PRIMITIVE STABILITY AND THE BOWDITCH CONDITIONS REVISITED

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Abstract. The equivalence of two conditions on the primitive elements in an $SL(2,\mathbb{C})$ representation of the free group $F_2 =< a, b >$, namely Minsky's condition of primitive stability and the BQ-conditions introduced by Bowditch and generalised by Tan, Wong and Zhang, has been proved by Lee and Xu and independently by the author in arXiv:1901.01396. This note is a revised version of our original proof, which is greatly simplified by incorporating some of the ideas introduced by Lee and Xu, combined with the language of the Bowditch tree.

Keywords: Free group on two generators, Kleinian group, non-discrete representation, palindromic generator, primitive stable

To Ser Peow Tan on his 60th birthday.

MSC classification: 30F40 (primary), 57M50 (secondary).

1. Introduction

In this note we show the equivalence of two conditions on the primitive elements in an $SL(2,\mathbb{C})$ representation ρ of the free group $F_2 =< a, b >$ on two generators, which may hold even when the image $\rho(F_2)$ is not discrete. One is the condition of primitive stability PS introduced by Minsky [13] and the other is the so-called BQ-conditions introduced by Bowditch [3] and generalised by Tan, Wong and Zhang [18]. This result was proved in [11] and independently in [15]. This note is a revised version of [15], which can be greatly simplified by incorporating the elegant estimates and ideas in [11]. The reason for writing it is to give a concise presentation using the language of the Bowditch tree developed in [3] and [18] and used in [15].

Both [11] and [15] introduced a third condition which we call the *bounded intersection property* BIP, which they showed was implied by but may not imply the other two (depending on the precise definition, see below). We also explain this condition and prove the implication here.

We begin by explaining these three conditions one by one. Recall that an element $u \in F_2$ is called *primitive* if it forms one of a generating pair (u, v) for F_2. Let $\mathcal{P}$ denote the set of primitive elements in F_2. It is well known that up to inverse and conjugacy, the primitive elements are enumerated by the rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \infty$, see Section 2 for details.

1.1. The primitive stable condition PS.

The notion of primitive stability was introduced by Minsky in [13] in order to construct an $Out(F_2)$-invariant subset of the $SL(2,\mathbb{C})$ character variety $\chi(F_2)$ strictly larger than the set of discrete free representations.

Let $d(P, Q)$ denote the hyperbolic distance between points P, Q in hyperbolic 3-space $\mathbb{H}^3$. Recall that a path $t \mapsto \gamma(t) \subset \mathbb{H}^3$ for $t \in I$ (where I is a possibly infinite interval in $\mathbb{R}$) is called

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a (K,ϵ)-quasigeodesic if there exist constants $K,\epsilon > 0$ such that

$$K^{-1}|s-t| - \epsilon \le d(\gamma(s),\gamma(t)) \le K|s-t| + \epsilon \quad \text{for all} \quad s,t \in I. \tag{1}$$

For a representation $\rho\colon F_2 \to SL(2,\mathbb{C})$, in general we will denote elements in F_2 by lower case letters and their images under ρ by the corresponding upper case, thus $X = \rho(x)$ for $x \in F_2$. In particular if (u,v) is a generating pair for F_2 we write $U = \rho(u), V = \rho(v)$.

Fix once and for all a basepoint $O \in \mathbb{H}^3$ and suppose that $w = e_1 \dots e_n, e_k \in \{u^\pm, v^\pm\}, k = 1,\dots,n$ is a cyclically reduced word in the generators (u,v). The *broken geodesic* $\mathbf{br}_\rho(w;(u,v))$ of w with respect to (u,v) is the infinite path of geodesic segments joining vertices

$$\dots,\,, E_n^{-1}E_{n-1}^{-1}E_{n-2}^{-1}O, E_n^{-1}E_{n-1}^{-1}O, E_n^{-1}O, O, E_1O, E_1E_2O, \dots, E_1E_2\dots E_nO, E_1E_2\dots E_nE_1O, \dots.$$

where $E_i = \rho(e_i)$.

Definition 1.1. *Let (u,v) be a fixed generating pair for F_2. A representation $\rho\colon F_2 \to SL(2,\mathbb{C})$ is* primitive stable*, denoted PS, if the broken geodesics $\mathbf{br}_\rho(w;(u,v))$ for all words $w = e_1 \dots e_n \in \mathcal{P}, e_k \in \{u^\pm, v^\pm\}, k = 1,\dots,n$, are uniformly (K,ϵ)-quasigeodesic for some fixed constants (K,ϵ).*

Remark 1.2. Notice that this definition is independent of the choice of basepoint O and makes sense since the change from $\mathbf{br}_\rho(w;(u,v))$ to $\mathbf{br}_\rho(w;(u',v'))$ for some other generator pair (u',v') changes all the constants for all the quasigeodesics uniformly. Notice also that if a broken geodesic is quasigeodesic, then it is within bounded distance of the corresponding geodesic axis. To see this, use the stability of quasigeodesics as for example in [2] Theorem III.H.1.7 to compare the broken geodesic segments between points $\rho(w^{-n})(O), \rho(w^n)(O)$ to the hyperbolic geodesic joining $\rho(w^{-n})(O)$ to $\rho(w^n)(O)$, and note that $\rho(w^{-n})(O), \rho(w^n)(O)$ converge to the (necessarily distinct) fixed points of $\rho(w)$. In particular, if ρ is PS then the images of all primitive elements must be loxodromic, moreover $\mathrm{Ax}\,\rho(w)$ is at uniformly bounded distance from $\mathbf{br}_\rho(w;(u,v))$, independent of u,v or w.

For $g \in F_2$ write $||g||$ or more precisely $||g||_{u,v}$ for the word length of g, that is the shortest representation of g as a product of generators (u,v). It is easy to see that for fixed generators, the condition PS is equivalent to the existence of $K,\epsilon > 0$ such that

$$K^{-1}||g'|| - \epsilon \le d(O,\rho(g')O) \le K||g'|| + \epsilon \tag{2}$$

for all finite subwords g' of the infinite reduced word $\dots e_1 \dots e_n \dots e_1 \dots e_n \dots$

Recall that an irreducible representation $\rho\colon F_2 \to SL(2,\mathbb{C})$ is determined up to conjugation by the traces of $U = \rho(u), V = \rho(v)$ and $UV = \rho(uv)$ where (u,v) is a generator pair for F_2. More generally, if we take the GIT quotient of all (not necessarily irreducible) representations, then the resulting $SL(2,\mathbb{C})$ character variety of F_2 can be identified with $\mathbb{C}^3$ via these traces, see for example [9] and the references therein. (The only non-elementary (hence reducible) representation occurs when $\mathrm{Tr}[U,V] = 2$. We exclude this from the discussion, see for example [17] Remark 2.1.)

Proposition 1.3 ([13] Lemma 3.2)**.** *The set of primitive stable $\rho\colon F_2 \to SL(2,\mathbb{C})$ is open in the $SL(2,\mathbb{C})$ character variety of F_2.*

Minsky showed that not all PS representations are discrete.

1.2. **The Bowditch BQ-conditions.** The BQ-conditions were introduced by Bowditch in [3] in order to give a purely combinatorial proof of McShane's identity.

Again let (u, v) be a generator pair for F_2 and let $\rho\colon\ F_2 \to SL(2,\mathbb{C})$.

Definition 1.4. *Following* [18], *an irreducible representation $\rho\colon\ F_2 \to SL(2,\mathbb{C})$ is said to satisfy the BQ-conditions if*

$$\begin{aligned} &\operatorname{Tr}\rho(g) \notin [-2,2] \quad \forall g \in \mathcal{P} \quad \text{and} \\ &\{g \in \mathcal{P} : |\operatorname{Tr}\rho(g)| \le 2\} \text{ is finite.} \end{aligned} \tag{3}$$

We denote the set of all representations satisfying the BQ-conditions by $\mathcal{B}$.

Proposition 1.5 ([3] Theorem 3.16, [18] Theorem 3.2)**.** *The set $\mathcal{B}$ is open in the $SL(2,\mathbb{C})$ character variety of F_2.*

Bowditch's original work [3] was on the case in which the commutator $[X, Y] = XYX^{-1}Y^{-1}$ is parabolic and $\operatorname{Tr}[X, Y] = -2$. He conjectured that all representations in $\mathcal{B}$ of this type are quasifuchsian and hence discrete. While this question remains open, it is shown in [17] that without this restriction, there are definitely representations in $\mathcal{B}$ which are not discrete.

