi/2$.

9. A *geodesic quadrilateral* on the unit sphere is a four-sided figure, with each side an arc of a geodesic (great circle). What is the angle-sum in a geodesic quadrilateral? What is the angle-sum in a geodesic n-gon (n-sided figure) on the unit sphere? Hint: subdivide it into geodesic triangles.

Section C

10. (i) Prove that Pythagoras's theorem (the square on the hypotenuse of a right-angled triangle ABC is equal to the sum of the squares on the other two sides) remains true if we replace "square" "by equilateral triangle".

(ii) Ditto, but with the equilateral triangles on the 3 sides replaced by arbitrary triangles ABZ, BCX, CAY all similar to one another.

(iii) Ditto, but with the arbitrary triangles on the three sides replaced by arbitrary shapes, all similar to one another in the appropriate sense. A good answer here need not be absolutely complete: pointing out gaps that need to be filled is itself valuable.

11. Are the three medians of a spherical triangle concurrent?

12. The *geodesic radius* of a circle on the sphere is the length of a geodesic arc from its centre to any point on its circumference. What is the relation between the area and the geodesic radius of a circle on the unit sphere?

13. Let XYZ be a triangle with its vertices on a circle of radius 1. Show that it is not possible for all three edges to pass through the interior of a concentric circle of radius 1/2.