Notes on Logarithmic Vector Fields, Logarithmic Differential Forms and Free Divisors

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1 Introduction

Let $D = \{h = 0\} \subset \mathbb{C}^n$, where h is a polynomial or a germ of holomorphic function. Then

$$\begin{aligned}
\mathrm{Der}(-\log D) &:= \{\chi \in \mathrm{Der}_{\mathbb{C}^n} : \chi \text{ is tangent to } D_{\mathrm{reg}}\} && (1.1)\\
&= \{\chi \in \mathrm{Der}_{\mathbb{C}^n} : \chi \cdot h \in (h)\} && (1.2)\\
\Omega^p(\log D) &:= \{\omega \in \Omega^p_{\mathbb{C}^n}(*D) : h\omega \text{ and } hd\omega \text{ are both regular }\} && (1.3)\\
&:= \{\omega \in \Omega^p_{\mathbb{C}^n}(*D) : h\omega \text{ and } dh \wedge \omega \text{ are both regular }\} && (1.4)
\end{aligned}$$

These definitions really refer to sheaves of germs of vector fields and differential forms, but in these notes I will not be careful to indicate this.

D is a *free divisor* if $\mathrm{Der}(-\log D)$ is a locally free $\mathcal{O}_{\mathbb{C}^n}$-module. Outside D, $\mathrm{Der}(-\log D)$ coincides with $\mathrm{Der}_{\mathbb{C}^n}$, and therefore has rank n. So if D is a free divisor, it follows that at each point there is a (local) basis for $\mathrm{Der}(-\log D)$ consisting of exactly n logarithmic vector fields.

The definitions and theorems in this section and the next three are due to Kyoji Saito, [9]. For background on commutative algebra I recommend the book of Matsumura, [7].

2 Singular free divisors are singular in codimension 1

Obviously, smooth hypersurfaces are free divisors. But suppose that D is a free divisor and that x_0 is a singular point of D. From the exact sequence

$$0 \longrightarrow \mathrm{Der}(-\log D) \xrightarrow{i} \theta_{\mathbb{C}^n} \xrightarrow{dh} J_h\,\mathcal{O}_D \longrightarrow 0, \qquad (2.1)$$

where i is inclusion, together with the Depth Lemma [1] we deduce that

$$D \text{ is free} \quad \Leftrightarrow \quad \text{depth } J_h\,\mathcal{O}_D = n-1, \qquad (2.2)$$

where J_h is the Jacobian ideal of h.

The Depth Lemma, applied now to the short exact sequence

$$0 \to J_h\,\mathcal{O}_D \to \mathcal{O}_D \to \mathcal{O}_D / J_h\,\mathcal{O}_D \to 0,$$

[1] If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules then depth $B \geq \min\{\text{depth } A, \text{depth } B\}$, and if the inequality is strict then depth $A = \text{depth } B + 1$.

shows that

$$\text{depth } J_h \mathcal{O}_D = n-1 \quad \Leftrightarrow \quad \text{depth } \mathcal{O}_D / J_h \mathcal{O}_D = n-2. \tag{2.3}$$

From (2.2) and (2.3) we deduce

Proposition 2.1. *Let the divisor $D = \{h = 0\} \subset \mathbb{C}^n$ be singular at x. Then*

$$D \text{ is free} \quad \Leftrightarrow \quad \mathcal{O}_D / J_h \mathcal{O}_D \text{ is Cohen-Macaulay of dimension } n-2. \tag{2.4}$$

3 Reflexivity

$\Omega^1(\log D)$ and $\mathrm{Der}(-|logD)$ are naturally paired. If $\omega \in \Omega^k(\log D)$ and $\chi \in \mathrm{Der}(-\log D)$ then I claim $\iota_\chi(\omega)$ (the contraction of ω by χ) lies in $\Omega^{k-1}(\log D)$. Certainly $h\iota_\chi(\omega) = \iota_\chi(h\omega)$ is regular. And as $dh \wedge \omega$ is regular, so $\iota_\chi(dh \wedge \omega)$ is regular. As

$$\iota_\chi(dh \wedge \omega) = \iota_\chi(dh) \wedge \omega - dh \wedge \iota_\chi(\omega),$$

$\iota_\chi(dh) \in (h)$ and $h\omega$ is regular, it follows also that $dh \wedge \iota_\chi(\omega)$ is regular. Thus $\iota_\chi(\omega) \in \Omega^{p-1}(\log D)$.