1.3. **The bounded intersection property BIP.** Recall that a word $w = e_1e_2 \dots e_n$ in generators (u, v) of F_2 is *palindromic* if it reads the same forwards and backwards, that is, if $e_1e_2 \dots e_n = e_ne_{n-1} \dots e_1$. Palindromic words have been studied by Gilman and Keen in [6, 7].

Suppose that $\rho\colon\ F_2 \to SL(2,\mathbb{C})$ and let (u, v) be a generating pair, and suppose that the images $\rho(u), \rho(v)$ are not parabolic, so they have well defined axes. Denote the extended common perpendicular of the axes of $U = \rho(u), V = \rho(v)$ by $\mathcal{E}(U, V)$. By applying the π rotation about $\mathcal{E}(U, V)$, it is not hard to see that if a word w is palindromic in a generator pair (u, v) then, provided $W = \rho(w)$ is not parabolic, its axis intersects $\mathcal{E}(U, V)$ perpendicularly, see for example [1]. (See [11] Remark 6.9 for an interesting remark on the failure of the converse.) For the case of parabolic elements see Remark 1.7 below.

Fix generators (a, b) for F_2. We call the pairs $(a, b), (a, ab)$ and (b, ab) the *basic generator pairs*. Assume given $\rho\colon\ F_2 \to SL(2,\mathbb{C})$ for which none of $A = \rho(a), B = \rho(b)$ and $\rho(ab)$ are parabolic, and consider the three common perpendiculars $\mathcal{E}(A, B), \mathcal{E}(A, AB)$ and $\mathcal{E}(B, AB)$. (We could equally well chose to use BA in place of AB; the main point is that the choice is fixed once and for all.) We call these lines the *special hyperelliptic axes*.

Definition 1.6. *Fix a basepoint $O \in \mathbb{H}^3$. A representation $\rho\colon\ F_2 \to SL(2,\mathbb{C})$ satisfies the* bounded intersection property *BIP if no primitive elements have parabolic images and there exists $D > 0$ so that if a generator w is palindromic with respect to one of the three basic generators pairs, then its axis intersects the corresponding special hyperelliptic axis in a point at distance at most D from O. Equivalently, the axes of all palindromic primitive elements intersect the appropriate hyperelliptic axes in bounded intervals.*

It is not hard to see that this definition is independent of the choice of O. We prove that it is independent of the choice of generators in Proposition 6.3.

Remark 1.7. We remark that, in contrast to [11] Definition 6.10, we do not assume in the definition of BIP that the images of primitive elements are necessarily loxodromic. (The statement in a previous version of this paper that the second statement of Theorem II in [11] is incorrect was wrong and followed from a misreading of this point.) Our version of the condition rules out parabolicity (consider the fixed point of a palindromic parabolic element to be a degenerate axis which clearly meets the relevant hyperelliptic axis at infinity). However BIP does not obviously rule out elliptic elements in $\rho(\mathcal{P})$. In particular, consider any $SO(3)$ representation, discrete or otherwise. Here all axes are elliptic and all pass through a central fixed point which is also at the intersection of all three hyperelliptic axes. Such a representation clearly satisfies BIP.

A similar condition but related to *all* palindromic axes was used in [7] to give a condition for discreteness of geometrically finite groups.

In Section 6 we show that every generator is conjugate to one which is palindromic with respect to one of the three basic generator pairs. In fact each primitive element can be conjugated (in different ways) to be palindromic with respect to two out of the three possible basic pairs. For a more precise statement see Proposition 6.2.

1.4. **The main result.** The main results of this paper are:

Theorem A. *The conditions BQ and PS are equivalent.*

Theorem B. *The conditions BQ and PS both imply, but are not implied by, the condition BIP.*

In the case of real representations, Damiano Lupi [12] showed by case by case analysis following [8] that the conditions BQ and PS are equivalent.

To see that BIP does not imply the other conditions, first note that conditions PS and BQ both imply that no element in $\rho(\mathcal{P})$ is elliptic or parabolic. However, as explained in Remark 1.7, it is possible that all axes in a $SO(3)$ representation are elliptic and satisfy BIP. If one excludes elliptics from BIP as in [11], as far as we know the equivalence of the other conditions with BIP is not known.

The plan of the paper is as follows. The hardest part of the work is to prove Theorem 5.3, that if ρ satisfies the BQ-conditions then ρ is primitive stable. In [15] this was done by first showing that if ρ satisfies the BQ-conditions then ρ has the bounded intersection property, and using this to deduce PS. However, as explained in Section 4, this is shown to be unnecessarily complicated by the improved estimates and methods of [11].

In Section 2 we present background on the Farey tree and also introduce Bowditch's condition of Fibonacci growth. In Section 3, we summarise Bowditch's method of assigning an orientation to the edges of the Farey tree (T-arrows) and, subject to the BQ-conditions, the existence of a finite attracting subtree. In 3.1 we introduce a second way of orienting edges based on word length (W-arrows), and show that for all but finitely many words these two orientations coincide.

In Section 4 we collect the background and estimates used to prove Theorem A. This is based almost entirely on [11], in particular we need the amplitude of a right angle hexagon whose three alternate sides correspond to the axes of a generator triple (u, v, uv). As we shall explain, this quantity defined in [5] is an invariant of the representation ρ and plays a crucial part what follows. We then continue following [11] to get the crucial result Proposition 4.11.

Theorem A is proved in Section 5. That PS implies BQ follows easily from the condition of Fibonacci growth (see Definition 2.2). This was proved in [12]. Proposition 4.11 and the results of Section 3 then lead to the proof of Theorem 5.3, that BQ implies PS.

In Section 6 we discuss the condition BIP. We begin with a result which may be of independent interest on the palindromic representation of primitive elements, Proposition 6.2. Theorem B, that BQ implies BIP, is then easily deduced from Theorem 5.3. In Theorem 6.4 we give an alternative direct proof using Inequality (7) in the proof of Proposition 4.9, which uses the invariance of the amplitude of ρ to give an improved version of the estimates in [15].

We would like to thank Ser Peow Tan and Yasushi Yamashita for initial discussions about the original version [15] of this paper. The work involved in Lupi's thesis [12] also made a significant contribution. We also thank Tan for pointing us to the work of Lee and Xu, and for a careful reading of this paper. The idea of introducing the condition BIP arose while trying to interpret some very interesting computer graphics involving non-discrete groups made by Yamashita. We hope to return to this topic elsewhere.

As we hope we have made clear above, there is little in this revised version of [15] which is not essentially contained in [11] and we wish to fully acknowledge the elegance and ingenuity of their method.

I would also like to thank the referee for his or her exceptionally careful reading of the text and pointing out several non-trivial errors.

2. Primitive elements, the Farey tree and Fibonacci growth

The Farey tessellation $\mathcal{F}$ as shown in Figures 1 and 2 consists of the images of the ideal triangle with vertices at $1/0, 0/1$ and $1/1$ under the action of $SL(2,\mathbb{Z})$ on the upper half plane, suitably conjugated to the position shown in the disk. The label p/q in the disk is just the conjugated image of the actual point $p/q \in \mathbb{R}$.

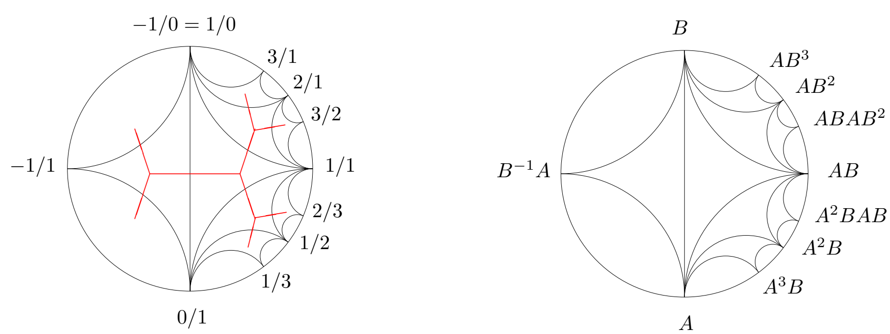

Figure 1. The Farey diagram, showing the arrangement of rational numbers on the left with the corresponding primitive words on the right. The dual graph shown on the left is the Farey tree $\mathcal{T}$.

