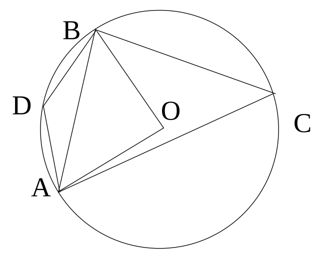
Introduction to Geometry   Exercises 1

Section A: everyone should do these

1. Using the criteria for congruence of triangles, show that

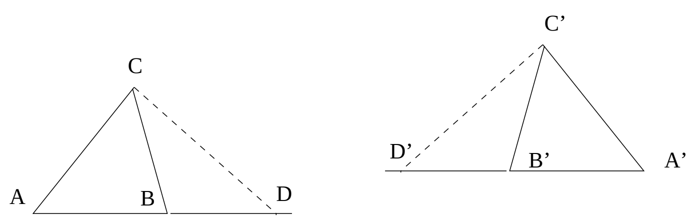
1. reflection in a line is an isometry (see the hint following this exercise (3.7) in the Lecture Notes)
2. rotation about a point is an isometry
3. translation is an isometry.

2. In the diagram below, A, B, C and D lie on the circumference and O is the centre of the circle. Assuming that the sum of the angles in a triangle is π (which we have not yet proved), show that $A\hat{O}B = 2A\hat{C}B$. Which angle in the diagram is equal to $2A\hat{D}B$?



3. Show that a reflection in a line ℓ maps any circle with centre on ℓ to itself.

4. Suppose $AB = A'B', BC = B'C', CA = C'A'$ and $AD = A'D'$ in the diagram below. Show that $CD = C'D'$.



5. Suppose that in the triangle ABC, the circumcentre (the centre of the circle passing through all three vertices) coincides with the incentre (the centre of the circle inside the triangle which is tangent to all three sides). Show that ABC is an equilateral triangle. Hint: Show first that all three angles are equal.

Section B: a little harder. Try them all, perhaps working with others.

5. Suppose that ABC and PQR are congruent triangles. Show that you can map ABC to PQR by successively performing no more than three suitably-chosen reflections. (This is best done step-by-step, with the help of drawings. First map A to P by a suitable reflection ....)

6. Suppose that an isometry f of the plane fixes three non-collinear points (i.e. maps them to themselves). Show that f must be the identity (i.e. fix every point). Hint: Theorem 3.18.

7. Algebra of isometries: The *composite* of two maps f and g, written $f \circ g$, is the map you get by first doing g and then doing f to the result:

$$f \circ g\ (P) = f(g(P)).$$

Note that the order may be important here: in general $f \circ g$ and $g \circ f$ are not the same map.
(i) Give an example of two isometries f, g such that $f \circ g \neq g \circ f$.
A map f is 1-1, or *injective*, if two distinct points never have the same image.
(ii) Show that every isometry is 1-1.
(iii) Suppose that f, g and h are maps, with f an isometry, and that $f \circ g = f \circ h$ (i.e. $f \circ g$ and $f \circ h$ are the same map). Show that necessarily $g = h$.
(iv) Suppose that f and g are maps, and r_ℓ is reflection in the line ℓ. Show that if $f = r_\ell \circ g$ then $r_\ell \circ f = g$. Hint: what is $r_\ell \circ r_\ell$?

8. Show that every isometry of the plane is the composite of one, two or three reflections. Hint: let PQR be any triangle, and let ABC be its image under some isometry f. Exercise 4 says that by a product $r_1 \circ \cdots \circ r_k$ of reflections (with $k \leq 3$), we can map ABC back to PQR. Now what can you say about $r_1 \circ \cdots \circ r_k \circ f$? Can you express f as a composite of reflections? Hint: Exercise 7(iv). A consequence of this exercise is that every isometry has an inverse. Why?

Section C. These ones are optional, and I'll be happy to discuss or mark your answers any of them

9. How can one obtain a rotation through θ about a point O as the composite of reflections? How can one obtain a translation as the composite of reflections?

10. (i) Find a formula in terms of a, b and c for reflection in the line $ax + by + c = 0$ in $\mathbb{R}^2$.

(ii) Using trigonometry, find a formula for anti-clockwise rotation through an angle of θ about

1. the origin $(0, 0)$ in $\mathbb{R}^2$
2. the point (a, b) in $\mathbb{R}^3$.

11. Find the centre and radius of the circle through the three points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3). This looks quite tough, since the equations are non-linear. However, even though the point of this material was to convince you that sometimes numbers are not the best way forward, honesty compels me to add that the following very clever trick does considerably simplify the business of obtaining the answer: write $R = r^2 - p^2 - q^2$. Then each equation

$$(x_i - p)^2 + (y_i - q)^2 = r^2$$

can be rewritten

$$2x_i p + 2y_i q + R = x_i^2 + y_i^2$$

which is linear in p, q and R. If you can find p, q and R then of course you can easily find r. So the system of quadratic equations (1) on page 2 of the lecture-notes can be reduced to a system of three linear equations in p, q and R, and solved using, for example, Cramer's rule. *To do:* Follow this procedure through to obtain a formula for p, q and r.

12. What is the correct version of Exercise 8 for three dimensional space?

13. Let Δ be a tetrahedron. Let C_1, C_2, C_3 and C_4 be the circumcentres of its four faces, and let L_1, L_2, L_3 and L_4 be the perpendiculars to the faces passing through O_1, O_2, O_3 and O_4 respectively. Show that the L_i all meet at one point.