From this we deduce that the natural pairing of vector fields and 1-forms induces inclusions

$$\Omega^1(\log D) \subset \mathrm{Der}(-\log D)^*$$

and

$$\mathrm{Der}(-\log D) \subset (\Omega^1(\log D))^*.$$

Now we show the opposite inclusions. The chain of inclusions

$$\Omega^1_X \subset \Omega^1(|logD) \subset \frac{1}{h}\Omega^1_X$$

dualises to a chain of inclusions

$$\mathrm{Der}_X \supset (\Omega^1(\log D))^* \supset h\mathrm{Der}_X$$

which shows that we can regard the elements of $\Omega^1(\log D)^*$ as holomorphic vector fields. Any vector field $\delta \in \Omega^1(\log D)^*$ in particular sends $\frac{dh}{h}$ to something in $\mathcal{O}_X$. This means $\iota_\delta(dh) \in (h)$, and thus that $\Omega^1(\log D)^* \subset \mathrm{Der}(-\log D)$.

Dually, the inclusion

$$h\mathrm{Der}_X \subset \mathrm{Der}(-\log D) \subset \mathrm{Der}_X$$

dualises to

$$\frac{1}{h}\Omega^1_X \supset \mathrm{Der}(-\log D)^* \supset \Omega^1_X,$$

so that we can regard the elements of $\mathrm{Der}(-\log D)^*$ as 1-forms with a simple pole along D. As

$$\frac{\partial h}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j}\frac{\partial}{\partial x_i} \in \mathrm{Der}(-\log D),$$

for any $\omega = \frac{\sum_k \alpha_k dx_k}{h} \in \mathrm{Der}(-\log D)^*$ we find that

$$\frac{\partial h}{\partial x_i}\alpha_j - \frac{\partial h}{\partial x_j}\alpha_i \in (h)$$

for all i, j, which means that $dh \wedge \omega$ is regular. Since we already know that $h\omega$ is regular, we conclude that $\omega \in \Omega^1(\log D)$. We have proved

Proposition 3.1. *For any divisor $D \subset \mathbb{C}^n$, $\Omega^1(\log D)$ is the $\mathcal{O}_{\mathbb{C}^n}$-dual of Der$(-\log D)$ and vice versa.*

It immediately follows that $\Omega^1(\log D)$ is free if and only if D is a free divisor.

Proposition 3.2. *If D is a free divisor then $\Omega^p(\log D) = \bigwedge^p \Omega^1(\log D)$ and is also free, for all $p \geq 0$.*

4 Plane curves are free divisors

Lemma 4.1. *If R is a local ring with depth ≥ 2 and $0 \neq M^*$ for some finitely generated R-module M then depth$_R\, M^* \geq 2$.*

Proof. A presentation

$$R^p \xrightarrow{\lambda} R^q \longrightarrow M \longrightarrow 0$$

dualises to

$$R^p \xleftarrow{\lambda^t} R^q \longleftarrow M^* \longleftarrow 0\ .$$

This gives rise to short exact sequences:

$$0 \longleftarrow \operatorname{im}(\lambda^t) \xleftarrow{\lambda^t} R^q \longleftarrow M^* \longleftarrow 0 \tag{4.1}$$

and

$$0 \longleftarrow \operatorname{coker}(\lambda^t) \longleftarrow R^p \longleftarrow \operatorname{im}(\lambda^t) \longleftarrow 0\ . \tag{4.2}$$

Suppose that the depth of M^* is less than 2, and $M^* \neq 0$. As depth $R \geq 2$, applying the depth lemma[2] to (4.1) we deduce that depth $\operatorname{im}(\lambda^t) = 0$. Now applying the depth lemma to (4.2) shows that this is impossible. □

Corollary 4.2. *Every plane curve is a free divisor.*

Proof. The depth of Der$(-\log D)$ is 2 because Der$(-\log D) = \Omega^1(\log D)^*$. So Der$(-\log D)$ is free, by the Auslander-Buchsbaum theorem[3]. □

[2]This says that if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated modules over the noetherian local ring R, then

$$\operatorname{depth}_R B \geq \min\{\operatorname{depth}_R A,\ \operatorname{depth}_R C\},$$

and if the inequality is strict then $\operatorname{depth}_R A = \operatorname{depth}_R C + 1$.