Since the rational points in $\hat{\mathbb{Q}} = \mathbb{Q} \cup \infty$ are precisely the images of ∞ under $SL(2,\mathbb{Z})$, they correspond bijectively to the vertices of $\mathcal{F}$. A pair $p/q, r/s \in \hat{\mathbb{Q}}$ are the endpoints of an edge if and only if $pr - qs = \pm 1$; such pairs are called *neighbours*. A triple of points in $\hat{\mathbb{Q}}$ are the vertices of a triangle precisely when they are the images of the vertices of the initial triangle $(1/0, 0/1, 1/1)$; such triples are always of the form $(p/q, r/s, (p+r)/(q+s))$ where $p/q, r/s$ are neighbours. In other words, if $p/q, r/s$ are the endpoints of an edge, then the vertex of the triangle on the side away from the centre of the disk is found by 'Farey addition' to be $(p+r)/(q+s)$. Starting from $1/0 = -1/0 = \infty$ and $0/1$, all points in $\hat{\mathbb{Q}}$ are obtained recursively in this way. Note we need to start with $-1/0 = \infty$ to get the negative fractions on the left side of the left hand diagram in Figure 1.

As noted in the introduction, up to inverse and conjugation, the equivalence classes of primitive elements in F_2 are enumerated by $\hat{\mathbb{Q}}$. Formally, we set $\overline{\mathcal{P}}$ to be the set of equivalence classes of cyclically reduced primitive elements under the relation $u \sim v$ if and only if either $v = gug^{-1}$ or $v = gu^{-1}g^{-1}, g \in F_2$. We call the equivalence classes, *extended conjugacy classes* and denote the equivalence class of $u \in \mathcal{P}$ by $\mathbf{u}$. In particular, the set of all cyclic permutations of a given word are in the same extended class. A word is *cyclically reduced* if it, together with all its cyclic permutations, is reduced, that is, contains no occurrences of x followed by $x^{-1}, x \in \{a^{\pm}, b^{\pm}\}$. Such a word is cyclically shortest, meaning that it together with all its cyclic permutations is shortest.

The right hand picture in Figure 1 shows an enumeration of representative elements from $\overline{\mathcal{P}}$, starting with initial triple (a, b, ab). Each vertex is labelled by a certain cyclically reduced generator $w_{p/q}$. Corresponding to the process of Farey addition, the words $w_{p/q}$ can be found by juxtaposition as indicated on the diagram. Note that for this to work it is important to preserve the order: if u, v are the endpoints of an edge with u before v in the anti-clockwise order round the circle, the correct concatenation is uv, see Figure 3. Note also that the words on the left side of the diagram involve b^{-1} and a, rather than b and a, corresponding to starting with $\infty = -1/0$. It is not hard to see that pairs of primitive elements form a generating pair if and only if they are at the two endpoints of an edge of the Farey tessellation, while the words at the vertices of a triangle correspond to a generator triple of the form (u, v, uv).

The word $w_{p/q}$ is a representative of the extended conjugacy class identified with $p/q \in \hat{\mathbb{Q}}$. It is almost but not exactly the same as the *Christoffel word* as described [11]. We denote this class by $[p/q]$ and call $w_{p/q}$ the *Farey representative* of $[p/q]$. Likewise if $p/q, r/s \in \hat{\mathbb{Q}}$ are neighbours and if p/q is before r/s in the anticlockwise order, we call $(w_{p/q}, w_{r/s})$ the *Farey generator pair* corresponding to $p/q, r/s$. With this arrangement, note that $w_{p/q}w_{r/s} = w_{(p+r)/(q+s)}$ and $w_{p/q}w_{r/s}^{-1} = w_{(p-r)/(q-s)}$ so that $||w_{p/q}|| + ||w_{r/s}|| = ||w_{(p+r)/(q+s)}||$ and $||w_{p/q}w_{r/s}^{-1}|| \leq ||w_{p/q}w_{r/s}||$ with equality if and only if $p/q = 0/1, r/s = 1/0$. It is also easy to see that $e_b(w_{p/q})/e_a(w_{p/q}) = p/q$, where $e_a(w_{p/q}), e_b(w_{p/q})$ are the sum of the exponents in $w_{p/q}$ of a, b respectively. All other cyclically shortest words in $[p/q]$ are cyclic permutations of $w_{p/q}$ or its inverse. For more details on primitive words in F_2, see for example [16] or [4].

Later it will be essential to distinguish between a primitive element and its inverse, while for an arbitrary generator pair (u, v) we need to distinguish between uv (or its cyclic conjugate vu), and uv^{-1} (or its cyclic conjugate $v^{-1}u$).

Definition 2.1. *The word $w \in F_2 =< a, b| >$ is* positive *if it is cyclically reduced and if all exponents of a in w are positive. A generator pair (u, v) is* proper *if each of u, v is positive and (u, v) is conjugate to some Farey generator pair $(w_{p/q}, w_{r/s})$.*

Note that the definitions of positive and proper refer to words written in the generators (a, b). In particular, the Farey word $w_{p/q}$ constructed as indicated in Figure 1 is positive, as is the Farey generator pair $(w_{p/q}, w_{r/s})$, see also Figure 3. Also note that if (u, v) is proper then $||uv||_{a,b} = ||u||_{a,b} + ||v||_{a,b}$ and $||uv^{-1}||_{a,b} \leq ||uv||_{a,b}$ with equality if and only if $(u, v) = (a, b)$.

2.1. **Fibonacci growth.** Since all words in an extended conjugacy class have the same length, and since $w_{p/q}$ can found by concatenation starting from the initial generators (a, b), it follows that $||w||_{(a,b)} = p + q$ for all $w \in [p/q]$. This leads to the following definition from [3]:

Definition 2.2. *A representation $\rho\colon F_2 \to SL(2,\mathbb{C})$ has* Fibonacci growth *if there exists $c > 0$ such that for all cyclically reduced words $w \in \mathcal{P}$ we have $\log^+ |\operatorname{Tr}\rho(w)| < c||w||_{(a,b)}$ and $\log^+ |\operatorname{Tr}\rho(w)| > ||w||_{(a,b)}/c$ for all but finitely many cyclically reduced $w \in \mathcal{P}$ where $\log^+ x = \max\{0, \log|x|\}$.*

Notice that although the definition is made relative to a fixed pair of generators for F_2, it is in fact independent of this choice.

The following result is fundamental. It is proved using the technology described in the next section.

Proposition 2.3 ([3] Proof of Theorem 2, [18] Theorem 3.3)**.** *If $\rho\colon F_2 \to SL(2,\mathbb{C})$ satisfies the BQ-conditions then ρ has Fibonacci growth.*

3. More on the Bowditch condition

In this section we explain some further background to the BQ-conditions. For more detail see [3] and [18], and for a quick summary [17]. The *Farey tree* $\mathcal{T}$ is the trivalent dual tree to the tessellation $\mathcal{F}$, shown superimposed on the left in Figure 1. As above, $\overline{\mathcal{P}}$ is identified $\hat{\mathbb{Q}}$ and hence with the set Ω of complementary regions of $\mathcal{T}$. We label the region associated to a generator u by $\mathbf{u}$, thus $\mathbf{u}' = \mathbf{u}$ for all $u' \sim u$. If e is an edge of $\mathcal{T}$ we denote the adjacent regions by $\mathbf{u}(e), \mathbf{v}(e)$.

For a given representation $\rho\colon F_2 \to SL(2,\mathbb{C})$, note that $\operatorname{Tr}[U,V]$ and hence $\mu = \operatorname{Tr}[A,B]+2$ is independent of the choice of generators of F_2, where as usual $U = \rho(u)$ and so on. Since $\operatorname{Tr} U$ is constant on extended equivalence classes of generators, for $\mathbf{u} \in \Omega$ we can define $\phi(\mathbf{u}) = \phi_\rho(\mathbf{u}) = \operatorname{Tr} U$ for any $u \in \mathbf{u}$. For notational convenience we will sometimes write $\hat{\mathbf{u}}$ in place of $\phi(\mathbf{u})$.

For matrices $X, Y \in SL(2,\mathbb{C})$ set $x = \operatorname{Tr} X, y = \operatorname{Tr} Y, z = \operatorname{Tr} XY$. Recall the trace relations:

$$\operatorname{Tr} XY^{-1} = xy - z \tag{4}$$

and

$$x^2 + y^2 + z^2 = xyz + \operatorname{Tr}[X,Y] + 2. \tag{5}$$

Setting $\mu = \operatorname{Tr}[X,Y] + 2$, this last equation takes the form

$$x^2 + y^2 + z^2 - xyz = \mu.$$

As is well known and can be proven by applying the above trace relations inductively, if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a triple of regions round a vertex of $\mathcal{T}$, then $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ satisfy (5) with $x = \hat{\mathbf{u}}$ and so on. Likewise if e is an edge of $\mathcal{T}$ with adjacent regions $\mathbf{u}, \mathbf{v}$ and if $\mathbf{w}, \mathbf{z}$ are the third regions at either end of e, then $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}, \hat{\mathbf{z}}$ satisfy (4), that is, $\hat{\mathbf{z}} = \hat{\mathbf{u}}\hat{\mathbf{v}} - \hat{\mathbf{w}}$. (A map $\phi : \Omega \to \mathbb{C}$ with this property is called a *Markoff map* in [3].)

