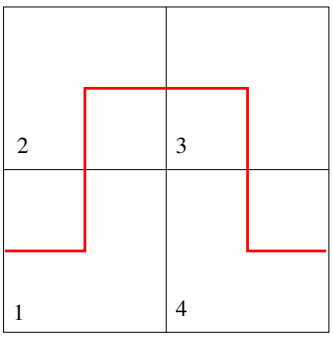
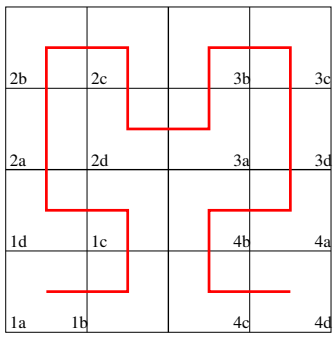
Draw a space-filling path

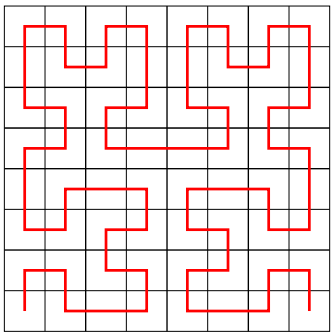
The three drawings below show three stages in the construction of a path that winds its way through every point of the unit square. At stage 1, the square is divided into four parts, and the red path passes through each one.



