Exercises on Abelian Groups and Quotients

1. If $\phi : G_1 \to G_2$ is a group homomorphism, then by definition $\ker(\phi)$ is equal to the preimage of the neutral element of G_2. Can you describe the *cosets* of $\ker(\phi)$ in G_1 in an analogous way?

2. Let A and B be subgroups of a abelian group C, with $A \cap B = 0$. Show that the map $\phi : A \oplus B \to C$ sending (a, b) to $a + b$ is an injective homomorphism.

3. In the group $G = \mathbb{Z} \times \mathbb{Z}$, consider the subgroup H generated by $(-5, 1)$ and $(1, -5)$. Show that G/H is cyclic. Which of the standard cyclic groups is it isomorphic to?

4. In the group $\mathbb{Z}^2$, consider the subgroup H generated by (a, b) and (c, d).

(i) Show that if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

then $\mathbb{Z}^2/H$ is a finite group. Hint: suppose the determinant is equal to n. Show that $(n, 0)$ and $(0, n)$ both lie in H.

(ii) Show the converse of (i): if G/H is a finite group then the determinant is non-zero. Hint: show that if G/H is finite then for some $m, n \in \mathbb{Z}$, we must have $(m, 0) \in H$ and $(0, n) \in H$. Hence there exist integers p, q, r, s such that

$$(m, 0) = p(a, b) + q(c, d), \quad (0, n) = r(a, b) + s(c, d).$$

Rewrite these two equations as as a single matrix equation.

(ii) In each of the following cases, decide whether G/H is cyclic. If it is cyclic, determine which of the standard cyclic groups it is isomorphic to:

1. $(a, b) = (3, 4), (c, d) = (6, 7)$
2. $(a, b) = (3, 4),\ (c, d) = (5, 7)$

5. Suppose that A, B are subgroups of the abelian group C. We define

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

(i) Show that $A + B$ is a subgroup of C.

(ii) There is a natural homomorphism $\phi : A \times B \to A + B$, defined by $\phi(a, b) = a + b$. Show that ϕ is surjective, and show that ϕ is an isomorphism if and only if $A \cap B = \{0\}$. If $A \cap B \neq \{0\}$, what is $\ker\ \phi$?

(iii) Now assume that A and B are both finite. If $A \cap B$ is bigger than just $\{0\}$, how many elements does the subgroup $A + B$ have? Hint: Use the first isomorphism theorem.

6. (i) Suppose that A is an abelian group. Show that the set $T = \{a \in A \mid a \text{ has finite order}\}$ is a subgroup. It is known as the "torsion subgroup" of A.

(ii) Show that in the quotient group A/T, every non-zero element has infinite order. So A/T is "torsion-free".

(iii) Let A be an abelian group and let n be a positive integer. Explain how to define a subgroup H such that in the quotient A/H, every element $\bar{a}$ satisfies $n\bar{a} = \bar{0}$. Is there a smallest such subgroup? (Here $\bar{a}$ means $a + H$ and $\bar{0}$ means H itself.)

7. Let A and B be subgroups of an abelian group C. This exercise examines the subgroup $A + B$ and its quotient $(A + B)/B$.

(i) Show that every element in $(A + B)/B$ can be written in the form $a + B$ for some $a \in A$.

(ii) Construct a surjective homomorphism $A \to (A + B)/B$.

(iii) Prove that there is an isomorphism

$$A/(A \cap B) \to (A + B)/B.$$

8. Suppose that $C \subset B \subset A$ are abelian groups. Prove the "Third Isomorphism Theorem", that

$$\frac{A/C}{B/C} \simeq \frac{A}{B}.$$

Hint: there is a natural choice of map from left to right.