Given $\rho\colon F_2 \to SL(2,\mathbb{C})$, let e be an edge of $\mathcal{T}$ and suppose that the regions meeting its two end vertices are $\mathbf{w}, \mathbf{z}$. Following Bowditch [3], orient e by putting an arrow from $\mathbf{z}$ to $\mathbf{w}$ whenever $|\hat{\mathbf{z}}| > |\hat{\mathbf{w}}|$. If both moduli are equal, make either choice; if the inequality is strict, say that the edge is *oriented decisively*. We denote the oriented edge by $\vec{e}$ and refer to this oriented tree as the *Bowditch tree*, denoted $\mathcal{T}_\rho$. If $\vec{e}$ is a directed edge then its *head* and *tail* are its two ends, chosen so that the arrow on $\vec{e}$ points towards its head.

We say a path of oriented edges $\vec{e}_r, 1 \leq r \leq m$ is *descending to* $\vec{e}_m$ if the head of $\vec{e}_r$ is the tail of $\vec{e}_{r+1}$ for $r = 1, \ldots, m-1$. It is *strictly descending* if each arrow is oriented decisively. A vertex at which all three arrows are incoming is called a *sink*.

For any $m \geq 0$ and $\rho \colon F_2 \to SL(2, \mathbb{C})$ define $\Omega_\rho(m) = \{\mathbf{u} \in \Omega : |\phi_\rho(\mathbf{u})| \leq m\}$. From the definition, if $\rho \in \mathcal{B}$ then $\Omega_\rho(2)$ is finite and $\phi(\mathbf{u}) \notin [-2, 2]$ for $\mathbf{u} \in \Omega$.

The first two of the following lemmas show that starting from any directed edge $\vec{e}_1$, there is a unique descending path to an edge $\vec{e}_m$ which is adjacent to a region in $\Omega(2)$.

Lemma 3.1 ([18, Lemma 3.7])**.** *Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \Omega$ *meet at a vertex* q *of* $\mathcal{T}_\rho$ *with the arrows on both the edges adjacent to* $\mathbf{u}$ *pointing away from* q*. Then either* $|\phi(\mathbf{u})| \leq 2$ *or* $\phi(\mathbf{v}) = \phi(\mathbf{w}) = 0$*. In particular, if* $\rho \in \mathcal{B}$ *then* $|\phi(\mathbf{u})| \leq 2$*.*

Lemma 3.2 ([18, Lemma 3.11] and following comment)**.** *Suppose* β *is an infinite ray consisting of a sequence of edges of* $\mathcal{T}_\rho$ *all of whose arrows point away from the initial vertex. Then* β *meets at least one region* $\mathbf{u} \in \Omega$ *with* $|\phi(\mathbf{u})| < 2$*.*

Lemma 3.3. *For any* $m \geq 2$*, the set* $\Omega_\rho(m)$ *is connected. Moreover if* $\rho \in \mathcal{B}$ *then* $|\Omega_\rho(m)| < \infty$*.*

Proof. The first statement is [18] Theorem 3.1(2). That $\Omega_\rho(m)$ is finite follows from Proposition 2.3, see [18] P. 773. $\square$

The result which we mainly use is the following:

Theorem 3.4. *Suppose* $\rho \in \mathcal{B}$*. Then there is a constant* $M_0 \geq 2$ *and a finite connected non-empty subtree tree* T_F *of* $\mathcal{T}_\rho$ *so that for every edge* $\vec{e}$ *not in* T_F*, there is a strictly descending path from* $\vec{e}$ *to an edge of* T_F*. Moreover if regions* $\mathbf{u}, \mathbf{v}$ *are adjacent to an edge of* $\mathcal{T}$*, then* $|\operatorname{Tr} U|, |\operatorname{Tr} V| \leq M_0$ *implies* $e \in T_F$*. For any* $M \geq M_0$*, the tree* $T_F = T_F(M_0)$ *can be enlarged to a larger tree* $T_F(M)$ *with similar properties, and in addition* T_F *can be enlarged to include any finite set of edges.*

Proof. Most of the assertions are proved on p. 782 of [18], see also Corollary 3.12 of [3]. To see that T_F can always be enlarged to a tree $T_F(M)$ with similar properties, see the proofs of Theorem 3.2 of [18] and Theorem 3.16 of [3]. (In fact there is a precise condition to determine which edges are in T_F, see [18] Lemma 3.23.) Finally, let $\mathcal{K}$ be any finite subset of $\mathcal{T}$ and let $M = \max\{\phi(\mathbf{u}), \phi(\mathbf{v}) : \mathbf{u}, \mathbf{v} \text{ are adjacent to an edge in } \mathcal{K}\}$. Enlarging T_F to $T_F(M)$ the result is clear. $\square$

Definition 3.5. *Let* $\vec{e}$ *be a directed edge. The* wake *of* $\vec{e}$*, denoted* $\mathcal{W}(\vec{e})$*, is the set of regions whose boundaries are contained in the component of* $\mathcal{T} \setminus \{\vec{e}\}$ *which contains the tail of* $\vec{e}$*, together with the two regions adjacent to* $\vec{e}$*.*

We remark that the wake $\mathcal{W}(\vec{e})$ is the subset of Ω denoted $\Omega^{0-}(\vec{e})$ in [3] and [18]. Also denote by $\mathcal{W}_{\mathcal{E}}(\vec{e})$ the set of edges which are adjacent to two regions in $\mathcal{W}(\vec{e})$.

Theorem 3.4 says that if $\vec{e} \notin T_F$ then the arrow on $\vec{e}$ points towards T_F. We note the following slight variation:

Lemma 3.6. *If* $\vec{e} \notin T_F$ *then every edge in* $\mathcal{W}_{\mathcal{E}}(\vec{e})$ *is oriented towards* $\vec{e}$*.*

Proof. This follows easily from the definitions. In detail, let $\partial(T_F)$ be the boundary of T_F, that is, the set of edges in T_F whose tails meet the head of an edge not in T_F. If $\vec{e} \in \partial(T_F)$ then by Theorem 3.4 the arrow on every edge in $\mathcal{W}_{\mathcal{E}}(\vec{e})$ points towards $\vec{e}$. Now suppose that $\vec{e} \notin \partial(T_F)$ and that $\vec{f} \in \mathcal{W}_{\mathcal{E}}(\vec{e})$. Suppose that the descending path $\beta(e)$ from $\vec{e}$ lands on $\vec{g} \in \partial(T_F)$ while the descending path $\beta(f)$ from $\vec{f}$ lands on $\vec{h} \in \partial(T_F)$. Then $\beta(e) \subset \mathcal{W}_{\mathcal{E}}(\vec{g})$ while $\vec{f} \in \beta(f) \subset \mathcal{W}_{\mathcal{E}}(\vec{h})$. Since $\mathcal{W}_{\mathcal{E}}(\vec{g})$ and $\mathcal{W}_{\mathcal{E}}(\vec{h})$ are disjoint unless $g = h$ and $\vec{f} \in \mathcal{W}_{\mathcal{E}}(\vec{e}) \subset \mathcal{W}_{\mathcal{E}}(\vec{g})$ this gives the result. □

Finally, for the proof of Theorem 6.4 we need the following refinement of Proposition 2.3, which is a minor variation of Lemmas 3.17 and Lemma 3.19 of [18]. For $\mathbf{u} \in \mathcal{W}(\vec{e})$ let $d(\mathbf{u})$ be the number of edges in the shortest path from $\mathbf{u}$ to the head of $\vec{e}$. Following [18] P.777, define the *Fibonacci function* $F_{\vec{e}}$ on $\mathcal{W}(\vec{e})$ as follows: $F_{\vec{e}}(\mathbf{w}) = 1$ if $\mathbf{w}$ is adjacent to $\vec{e}$ and $F_{\vec{e}}(\mathbf{u}) = F_{\vec{e}}(\mathbf{v}) + F_{\vec{e}}(\mathbf{w})$ otherwise, where $\mathbf{v}, \mathbf{w}$ are the two regions meeting $\mathbf{u}$ and closer to $\vec{e}$ than $\mathbf{u}$, that is, with $d(\mathbf{v}) < d(\mathbf{u}), d(\mathbf{w}) < d(\mathbf{u})$.