[3]This says that if M is a finitely generated module over the noetherian local ring R, and if M has a finite free resolution, then the *projective dimension* of M, $\operatorname{pd}_R M$ (the length of a minimal free resolution of M), is related to $\operatorname{depth}_R M$ by the formula

$$\operatorname{pd}_R M + \operatorname{depth}_R M = \operatorname{depth}_R R.$$

Because $\mathcal{O}_{\mathbb{C}^2,x}$ is a regular local ring for every point x, the module Der$(-\log D)$ has a finite free resolution, which, if minimal, must have length 0 by the theorem (note that depth$(\mathcal{O}_{\mathbb{C}^n,x}) = n$). This means that Der$(-\log D)$ is free.

5 Saito's criterion

If $D \subset \mathbb{C}^n$ is a free divisor then $\mathrm{Der}(-\log D)$ necesarily has rank n, since outside D it coincides with $\mathrm{Der}_{\mathbb{C}^n}$. If $\chi_1, \ldots, \chi_n$ is a basis for $\mathrm{Der}(-\log D)$ with $\chi_i = \sum_j \chi_{ij} \frac{\partial}{\partial x_j}$ then the $n \times n$ matrix of coefficients (χ_{ij}) is called a *Saito matrix* for χ. A Saito matrix contains a lot of information.

The following theorem is known as *Saito's criterion for freeness.*

Theorem 5.1. *(1) If D is a divisor and $\chi_1, \ldots, \chi_n$ generate Der$(-\log D)$ then the determinant of a Saito matrix is a reduced equation for D.*
(2) If $\chi_1, \ldots, \chi_n \in$ Der$(-\log D)$ and the determinant of their matrix of coefficients is a reduced equation for D then D is a free divisor and $\chi_1, \ldots, \chi_n$ are a basis for Der$(-\log D)$.

Proof. (1) Suppose that $p \notin D$. Then the linear span of the vectors $\chi_1(p), \ldots, \chi_n(p)$ is equal to $\mathbb{C}^n$, and so the determinant of the Saito matrix is not 0. On the other hand, if p is in D, and is a regular point, then because the n vectors $\chi_1(p), \ldots, \chi_n(p)$ lie in the $n-1$-dimensional space T_pD, they are linearly dependent, so the determinant of the Saito matrix must vanish. From these two statements we conclude that up to multiplication by a unit, the determinant is equal to a (possibly non-reduced) product of irreducible defining equations of the irreducible components of D. We have to show that the power of each irreducible factor is 1. This is achieved by looking at a smooth point on D, where, evidently, only one factor vanishes. After a change of coordinates we may suppose that this factor is x_1. Then the vector fields $x_1\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ all lie in $\mathrm{Der}(-\log D)_p$. The determinant of *their* Saito matrix is x_1; since this Saito matrix is related to the Saito matrix with which we started by multiplication by an invertible matrix, we conclude that the determinant of our original Saito matrix also is a reduced defining equation.

(2) Suppose that ξ is any other logarithmic vector field. To express ξ as a linear combination of the χ_i we have to solve the $n \times n$ system of equations

$$\xi = \alpha_1\chi_1 + \cdots + \alpha_n\chi_n$$

in the unknowns $\alpha_1, \ldots, \alpha_n$. If we solve by Cramer's rule, we obtain solutions which are in principal meromorphic. But closer inspection shows that in the equation

$$\alpha_i = \frac{\det[\chi_1 \cdots \chi_{i-1}\ \xi\ \chi_{i+1} \cdots \chi_n]}{\det[\chi_1 \cdots \chi_n]},$$

the denominator is a reduced equation for D, by hypothesis, while the first argument of (1) shows that the numerator is divisible by this equation; it follows that α_i is regular, and ξ *is* an $\mathcal{O}_{\mathbb{C}^n}$-linear combination of the χ_i. □

Example 5.2. Deduce from 3.1 and 3.2 that the module $\Omega^p(\log D)$ has a basis consisting of the differential forms ω_I for $I \subset \{1, \ldots, n\}$ with $I = p$, defined by

$$\omega_I = \frac{1}{h} \sum_{|J|=p} \mathrm{sign}(J)\alpha_I^J dx_{j_1} \wedge \cdots \wedge dx_{j_p}$$

where α_I^J is the minor determinant of the Saito matrix obtained by omitting columns with indices in I and rows with indices in J, and $\mathrm{sign}(J)$ is the sign of the permutation (J, J^c) of $\{1, \ldots, n\}$.

In particular, $\Omega^{n-1}(\log D)$ has basis

$$\iota_{\chi_i}\left(\frac{dx_1 \wedge \cdots \wedge dx_n}{h}\right)$$

for $i = 1, \ldots, n$.