Stage 1



Stage 2



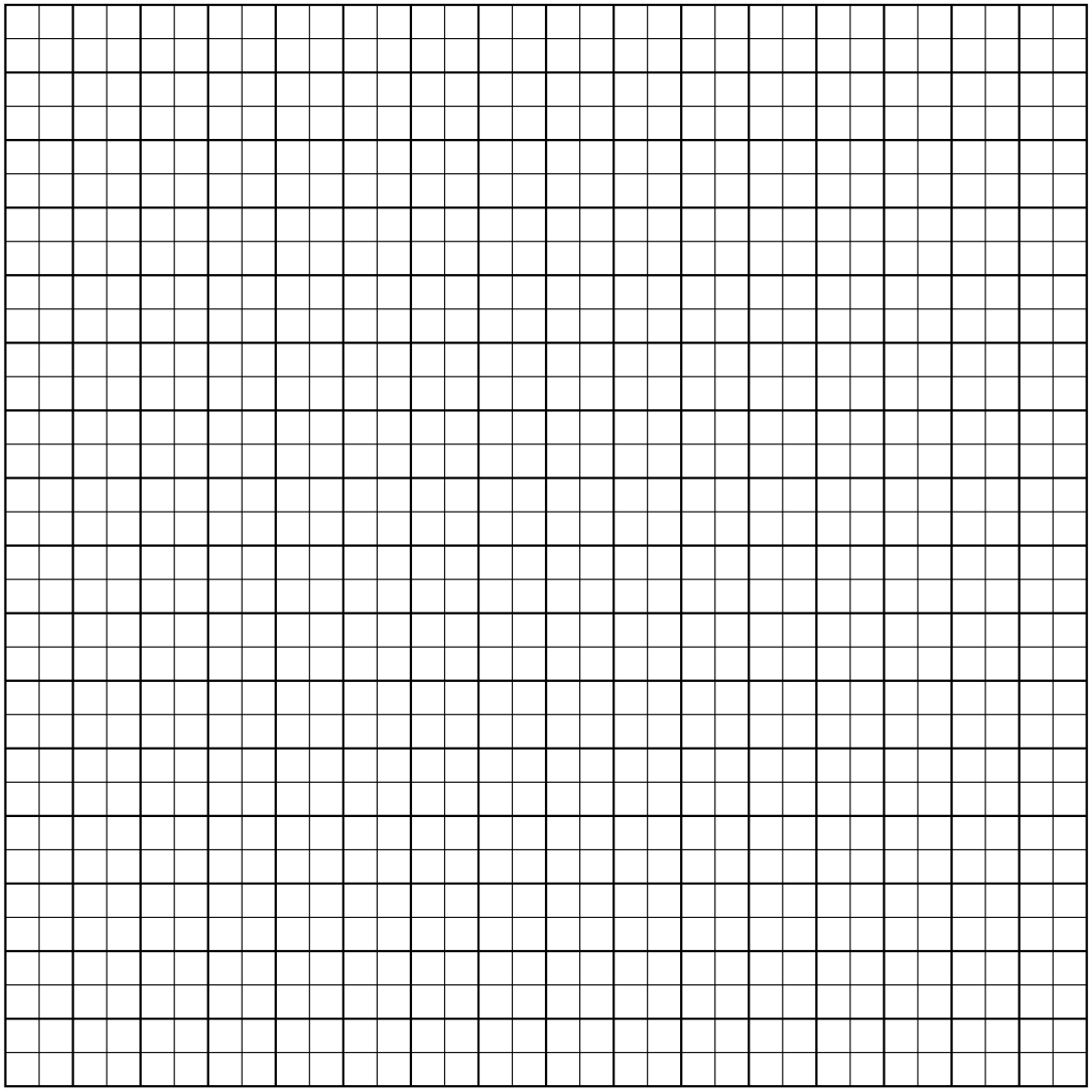
Stage 3

At stage 2, each little square has been subdivided into four, and once agan the red path passes through each one. But not in just any old way, as I will now explain. Go back to Stage 1. The red path spends the first quarter of its time in Square 1. In the second picture, of Stage 2, the red path still spends the first quarter of its time in Square 1 - but it wiggles about a bit while it does so, passing in turn through squares 1a, 1b, 1c and 1d. It spends the second quarter of its time in Square 2, just as it did in Stage 1, passing through little squares 2a, 2b, 2c and 2d. And so on.

The same guiding principle governs the behaviour of the red path at Stage 3. The little square 1a from Stage 2 has been subdivided into four. The red path spends the first 1/16th of its time in little square 1a, just as it did at Stage 2, but now winding its way through each of the four smaller squares into which 1a has been divided.

The next pages give you some grids on which you can draw Stage 4 and Stage 5. Make sure you follow the guiding principle! It's more or less essential to look at Stage 3 while you draw Stage 4, and to look at Stage 4 while you draw Stage 5.

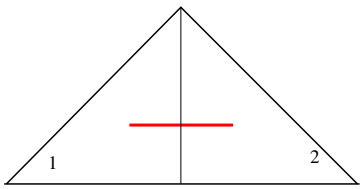
After the grids, there is a little more explanation. Draw at least stage 4 before reading more of the explanation.

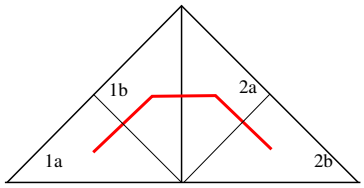


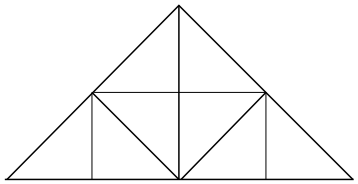
It is clear that at each stage, more of the square is covered by the red path. The final path, that I claim covers every point in the unit square, is the limit of this infinite sequence of paths. Why do we call this limit a "path"? Here is why: by the term "path" we mean a rule that tells you exactly where to be in the square, at each instant of the period during which you trace out the path. The path at each stage is a slight distortion of the path at the previous stage. Although the distortion greatly increases the overall length of the path, *each point on it doesn't have to move very far to get to its new position.* You can see this, because in passing from one stage to the next, *each point on the path never moves from the square that it is in.* The size of the new squares we draw is halved each time, so at each stage, the distance the point has to move gets less and less. As we go further and further on in the sequence of paths, each point settles down and gradually approaches a final position. This final position is its place on the limit path.

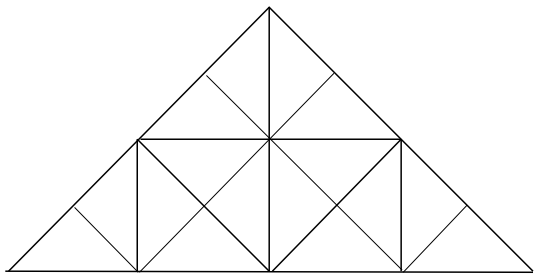
Another explanation is needed, as to why every point in the square lies on the limit path. I've written something about this at the end.

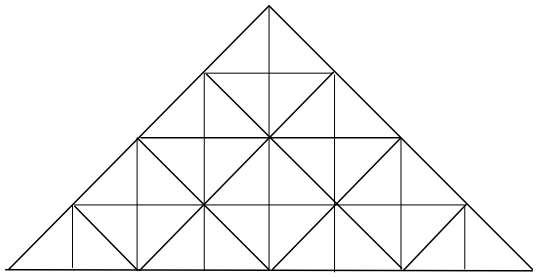
If this explanation is making your head hurt, it may be more fun to draw another sequence of paths, this time using the triangular grids on the next page. I've drawn the first two, to get you started. But remember the guiding principle: when, in going from one stage to the next, we subdivide the triangles and deform the old path to the new, *each point on the old path does not leave the old triangle it was in.*