Lemma 3.7. *Suppose that $\rho \in \mathcal{B}$ and that $\vec{e}$ is a directed edge such at most one of the adjacent regions is in $\Omega(2)$. Suppose also that no edge in $\mathcal{W}_{\mathcal{E}}(\vec{e})$ is adjacent to regions in $\Omega(2)$ on both sides. Then there exist $c > 0, n_0 \in \mathbb{N}$, independent of $\vec{e}$ (but depending on ρ), so that $\log|\phi_\rho(\mathbf{u})| \geq cF_{\vec{e}}(\mathbf{u})$ for all but at most n_0 regions $\mathbf{u} \in \mathcal{W}(\vec{e})$.*

Proof. This essentially Lemmas 3.17 and 3.19 of [18], see also Corollary 3.6 of [3].

Since $\Omega(M)$ is finite for any $M > 2$, the set $\{\log|\phi(\mathbf{u})| : \mathbf{u} \notin \Omega(2)\}$ has a minimum $m > \log 2$. By Lemma 3.17, if neither adjacent region to $\vec{e}$ is in $\Omega(2)$, we can take $c = m - \log 2$ and $n_0 = 0$.

Suppose then that exactly one of the adjacent regions $\mathbf{x}_0$ to $\vec{e}$ is in $\Omega(2)$. To apply Lemma 3.19, we need to verify that $\mathcal{W}(\vec{e}) \cap \Omega(2) = \{\mathbf{x}_0\}$. Note that no region which meets the boundary $\partial\mathbf{x}_0$ of $\mathbf{x}_0$ can be in $\Omega(2)$ by hypothesis. Let $\vec{\epsilon}_n, n \in \mathbb{N}$ be the oriented edges whose heads meet $\partial\mathbf{x}_0$ but which are not contained in $\partial\mathbf{x}_0$, numbered so that $\vec{\epsilon}_1$ is the edge not contained in $\partial\mathbf{x}_0$ whose head meets $\vec{e}$. Then neither of the two adjacent regions to $\vec{\epsilon}_n$ are in $\Omega(2)$ for any n. It follows from Lemma 3.17 that $\mathcal{W}(\vec{\epsilon}_n) \cap \Omega(2) = \emptyset$ for $n \in \mathbb{N}$. Since clearly $\mathcal{W}(\vec{e}) = \{\mathbf{x}_0\} \cup \bigcup_{n \in \mathbb{N}} \mathcal{W}(\vec{\epsilon}_n)$ the claim follows.

Now Lemma 3.19 gives $c > 0$ and $n_0 \in \mathbb{N}$, depending only on $\mathbf{x}_0$, so that $\log|\phi_\rho(\mathbf{u})| \geq cF_{\vec{e}}(\mathbf{u})$ for all but at most n_0 regions $\mathbf{u} \in \mathcal{W}(\vec{e})$. Since $\Omega(2)$ is finite and $\mathbf{x}_0 \in \Omega(2)$, we can adjust the constants so as to be uniform independent of $\vec{e}$. □

3.1. **The W-arrows.** There is another way to orient the edges of $\mathcal{T}$, this time in relation to word length. For $\mathbf{u} \in \Omega$, define $||\mathbf{u}|| = ||u||_{(a,b)}$ for any cyclically reduced positive word $u \in \mathbf{u}$; clearly this is independent of the choice of u. Provided e is not the edge e_0 separating the regions $(\mathbf{a}, \mathbf{b})$, then if $\mathbf{z}, \mathbf{w}$ are the regions at the two ends of $e \in \mathcal{T}$, put an arrow pointing from $\mathbf{z}$ to $\mathbf{w}$ whenever $||\mathbf{z}||_{a,b} > ||\mathbf{w}||_{a,b}$. We call these arrows, W-arrows, while the previously assigned arrows defined by the condition $|\phi(\mathbf{z})| \geq |\phi(\mathbf{w})|$ we refer to as T-arrows (for word length and trace respectively). Clearly every edge is connected by a strictly descending path of W-arrows to one of the two vertices at the ends of the edge e_0. *We retain the notation $\vec{e}$ exclusively to refer to the orientation of the T-arrow, likewise the terms head and tail.*

If e is an edge of $\mathcal{T}$, as usual denote by $\mathbf{u}(e), \mathbf{v}(e)$ the regions adjacent to e. Notice that if $u \in \mathbf{u}(e), v \in \mathbf{v}(e)$ are a proper generator pair, then, since $e \neq e_0$, we have $||\mathbf{uv}|| > ||\mathbf{uv^{-1}}||$ so that the W-arrow points from $\mathbf{uv}$ to $\mathbf{uv^{-1}}$.

For $N \in \mathbb{N}$ let $B((a,b),N) = \{e \in \mathcal{T} : \max\{||\mathbf{u}(e)||_{a,b}, ||\mathbf{v}(e)||_{a,b}\} \leq N\}$. The next proposition shows that for all but finitely many arrows, the W- and T- arrows point in the same direction.

Proposition 3.8. *Suppose $\rho \in \mathcal{B}$. Then there exists $N_0 > 0$ such that if $\vec{e} \notin B((a,b),N_0)$ is an oriented edge of $\mathcal{T}_\rho$ with regions $\mathbf{z}, \mathbf{w}$ at its tail and head respectively, then $||\mathbf{z}|| > ||\mathbf{w}||$.*

Proof. This is a general result about attracting trees. Enlarge the finite sink tree T_F of Theorem 3.4 if necessary so that $e_0 \in \mathcal{T}_F$. Choose N_0 large enough that $T_F(M_0) \subset B = B((a,b),N_0)$. Then every edge not in B is connected by a path of decreasing T-arrows to an edge of T_F.

If the result is false, there is an edge $\vec{e}$ not in B with regions $\mathbf{z}, \mathbf{w}$ at its tail and head respectively such that $||z||_{a,b} < ||w||_{a,b}$ for $z \in \mathbf{z}, w \in \mathbf{w}$. By Lemma 3.6, every edge in $\mathcal{W}_{\mathcal{E}}(\vec{e})$ is connected by a strictly descending path of T-arrows to the tail of $\vec{e}$. On the other hand, $\vec{e}$ is connected by a strictly descending path of W-arrows to one of the two vertices at the ends of e_0. But these W-arrows are contained in $\mathcal{W}(\vec{e})$ and, following on from the initial edge e, must all point in the opposite direction to the T-arrows. Thus one of the two vertices at the ends of e_0 is outside B, which is impossible. □

Corollary 3.9. *If $\rho \in \mathcal{B}$, there exists $N_0 \in \mathbb{N}$ such that if $\vec{e}$ is an edge outside $B(N_0)$, then every edge $\vec{f} \in \mathcal{W}(\vec{e})$ has head $\mathbf{uv^{-1}}$ and tail $\mathbf{uv}$ whenever $u \in \mathbf{u}(f), v \in \mathbf{v}(f)$ are a proper generator pair associated to $\vec{f}$.*

4. Results from [11]

In this section we collect the main results from [11] needed to prove Theorem 5.3.

4.1. **The double cone lemma.** Suppose that H is a hyperbolic hyperplane and let $\hat{H}$ be one of the two closed half spaces defined by H. By an inward (resp. outward) pointing normal to $\hat{H}$ we mean a normal to H which points into (resp. out of) $\hat{H}$. If $\hat{H}'$ is another half space such that $\hat{H} \supset \hat{H}'$ and $d(H,H') > 0$ we say that $\hat{H}, \hat{H}'$ are *properly nested.*

Lemma 4.1. *Suppose $0 < \alpha < \pi/2$. Then there exists $L_0 > 0$ with the following property. Suppose that H, H' are hyperbolic hyperplanes defining half spaces $\hat{H}, \hat{H}'$. Let $\mathcal{M}$ be a line segment joining points $O \in H, P \in H'$ such that $Int\, \mathcal{M}$ is inside $\hat{H}$ and outside $\hat{H}'$. Suppose also that $\mathcal{M}$ is orthogonal to $\hat{H}'$, and makes an angle $0 \leq \theta < \alpha$ with the inward pointing normal to $\hat{H}$. Then $\hat{H} \supset \hat{H}'$ are properly nested whenever $d(O,P) > L_0$.*

Proof. Since by assumption $Int\, \mathcal{M}$ is inside $\hat{H}$ and outside $\hat{H}'$, the only possible nesting between the two half spaces is $\hat{H} \supset \hat{H}'$.