6 Discriminants

The first examples of free divisors, beyond the normal crossing divisor $\{x_1 \cdots x_n = 0\} \subset \mathbb{C}^n$, were the discriminants in the base of versal deformation of isolated hypersurface singularities. It was this discovery that motivated K. Saito's definition of free divisor and his development of the theory of logarithmic differential forms and logarithmic vector fields. Since then, the discriminants in the bases of versal deformations in many other contexts have been shown to be free – see e.g. [2], [4],[11]. Here we prove a version due to Looijenga, from which the freeness follows for the case of the discriminant in the base of a versal deformation of an isolated complete intersection singularity.

Theorem 6.1. *(Looijenga, [6]) Suppose that $F : (\mathbb{C}^n, 0)) \to (\mathbb{C}^p, 0)$ is a right-left stable map-germ, with $n \geq p$. Then the discriminant of F (the set of critical values of F) is a free divisor.*

Proof. Let $\theta(F) := F^*(\mathrm{Der}_{\mathbb{C}^k \times \mathbb{C}^d})$ denote the set of vector fields along F [4]. Right-left stability of F is equivalent to having

$$\theta(F) = tF(\mathrm{Der}_X) + \omega F(\mathrm{Der}_Y) \tag{6.2}$$

where $tF : \mathrm{Der}_X \to \theta(F)$ and $\omega F : \mathrm{Der}_Y \to \theta(F)$ are defined by composing sections of TX and TY, respectively, with dF and F.

Step 1: We show that $\theta(F)/tF(\mathrm{Der}_X)$ is Cohen-Macaulay of dimension $p-1$. The support of this module is the critical set Σ_F of F, the set of points where F is not a submersion. A standard argument[5] deduces Cohen-Macaulayness from a classical theorem of Buchsbaum and Rim, [1], provided that the support has the same codimension in X as the set of $p \times n$ matrices of non-maximal rank in $L(n,p)$, namely $n - p + 1$. Its codimension can be no *greater*, by a standard argument [6]. That is, its dimension is *at least* $\dim \Sigma_F = p - 1$. To prove equality, we first deduce

[4] $\theta(F)$ can helpfully be thought of as the set of mappings running from bottom left to top right in the diagram

$$\begin{tikzcd} TX \arrow[r, "dF"] \arrow[d, "\pi_X"'] & TY \arrow[d, "\pi_Y"] \\ X \arrow[r, "F"] & Y \end{tikzcd} \tag{6.1}$$

and making it commute. Here for brevity we use X and Y to denote the source and target of F.

[5] Suppose $f : X \to Y$, X and Y are smooth and $Z \subset Y$ is Cohen-Macaulay. If $\mathrm{codim}_X(f^{-1}(Z)) = \mathrm{codim}_Y(Z)$ then $f^{-1}(Z)$ also is Cohen-Macaulay. The argument is simply that $X \times Z$ is Cohen-Macaulay, and $f^{-1}(Z)$ is the image in X, under the isomorphic projection $\mathrm{graph}(f) \to X$, of $\mathrm{graph}(f) \cap (X \times Z)$. The hypothesis on the codimension means that the equations $y_i - f_i(x)$ defining the graph of f in $X \times Y$ form a regular sequence in $\mathcal{O}_{X \times Z}$, which means that $\mathrm{graph}(f) \cap (X \times Z)$ is Cohen-Macaulay. In our case this applies with $Z =$ set of $p \times n$ matrices of rank $< p$; this is Cohen-Macaulay by the Theorem of Buchsbaum-Rim

[6] If $f : X \to Y$ is a map of complex manifolds, Z is a closed complex subspace of Y, and $f^{-1}(Z) \neq \emptyset$, then

$$\text{codimension in } X \text{ of } f^{-1}(Z) \quad \leq \quad \text{codimension in } Y \text{ of } Z.$$

This applies in our situation because Σ_F is the preimage, under the map $j^1 F : X \to L(n,p)$, of the set of matrices of rank $< p$ in $L(n,p)$.