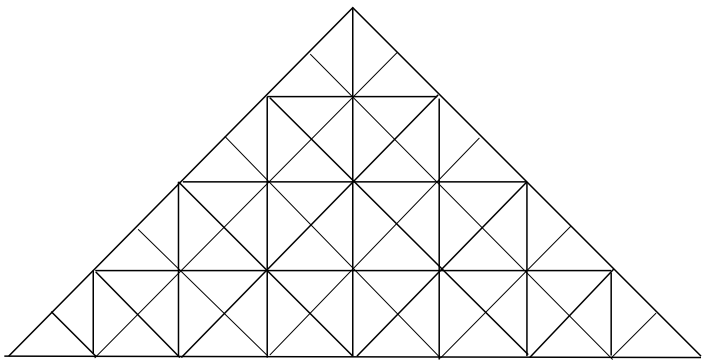


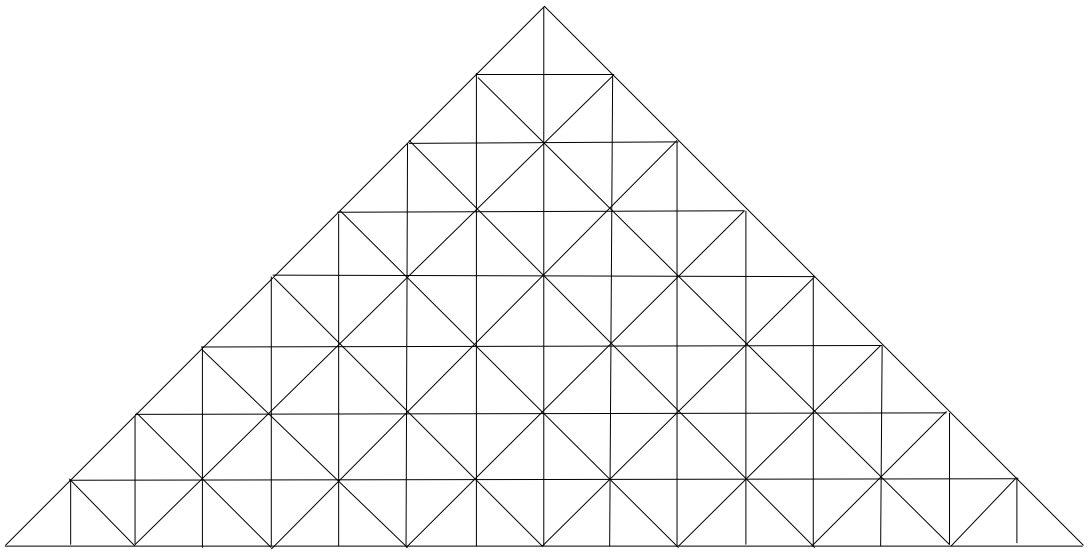










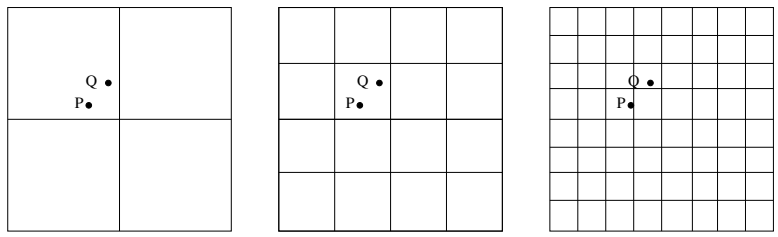


Let me make a comment about the enjoyment of mathematics: in my opinion, if you have enjoyed drawing one or more stages of the paths, then you have learned some mathematics, even if my explanations have not made much of an impression. A great deal of mathematics is about seeing

and appreciating patterns, and enjoyment is a sure sign that you are seeing and appreciating, even if you are not conscious of learning important stuff.

Why does the limit path cover the unit square?

Each point in the unit square we begin with is uniquely specified by the nested sequence of smaller and smaller squares it lies in, ("nested" means, with each square contained in the previous one). In the same way, *every* nested sequence of squares specifies a point in the unit square. Since the diameter of the squares gets smaller and smaller – indeed, tends to zero as we draw stage after stage – then two distinct points cannot lie in the same nested sequence of squares at every stage. For example, in the picture below, the two points P and Q lie in the same squares at stages 1 and 2, but at stage 3 they lie in different squares.



Every point in the unit square lies in a nested sequence of squares. The path at each stage passes through all of that stage's squares, so in every nested sequence there is a sequence of points on paths. The limit of this sequence of points is a point on the limit path. Thus the limit path passes through every point in the unit square.