Let L_0 be the length of the finite side of a triangle with angles $\pi/2-\alpha, \pi/2, 0$ and assume $d(O,P) > L_0$. Then if $\mathcal{L}'$ is a line through O making an angle $\psi > \pi/2-\alpha$ with OP, while $\mathcal{L}$ is a line through P and perpendicular to OP, then $\mathcal{L}, \mathcal{L}'$ do not meet.

On the other hand, if $\hat{H} \supset \hat{H}'$ are not properly nested then they meet in some point $Q \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$. Since the line PQ is perpendicular to OP, while the line OQ makes an angle at least $\psi = \pi/2-\theta > \pi/2-\alpha$ with OP, this is a contradiction. We conclude that if $d(O,P) > L_0$ then $\hat{H} \supset \hat{H}'$ are properly nested as claimed. □

Corollary 4.2. ([11] *Lemma 3.5) Suppose that H, H' are hyperbolic hyperplanes with corresponding half spaces $\hat{H}, \hat{H}'$ and let $\mathcal{M}$ be a line joining points $O \in H, P \in H'$ which makes* *angles* $0 \leq \theta, \theta' \leq \alpha$ *with the inward pointing normal to* $\hat{H}$ *and the outward pointing normal to* $\hat{H}'$ *respectively. Then* $\hat{H} \supset \hat{H}'$ *are properly nested provided* $d(O,P) > 2L_0$.

Proof. Let H'' be the plane perpendicular to $\mathcal{M}$ through its mid-point and apply Lemma 4.1 to H, H'' and H'', H'. □

4.2. **Generators and the amplitudes of a right angled hexagon.** Let $\mathcal{H}$ be a right angled hexagon with consistently oriented sides $s_1, \ldots, s_6$ and let σ_i be the complex distance between sides s_{i-1}, s_{i+1}. The amplitude $Am(\sigma_{i-2}, \sigma_i, \sigma_{i+2})$ introduced in [5] VI.5, is, up to sign, an invariant of the triple of alternate sides s_{i-2}, s_i, s_{i+2}. Its importance is that if $\mathcal{H}$ is constructed as described below from a generator pair (u, v), then the amplitude relative to the three sides $\mathrm{Ax}\, U, \mathrm{Ax}\, V, \mathrm{Ax}\, U^{-1}V^{-1}$ is closely related to the trace of the commutator and hence independent of the choice of generators, see Proposition 4.4 below or [11] Equation (3.13). This point was used crucially in [11].

Definition 4.3. *Let* $\mathcal{H}$ *be a consistently oriented right angled hexagon with oriented sides* $s_1, \ldots, s_6$ *and let* σ_i *be the complex distance between sides* s_{i-1}, s_{i+1}. *Define the* amplitude $Am(\sigma_1, \sigma_3, \sigma_5) = -i \sinh \sigma_2 \sinh \sigma_3 \sinh \sigma_4$.

See for example [5] or [14] for a discussion of complex length and hyperbolic right-angled hexagons.

Let σ_{14} be the complex distance between the oriented lines s_1 and s_4. Using the cosine formula in the oriented right angled pentagon with the sides $s_1, s_2, s_3, s_4, s_{14}$ (where s_{14} is the common perpendicular of s_1 and s_4, oriented from s_1 to s_4), we find $\cosh \sigma_{14} = -\sinh \sigma_2 \sinh \sigma_3$. Thus we can alternatively write the amplitude as $Am(\sigma_1, \sigma_3, \sigma_5) = i \cosh \sigma_{14} \sinh \sigma_4$.

We now fix a choice of lift $R \in SL(2, \mathbb{C})$ of the order two rotation about an oriented line using line matrices as described in [5] V.2. Denote the oriented line with endpoints $\zeta, \zeta' \in \hat{\mathbb{C}}$, oriented from ζ to ζ', by $[\zeta, \zeta']$. The *line matrix* $R([\zeta, \zeta']) \in SL(2, \mathbb{C})$ is a choice of matrix representing the π-rotation about $[\zeta, \zeta']$. If $\zeta, \zeta' \in \mathbb{C}$ then

$$R([\zeta, \zeta']) = \frac{i}{\zeta' - \zeta} \begin{pmatrix} \zeta + \zeta' & -2\zeta\zeta' \\ 2 & -\zeta - \zeta' \end{pmatrix},$$

while

$$R([\zeta, \infty]) = i \begin{pmatrix} 1 & -2\zeta \\ 0 & -1 \end{pmatrix}, \quad R([\infty, \zeta']) = -i \begin{pmatrix} 1 & -2\zeta \\ 0 & -1 \end{pmatrix}.$$

As shown in [5], this definition respects the orientation of lines and is invariant under conjugation in $SL(2, \mathbb{C})$.

If R_i is the line matrix associated to the oriented side s_i of $\mathcal{H}$ as above, then $R_i^2 = -id$ and $R_i R_{i+1} = -R_{i+1} R_i$. Moreover $R_{i-1} R_{i+1}$ is a loxodromic which translates by complex distance $2\sigma_i$ along an axis which extends s_i, moreover $\mathrm{Tr}\, R_{i-1} R_{i+1} = -2 \cosh \sigma_i$ and $\mathrm{Tr}\, R_{i-1} R_i R_{i+1} = -2i \sinh \sigma_i$, see [5] V.3. These formulae can be easily checked by letting $\zeta = e^{\sigma_i}$ and arranging s_{i-1}, s_i and s_{i+1} to be the oriented lines joining $[-1, 1], [0, \infty], [-\zeta, \zeta]$ respectively so that

$$R_{i-1} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; R_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; R_{i+1} = \begin{pmatrix} 0 & i\zeta \\ i/\zeta & 0 \end{pmatrix}.$$

Note in particular that if $X = \begin{pmatrix} \xi & 0 \\ 0 & 1/\xi \end{pmatrix}$ is translation along $[0, \infty]$ then $R_{i-1} R_{i+1} = -X$.

It follows from the above formulae, that we can alternatively define $Am(\sigma_1, \sigma_3, \sigma_5) = -\frac{1}{2}\operatorname{Tr}(R_5R_3R_1)$. Moreover this expression is unchanged under even cyclic permutations and changes sign under odd ones.

We now explain the invariance of the amplitude under change of generator. Suppose that (u, v) is a generator pair. Construct an oriented right angled hexagon $\mathcal{H} = \mathcal{H}(u, v)$ with the axes of $(U, V, U^{-1}V^{-1})$ oriented in their natural directions, i.e. pointing in their respective translation directions, forming three alternate sides. The orientations of the three remaining sides then follow. We call this the *standard hexagon* associated to (u, v).

Proposition 4.4. *Let $\mathcal{H} = \mathcal{H}(u, v)$ be the standard hexagon associated to the image of a generator pair (u, v). Let $s_2 = \operatorname{Ax} U, s_4 = \operatorname{Ax} V, s_6 = \operatorname{Ax} U^{-1}V^{-1}$ and label the other sides accordingly. Then up to sign, $Am(\sigma_1, \sigma_3, \sigma_5)$ is independent of the choice of (u, v).*

Proof. With $\mathcal{H} = \mathcal{H}(u, v)$ as defined in the statement, we have $R_3R_1 = -U$ and $R_5R_3 = -V$ so that $R_1R_5 = -R_1R_3R_3R_5 = -U^{-1}V^{-1}$. (It will be important for the proof of Proposition 4.9 below that we are working in $SL(2, \mathbb{C})$ not $PSL(2, \mathbb{C})$.) Hence

$$UVU^{-1}V^{-1} = R_3R_1R_5R_3R_1R_3R_3R_5 = -(R_3R_1R_5)^2.$$

On the other hand,

$$\operatorname{Tr}(R_5R_4R_3)\operatorname{Tr}(R_4R_1) = \operatorname{Tr}(R_5R_4R_3R_4R_1) + \operatorname{Tr}(R_5R_4R_3R_1R_4) = -2\operatorname{Tr}(R_5R_3R_1).$$

By the above, $\operatorname{Tr}(R_5R_4R_3)\operatorname{Tr}(R_4R_1) = -4i\sinh\sigma_4\cosh\sigma_{14} = 4\,Am(\sigma_1, \sigma_3, \sigma_5)$.