from (6.2), that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J_F + F^*(\mathfrak{m}_{Y,0})\,\mathcal{O}_X} < \infty;$$

this means that the restriction of F to its critical set $V(J_F)$ is finite. The argument is that

$$\frac{\mathcal{O}_X}{J_F + F^*(\mathfrak{m}_{Y,0})\,\mathcal{O}_X}$$

and

$$\frac{\theta(F)}{tF(\mathrm{Der}_X) + F^*\mathfrak{m}_{Y,0}\theta(F)} \tag{6.3}$$

have the same support, namely $F^{-1}(0) \cap V(J(F))$, and (6.2) implies that (6.3) has finite complex vector space dimension (in fact $\leq p$), so its support is just $\{0\}$. So the dimension of $V(J_F)$ is no greater than the dimension of its image in $D \subset Y$. This image is a closed variety, by finiteness, and cannot be all of Y, by Sard's theorem. Therefore it has dimension no *greater* than $p-1$. Hence $\dim V(J_F) = \dim D \leq p-1$, and $\theta(F)/tF(\mathrm{Der}_X)$ is Cohen Macaulay of dimension $p-1$ as required. For future use we note that J_F must in fact be radical: the condition on the codimension of the support of $\theta(F)/tF(\mathrm{Der}_X)$ guarantees that $\mathcal{O}_X\,/J_F$ also is Cohen-Macaulay; a Cohen-Macaulay space is reduced if and only if it is generically reduced (see e.g.[6, page 50]), so one can check reducedness at a generic point, i.e. by a local calculation, and this is easily done for example at a fold point.

Step 2: By (6.2), there is an epimorphism

$$\mathrm{Der}_Y \xrightarrow{\overline{\omega F}} \frac{\theta(F)}{tF(\mathrm{Der}_X)} \longrightarrow 0. \tag{6.4}$$

Because $\theta(F)/tF(\mathrm{Der}_X)$ has depth $p-1$ over $\mathcal{O}_X$, and is finite over $\mathcal{O}_Y$, its $\mathcal{O}_Y$-depth is also $p-1$. Therefore by Auslander-Buchsbaum its projective dimension is 1. It follows that the kernel of $\overline{\omega F}$ is free.

Step 3: We show that $\ker \overline{\omega F} = \mathrm{Der}(-\log D)$. The kernel of $\overline{\omega F}$ is the set of liftable vector fields,

$$\{\eta \in \mathrm{Der}_Y : \text{there exists } \xi \in \mathrm{Der}_X \text{ such that } tF(\xi) = \omega F(\eta)\}. \tag{6.5}$$

Suppose that $\eta \in \ker \overline{\omega F}$ and let ξ be a lift of η to X, as in (6.5). By integrating η and ξ we produce a pair of flows Ψ_t, Φ_t on Y and X respectively, such that

$$F \circ \Phi_t = \Psi_t \circ F. \tag{6.6}$$

Suppose $y \in D$, and let $x \in \Sigma_F$ satisfy $y = F(x)$. For every t, (6.6) shows that the germs

$$F : (X, \Phi_t(x)) \to (Y, \Psi_t(y)) \quad \text{and} \quad F : (X, x) \to (Y, y)$$

are left-right equivalent. Since x is a critical point of F, so is $\Phi_t(x)$, and therefore $\Psi_t(y)$, which is equal to $F(\Phi_t(x))$, lies in D. That is, we have shown that the flows Φ_t and Ψ_t preserve Σ_F and D respectively. It follows that the vector fields ξ and η are tangent to Σ_F and D. In particular, $\eta \in \mathrm{Der}(-\log D)$.

Reciprocally, if $\eta \in \mathrm{Der}(-\log D)$ then we can certainly lift $\eta|_D$ to a vector field ξ_0 on Σ_F. For Σ_F is the normalisation[7] of D, and vector fields lift to the normalisation by a theorem of Seidenberg[8]. Suppose ξ_0 is the restriction to Σ_F of a vector field $\xi \in \mathrm{Der}_X$. We have no guarantee that ξ is a lift of η – i.e. that $tF(\xi) = \omega F(\eta)$ – only that this equality holds on Σ_F. But because J_F is radical, the fact that $\xi|_{\Sigma_F}$ is a lift of $\eta|_D$ means that $tF(\xi) - \omega F(\eta) \in J_F\theta(F)$. By Cramer's rule,

$$J_F\theta(F) \subset tF(\mathrm{Der}_X), \tag{6.7}$$

and thus there exists a vector field $\xi_1 \in \mathrm{Der}_X$ such that

$$tF(\xi_1) = tF(\xi) - \omega F(\eta), \tag{6.8}$$

so that finally

$$tF(\xi - \xi_1) = \omega F(\eta), \tag{6.9}$$

showing that η is liftable and completing the proof that $\ker \overline{\omega F} = \mathrm{Der}(-\log D)$. □

Corollary 6.2. *The discriminant in the base space of a versal deformation of an isolated complete intersection singularity is a free divisor.*

Proof. If $F : X \to S$ is a deformation of an ICIS, then F is versal if and only if it is stable as a map-germ. □

Now we give an example of the application of Saito's criterion.