From this we check easily that $Am^2(\sigma_1, \sigma_3, \sigma_5) = (2 - \operatorname{Tr}[U, V])/4$. Since as we have seen the trace of the commutator is an invariant of generator triples, it follows that, up to sign, so is $Am(\sigma_1, \sigma_3, \sigma_5)$. □

Definition 4.5. *For a loxodromic element $X \in SL(2, \mathbb{C})$ let $\lambda(X) = (\ell(X) + i\theta(X))/2$ be* half *the complex length, oriented in the direction of positive translation. If U, V are two loxodromics, each oriented in the direction of positive translation, let δ_{UV} be the complex distance between their axes, oriented in the direction of from U to V.*

With this notation, we have

$$Am(\sigma_1, \sigma_3, \sigma_5) = -i\sinh\delta_{UV}\sinh\lambda(U)\sinh\lambda(V). \tag{6}$$

We refer to this as the *amplitude* of $\mathcal{H}(u, v)$.

4.3. **Some simple observations.** We need a few more simple observations.

Lemma 4.6. (See [3] P. 707.) Suppose that $\mathbf{u}, \mathbf{v} \in \Omega$ are adjacent to an oriented edge $\vec{e}$ of $\mathcal{T}$ with $\mathbf{w}, \mathbf{z}$ being the regions at the head and tail of $\vec{e}$ respectively. Then $\Re(\frac{\hat{\mathbf{z}}}{\hat{\mathbf{u}}\hat{\mathbf{v}}}) \geq 1/2$, where $\hat{\mathbf{z}} = \phi_\rho(\mathbf{z})$ and so on as in Section 3.

Proof. It is easy to check that if $\xi, \eta \in \mathbb{C}$ and $\xi + \eta = 1, |\eta| \leq |\xi|$, then $\Re\xi \geq 1/2$. With $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ as in the statement we have $\hat{\mathbf{z}} + \hat{\mathbf{w}} = \hat{\mathbf{u}}\hat{\mathbf{v}}$ and $|\hat{\mathbf{z}}| \geq |\hat{\mathbf{w}}|$. Now apply the above with $\xi = \frac{\hat{\mathbf{z}}}{\hat{\mathbf{u}}\hat{\mathbf{v}}}, \eta = \frac{\hat{\mathbf{w}}}{\hat{\mathbf{u}}\hat{\mathbf{v}}}$. □

Lemma 4.7. *If $\xi \in \mathbb{C}$, then $\Re\xi \geq 0$ if and only if $\Re(\tanh\xi) \geq 0$.*

Proof. If $\xi = x + iy$ then $\Re(\tanh \xi) = \dfrac{\sinh x \cosh x}{|\cosh x \cos y + i \sinh x \sin y|^2}$. □

We will also need a comparison of hyperbolic translation lengths and traces.

For a loxodromic element $X \in SL(2, \mathbb{C})$ let $\ell(X) > 0$ denote the (real) translation length and as in Definition 4.5 let $\lambda(X) = (\ell(X) + i\theta(X))/2$ be half the complex length, so that $\operatorname{Tr} X = \pm 2 \cosh \lambda(X)$.

Lemma 4.8. *There exists $L_0 > 0$ so that if $\xi + i\eta \in \mathbb{C}$ with $\xi > L_0$ then $\xi - \log 3 \le \log|\cosh(\xi + i\eta)| \le \xi$ and $|\sinh(\xi + i\eta)| \ge e^{\xi}/3$. In particular, for $X \in SL(2, \mathbb{C})$ we have $e^{\ell(X)/2}/3 \le |\operatorname{Tr} X|/2 \le e^{\ell(X)/2}$ and $|\sinh \lambda(X)| \ge e^{\ell(X)/2}/3$ whenever $\ell(X) > L_0$.*

Proof. For the right hand of the first inequality involving cosh, since $|\cosh(\xi + i\eta)| = e^{\xi}|(1 + e^{-2\xi-2i\eta})|/2$ we have

$$\log|\cosh(\xi + i\eta)| = \xi + \log|(1 + e^{-2\xi-2i\eta})|/2 \le \xi$$

since $|(1 + e^{-2\xi-2i\eta})|/2 \le 1$.

For the left hand inequality, since $\xi > L_0$ we have, choosing L_0 large enough, $|(1 + e^{-2\xi-2i\eta})|/2 \ge 1/3$ so that $\log(|(1 + e^{-2\xi-2i\eta})|/2) \ge -\log 3$ and hence $\log|\cosh(\xi + i\eta)| \ge \xi - \log 3$. The estimate on $\operatorname{Tr} X$ follows setting $\lambda(X) = \xi + i\eta$ and the estimates on $|\sinh(\xi + i\eta)|$ follows similarly. □

4.4. **The key step.** We now come to the key steps from [11] used to prove Theorem 5.3.

Proposition 4.9. *([11] Lemma 5.1) Suppose that $\rho \in \mathcal{B}$ and that $0 < \alpha < \pi/2$ is given. Suppose also that as in Lemma 4.6, $\mathbf{u}, \mathbf{v} \in \Omega$ are adjacent to an oriented edge $\vec{e}$ of $\mathcal{T}$. With N_0 as in Corollary 3.9, suppose $u \in \mathbf{u}, v \in \mathbf{v}$ are a proper generator pair and that $\max\{||u||, ||v||\} > N_0$. As in Definition 4.5, let δ_{UV} be the complex distance between the axes of $U = \rho(u), V = \rho(v)$, oriented in the direction of positive translation. Then there exists $L_1 > 0$ depending only on α and ρ such that $|\Im \delta_{UV}| \le \alpha$ whenever $\max\{\ell(U), \ell(V)\} > L_1$.*

Proof. Without loss of generality, suppose that $\ell(U) \ge \ell(V)$. Let $\delta_{UV} = d + i\theta$. By Lemma 3.3, $\Omega(m)$ is finite for any $m \ge 2$, moreover $\operatorname{Tr} \rho(g) \ne \pm 2$ for all $g \in \mathcal{P}$. Hence there exists $c > 0$ such that $|\operatorname{Tr} \rho(g) \pm 2| > c$ for all $g \in \mathcal{P}$. Hence $|\sinh \lambda(G)|$ is uniformly bounded away from 0 for all $g \in \mathcal{P}$, where $G = \rho(g)$. By Proposition 4.4 the absolute value of the amplitude of $\mathcal{H}(u, v)$, that is, $|\sinh \delta_{UV} \sinh \lambda(U) \sinh \lambda(V)|$, is independent of (u, v). Combined with Lemma 4.8, it follows that provided that $\ell(U) > L_0$ we have

$$|\sinh \delta_{UV}| = k'|\sinh \lambda(V)|^{-1}|\sinh \lambda(U)|^{-1} \le k e^{-\ell(U)/2} \tag{7}$$

for constants k', k which depends only on the representation ρ. Since $|\sinh \delta_{UV}|^2 = \cosh^2 d \sin^2 \theta + \sinh^2 d \cos^2 \theta$ we deduce that $d \to 0$ and either $\theta \to 0$ or $\theta \to \pi$ as $\ell(U) \to \infty$.

In the rest of the proof we show that in fact, $\theta \to 0$.

The cosine formula in $\mathcal{H}(u, v)$ (see for example [5] VI.2 (6)) gives

$$\cosh \delta_{UV} = \frac{\cosh \lambda(U^{-1}V^{-1}) - \cosh \lambda(U) \cosh \lambda(V)}{\sinh \lambda(U) \sinh \lambda(V)}$$

and hence

$$\cosh \delta_{UV} \tanh \lambda(U) \tanh \lambda(V) = \frac{\cosh \lambda(U^{-1}V^{-1})}{\cosh \lambda(U) \cosh \lambda(V)} - 1, \tag{8}$$

where we take a consistently oriented hexagon with sides of complex length σ_i as in the discussion on line matrices following Definition 4.3.

By Corollary 3.9 the T-and W-arrows on $\vec{e}$ agree. Hence by Lemma 4.6, uv represents the region at the tail of $\vec{e}$ and uv^{-1} the one at its head, so that $\Re\Big(\dfrac{\operatorname{Tr}\lambda(VU)}{\operatorname{Tr}\lambda(U)\operatorname{Tr}\lambda(V)}\Big) \geq 1/2$. We want to apply this result to $\Re\Big(\dfrac{\cosh\lambda(U^{-1}V^{-1})}{\cosh\lambda(U)\cosh\lambda(V)}\Big)$, so we need to take care with signs.