Proposition 6.3. *If $D \subset S$ is the discriminant of a versal deformation $X \xrightarrow{F} S$ of an ICIS curve singularity then $F^{-1}(D) \subset X$ is a free divisor.*

Proof. Note first that $\dim X = d + 1$. Let $\chi_1, \ldots, \chi_d$ be a basis for $\mathrm{Der}(-\log D)$. Each is liftable to a vector field $\tilde{\chi}_i \in \mathrm{Der}_X$. Thus we have a matrix equality

$$[dF][\tilde{\chi}_1 \cdots \tilde{\chi}_d] = [\chi_1 \cdots \chi_d] \circ F. \tag{6.10}$$

Apply the functor $\wedge^d$: we get

$$\bigwedge^d [dF] \times \bigwedge^d ([\tilde{\chi}_1 \cdots \tilde{\chi}_d]) = \det[\chi_1 \cdots \chi_d] \circ F \tag{6.11}$$

[7] Since Σ_F is Cohen Macaulay, it is normal if and only if it is non-singular in codimension 1 (i.e. it set of singular points has codimension at least 2 in Σ_F). Because j^1F is transverse to the stratification $\{\Sigma^k : k \in \mathbb{N}\}$ of $L(n,p)$,

$$(\Sigma_F)_{\mathrm{Sing}} = j^1F^{-1}((\overline{\Sigma_1})_{\mathrm{Sing}}) = j^1F^{-1}(\overline{\Sigma_2});$$

it therefore has codimension in Σ_F equal to $\operatorname{codim}\Sigma^2 - \operatorname{codim}\Sigma^1$, which is greater than 1.

[8] Sketched argument: the normalisation is unique up to isomorphism, so any automorphism of D lifts to an automorphism of its normalisation Σ_F; given a vector field on D, integrate it to get a 1-parameter family of automorphisms Ψ_t, lift the Ψ_t to a 1-parameter family of automorphisms Φ_t of Σ_F, then differentiate Φ_t with respect to t and set $t = 0$ to get a vector field on Σ_F lifting η.

The right hand side of (6.11) is an equation for $F^{-1}(D)$. A local calculation (analogous to the local calculation in the proof of 5.1(1)) shows that it is a *reduced* equation. The left hand side of (6.11) is equal to

$$\det\begin{pmatrix} \frac{\partial(F_1,\dots,F_d)}{\partial(x_2,\dots,x_{d+1})} & \tilde{\chi}_{1,1} & \cdots & \tilde{\chi}_{d,1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial(F_1,\dots,F_d)}{\partial(x_1,\dots,x_d)} & \tilde{\chi}_{1,d+1} & \cdots & \tilde{\chi}_{d,d+1} \end{pmatrix} \tag{6.12}$$

where the $\tilde{\chi}_{i,j}$ are the coefficients of the vector fields $\tilde{\chi}_i$. The first column of the matrix in (6.12) is a member (in fact Saito's criterion will show that it is a generator) of the $\mathcal{O}_X$-module of "vertical" vector fields – those that are annihilated by dF. In particular it belongs to $\mathrm{Der}(-\log F^{-1}(D))$. The remaining columns in this matrix are also members of $\mathrm{Der}(-\log F^{-1}(D))$. Since the determinant of this matrix is a reduced equation for $F^{-1}(D)$, it follows by Saito's criterion that $F^{-1}(D)$ is a free divisor and the listed vector fields form a basis for $\mathrm{Der}(-\log F^{-1}(D))$. □

Example 6.4. The previous statement does not hold for versal deformations of ICIS's of dimension greater than 1.

7 Linear Free Divisors

Another application of Saito's criterion is in the construction of *linear free divisors* (LFDs). An LFD $D \subset \mathbb{C}^n$ is a free divisor for which $\mathrm{Der}(-\log D)$ has a basis consisting of vector fields with linear coefficients ("linear vector fields"). An LFD necessarily has homogeneous equation of degree n, by Saito's criterion, since all the entries in the corresponding Saito matrix are linear forms. The most obvious example is the normal crossing divisor $D = \{(x_1,\dots,x_n) \in \mathbb{C}^n : x_1\cdots x_n = 0\}$, for which $x_1\partial/\partial x_1,\dots,x_n\partial/\partial x_n$ form a basis for $\mathrm{Der}(-\log D)$.

If D is an LFD then the group $G_D \subset \mathrm{Gl}_n(\mathbb{C})$ of linear isomorphisms preserving D is an algebraic group of dimension n, and the complement of D is an open orbit.

Example 7.1. For the normal crossing divisor, G_D is the semi-direct product of the symmetric group S_n which acts by permuting the coordinates, and the algebraic torus $\mathrm{Gl}_1(\mathbb{C})^n$ acting diagonally.

The argument is as follows: a linear vector field $\eta \in \mathrm{Der}_{\mathbb{C}^n}$ lifts, by the infinitesimal action of $\mathrm{Gl}_n(\mathbb{C})$, to an element of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of $\mathrm{Gl}_n(\mathbb{C})$: in other words, if

$$\alpha_p : \mathrm{Gl}_n(\mathbb{C}) \to \mathbb{C}^n \tag{7.1}$$

is the evaluation map determined by a point p,

$$\alpha_p(A) = Ap,$$

then there is a matrix $\hat{A} \in \mathfrak{gl}_n(\mathbb{C})$ such that

$$\eta(p) = d_I\alpha_p(\hat{A}). \tag{7.2}$$

This formula in fact shows how each $\hat{A} \in \mathfrak{gl}_n(\mathbb{C})$ gives rise to a vector field $\eta_{\hat{A}}$ on $\mathbb{C}^n$. The map $\hat{A} \mapsto \eta_{\hat{A}}$ is in fact a Lie algebra homomorphism $\mathfrak{gl}_n(\mathbb{C}) \to \mathrm{Der}_{\mathbb{C}^n}$, and is called the *infinitesimal action* of $\mathrm{Gl}_n(\mathbb{C})$ on $\mathbb{C}^n$.

The set of linear vector fields in $\mathrm{Der}(-\log D)$ is a Lie subalgebra, which we denote by $\mathscr{L}_D$. Lifting it by the homomorphism (7.2) we obtain an n-dimensional Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, which we denote by $\mathfrak{gl}_D$. For each $\hat{A} \in \mathfrak{gl}_D$, the automorphism $\exp(\hat{A}) = \sum_{k=0}^{\infty} \hat{A}^k$ preserves D, because the vector field $\eta_{\hat{A}}$ is tangent to D; thus $\mathfrak{gl}_D$ is contained in the Lie algebra of the group $G_D \subset \mathrm{Gl}_n(\mathbb{C})$ of linear automorphisms of $\mathbb{C}^n$ which preserve D. In fact one shows easily that the two are equal (see [5]), so G_D is an n-dimensional Lie subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Outside D, $\mathrm{Der}(-\log D) = \mathrm{Der}_{\mathbb{C}^n}$. Since $\mathscr{L}_D$ generates $\mathrm{Der}(-\log D)$, it follows that if $p \notin D$, the values at p of the vector fields in $\mathscr{L}_D$ span $\mathbb{C}^n$. So at such points *all* directions are tangent to the orbit of G_D. We conclude that $\mathbb{C}^n \smallsetminus D$ is a single orbit of G_D.

To find an LFD one can start by looking at an n-dimensional subgroup of $G \subset \mathrm{Gl}_n(\mathbb{C})$ and checking whether it has an open orbit; the complement of an open orbit in this case is necessarily a divisor D, and it is then only necessary to check that the vector fields coming from the infinitesimal action of G generate $\mathrm{Der}(-\log D)$. This can be done using Saito's criterion. Suppose that G is such a group. By choosing a basis $\hat{A}_1, \ldots, \hat{A}_n$ for its Lie algebra and applying the operator (7.2) to the $\hat{A}_i$, we obtain n linear vector fields $\eta_{\hat{A}_1}, \ldots, \eta_{\hat{A}_n}$. For each point $p \in \mathbb{C}^n$, the tangent space at p to the orbit $G \cdot p$ is spanned by the vectors $\eta_{\hat{A}_1}(p), \ldots, \eta_{\hat{A}_n}(p)$.

Let $[\eta_{\hat{A}_1} \ \cdots \ \eta_{\hat{A}_n}]$ denote the $n \times n$ matrix of coefficients of the $\eta_{\hat{A}_i}$.

Proposition 7.2. *(1) G has an open orbit if and only if $\det[\eta_{\hat{A}_1} \ \cdots \ \eta_{\hat{A}_n}]$ is not identically 0. In this case the open orbit is the complement of the divisor D defined by this determinant.*
(2) D is a linear free divisor if and only if $\det[\eta_{\hat{A}_1} \ \cdots \ \eta_{\hat{A}_n}]$ is reduced. □

We will refer to the determinant as the *discriminant determinant* and denote it by Δ.

Example 7.3.

1. Let $G := \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ embedded as the set of diagonal matrices in $\mathrm{Gl}_n(\mathbb{C})$. Its Lie algebra is the space of diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$, and as basis we can take the matrices

$$\hat{A}_1 := \mathrm{diag}(1, 0, \ldots, 0), \ldots, \hat{A}_n := \mathrm{diag}(0, \ldots, 1).$$

We have

$$\eta_i(\mathbf{x}) = (\partial_1, \ldots, \partial_n)\hat{A}_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i \partial_i$$

where ∂_i stands for $\partial/\partial x_i$. Thus the $n \times n$ matrix of coefficients of the η_i is

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & x_n \end{pmatrix}$$

whose determinant is $x_1 \cdots x_n$. We have recovered the normal crossing divisor.

2. **Exercise** What happens if G is the 3-dimensional group $O(3)$ acting on $\mathbb{C}^3$ in the usual representation?

3. A beautiful example is provided by the case of the Borel group B_n of lower triangular matrices, acting on the space Sym_n of symmetric $n \times n$ matrices by "transpose conjugation":

$$\alpha_S(B) = B^t \times S \times B.$$

The dimension of the group and the space are equal, so we can hope to find an LFD. Indeed, it turns out that each of the sets $D_i := \{S \in \mathrm{Sym}_n : \det_i(S) = 0\}$ is preserved by the action of B_n, where $\det_i(S)$ is the determinant of the top left $i \times i$ submatrix S_i of S (for $(B \cdot S)_i = B_i^t S_i B_i$). Thus their union D lies in the complement of an open orbit, if there is one. Note that the n varieties D_i are all irreducible, so no two have any irreducible component in common. We have

$$\deg(D) = \sum_i \deg(\det_i) = \frac{1}{2}n(n+1) = \dim \mathrm{Sym}_n = \dim B_n.$$

The discriminant determinant Δ, if it is not identically zero, has the same degree, and vanishes on D. It is thus necessary only to show that Δ is not identically 0. This can be easily checked at the identity matrix $I \in \mathrm{Sym}_n$: for any matrix B, the map α_I of (7.1) is given by

$$B \mapsto B^t I B = B^t B,$$

and its derivative at the neutral element $\mathbf{1} \in B_n$, $d_{\mathbf{1}}\alpha_I$, is given by

$$\hat{B} \mapsto \hat{B}^t + \hat{B}.$$

The kernel is evidently the set of skew-symmetric matrices. The only skew symmetric matrix in $T_{\mathbf{1}}B_n$ is the matrix 0. So $d_{\mathbf{1}}\alpha_I$ is injective, and $\Delta(I) \neq 0$.

Example 7.4. 1. Let $f : \mathrm{Mat}_{n\times n} \to \mathrm{Sym}_n$ be the map $f(A) = A^t A$. Show that the preimage under f of the divisor D of the previous example is a linear free divisor with group $O(n) \times B_n$. Note that if h is a reduced equation for D then the composed equation $h \circ f$ contains a repeated factor, which you have to remove.

Example 7.5. *(Questions I do not know the answer to:)* The free divisor $f^{-1}(D)$ of the previous example contains the divisor $D_n := \{\det = 0\}$, which is not free. What is the lowest degree of any homogeneous free divisor in $\mathrm{Mat}_{n\times n}$ containing D_n? Is it true that for *any* divisor $D_0 \subset \mathbb{C}^n$, there exists a divisor D_1 such that $D_0 + D_1$ is free? Masahiko Yoshinaga [12] has a proof that this is true if D_0 is a hyperplane arrangement (in which case D_1 may also be taken to be an arrangement).

Example 7.6. Let $D_n \subset \mathrm{Mat}_{n\times(n+1)}$ be the set defined by the vanishing of the product of the maximal minors. Show that D is a linear free divisor with group

$$\left(\mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_1(\mathbb{C})^{n+1}\right)/\mathrm{Gl}_1(\mathbb{C}).$$

Example 7.7. A systematic way of constructing examples of linear free divisors is provided by the theory of quiver representations. This is discussed in [3]; in fact Example 7.6 arises in this way.

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