As noted in the discussion following Definition 4.3, the signs of the traces in the hexagon $\mathcal{H}(u,v)$ are determined by the formula $\operatorname{Tr} R_{i-1}R_{i+1} = -\cosh\sigma_i$, moreover as elements in $SL(2,\mathbb{C})$ we have $R_3R_1 = -U, R_5R_3 = -V$ and $R_1R_5 = -U^{-1}V^{-1}$. Hence $\operatorname{Tr} U = 2\cosh\lambda(U)$, $\operatorname{Tr} V = 2\cosh\lambda(U)$ and $\operatorname{Tr} VU = \operatorname{Tr} U^{-1}V^{-1} = 2\cosh\lambda(U^{-1}V^{-1})$ so by Lemma 4.6, with the same choice of signs as in (8), $\Re\Big(\dfrac{\cosh\lambda(U^{-1}V^{-1})}{\cosh\lambda(U)\cosh\lambda(V)}\Big) \geq 1$.

On the other hand, $\ell(U) \to \infty$ so that $\tanh\lambda(U) \to 1$. By Lemma 4.7, $\Re\tanh\lambda(V) \geq 0$. Moreover, since $\rho \in \mathcal{B}$, $\Re\tanh\lambda(V) \neq 0$ and hence, since $\Omega(m)$ is finite, $\Re\tanh\lambda(V)$ is uniformly bounded away from 0. Now if $d \to 0$ and $\theta \to \pi$ then $\cosh\delta_{UV} \to -1$ so that $\Re(\cosh\delta_{UV}\tanh\lambda(U)\tanh\lambda(V)) < c < 0$ as $\ell(U) \to \infty$. Taking real parts in (8) gives a contradiction. We deduce that $\theta \to 0$ and the result follows. □

Proposition 4.10. *(*[11] *Theorem 5.4) Suppose that* $\mathbf{u}, \mathbf{v} \in \Omega$ *are adjacent to an edge* e *of* $\mathcal{T}$ *with associated proper generator pair* $(u,v), u \in \mathbf{u}, v \in \mathbf{v}$. *Then there is a half space* $\hat{H}$ *and* $L_2 > 0$ *so that if* $\max\{\ell(U), \ell(V)\} \geq L_2$, *then for any* $X, Y \in \{U, V\}$, *the half spaces* $X^{-1}\hat{H} \supset \hat{H} \supset Y\hat{H}$ *are properly nested.*

Proof. Suppose for definiteness that $\ell(U) \geq \ell(V)$. Let H be the hyperplane orthogonal to $\operatorname{Ax} V$ and containing the common perpendicular D to $\operatorname{Ax} U, \operatorname{Ax} V$. Let $\hat{H}$ be the half space cut off by H and containing the forward pointing unit tangent vector $\mathbf{t_V}$ to $\operatorname{Ax} V$ at $P = \operatorname{Ax} V \cap D$. Note that $V^{-1}\hat{H} \supset \hat{H} \supset V\hat{H}$ are properly nested since V is loxodromic and translates H disjointly from itself.

Now suppose $Y = U$. Note that for L sufficiently large, by Proposition 2.3, $\ell(U) > L$ implies that $||u||_{a,b} > N_0$ with N_0 as in Proposition 4.9. Hence by Proposition 4.9 we can choose $L = L_1(\pi/4)$ so that $|\Im\delta_{UV}| \leq \pi/4$ whenever $\ell(U) \geq L$. Let Q be the intersection point of $\operatorname{Ax} U$ with D and let $\mathbf{t_U}$ be the forward pointing unit tangent vector along $\operatorname{Ax} U$ at Q. Then $\mathbf{t_V}$ is translated by distance $\Re\delta_{UV}$ and rotated by angle $\Im\delta_{UV}$ along D to coincide with $\mathbf{t_U}$ at Q. Thus $\mathbf{t_U}$ makes an angle at most $\pi/4$ with the inward pointing normal $\mathbf{n_Q}$ to $\hat{H}$ at Q. Likewise $U(\mathbf{t_U})$ makes an angle at most $\pi/4$ with the inward pointing normal $U(\mathbf{n_Q})$ to $U(\hat{H})$. It follows by Corollary 4.2 that for $\ell(U)$ sufficiently large, the half planes $\hat{H} \supset U(\hat{H})$ are properly nested and hence so are $U^{-1}(\hat{H}) \supset \hat{H}$. This completes the proof. □

Proposition 4.11. *(*[11, Theorem 5.4]*) Suppose that* (u,v) *is a proper generator pair such that that* $\max\{\ell(U), \ell(V)\} > L_2$ *with* L_2 *as in Proposition 4.10. Let* $\mathcal{C}(u,v)$ *denote the set of all cyclically reduced words which are products of positive powers of u's and v's. Then the collection of broken geodesics* $\{\mathbf{br}_\rho(w;(u,v)), w \in \mathcal{C}(u,v)\}$ *is uniformly quasigeodesic.*

Proof. With the notation of Proposition 4.10, pick a basepoint O in the hyperplane H and let d be the minimum distance between any pair of the planes $H, U(H), V(H)$. Label the vertices of $\mathbf{br}_\rho(w;(u,v))$ in order as $P_n, n \in \mathbb{Z}$ with $O = P_0$ and denote the image of H containing P_n by H_n.

Any three successive vertices P_n, P_{n+1}, P_{n+2} are of the form $ZX^{-1}O, ZO, ZYO$ for some $X, Y \in \{U = \rho(u), V = \rho(v)\}, Z \in \rho(F_2)$. Therefore by Proposition 4.10 the corresponding half spaces $\hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+2}$ are properly nested. It follows that each consecutive pair of half spaces in the sequence $\ldots, \hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+2}, \ldots$ are properly nested and hence that $d(P_n, P_m) \geq d(\hat{H}_n, \hat{H}_m) = |n - m|d$ which proves the result. □

5. The Bowditch condition implies primitive stable

In this section we prove Theorem A, that a representation $\rho\colon F_2 \to SL(2, \mathbb{C})$ satisfies the BQ-conditions if and only if ρ is primitive stable.

The result in one direction is not hard, see for example [12].

Proposition 5.1. *The condition PS implies the Bowditch BQ-conditions.*

Proof. Let $u \in \mathcal{P}$. If the broken geodesic $\mathbf{br}(u; (a, b))$ is quasigeodesic then it is neither elliptic nor parabolic, so the first condition $\operatorname{Tr} U \notin [-2, 2]$ holds.

If the collection of broken geodesics $\mathbf{br}(u; (a, b)), u \in \mathcal{P}$ is uniformly quasigeodesic then $\mathbf{br}(u; (a, b))$ is at a uniformly bounded distance from $\operatorname{Ax} U$ for each $u \in \mathcal{P}$, see Remark 1.2. We deduce that

$$c'||u||_{a,b} - \epsilon \leq d_{\mathbb{H}}(O, UO) \leq c + \ell(U)$$

for uniform constants $c, c', \epsilon > 0$. Since only finitely many words have word length less than a given bound, this implies that only finitely many elements have hyperbolic translation lengths and therefore, by Lemma 4.8, traces, less than a give bound. □

It remains to prove the converse. The following lemma is well known.

Lemma 5.2. *Let w be a cyclically reduced word in a generator pair (u, v) and let $\rho\colon F_2 \to SL(2, \mathbb{C})$. Suppose that the image $W = \rho(w)$ is loxodromic. Then the broken geodesic $\mathbf{br}_\rho(w; (u, v))$ is quasigeodesic with constants depending only on $\rho, w,$ and (u, v).*

Proof. Suppose that $||w||_{(u,v)} = k$ and number the vertices $P = \rho(x)O, x \in F_2$ of $\mathbf{br}_\rho(w; (u, v))$ in order as $P_r, r \in \mathbb{Z}$ with $P_0 = O$. We have to show that there exist constants $K, \epsilon > 0$ so that if $n < m$ then

$$(m - n)/K - \epsilon \leq d(P_n, P_m) \leq K(m - n) + \epsilon.$$

Pick $c > 0$ so that $d(O, \rho(h)O) \leq c$ for $h \in \{u, v\}$. Clearly $d(P_n, P_m) \leq c(m - n)$. For the lower bound, write $m - n = rk + k_1$ for $r \geq 0, 0 \leq k_1 < k$. Then for some cyclic permutation of w, say w', setting $W' = \rho(w')$ we have $W'^r(P_n) = P_{n+rk}$ so that $d(P_n, P_{n+rk}) \geq r\ell(W)$. Thus

$$d(P_n, P_m) \geq d(P_n, P_{n+rk}) - d(P_{n+rk}, P_m) \geq (m - n)\ell(W)/k - kc - \ell(W)/k.$$

□

Theorem 5.3. *The Bowditch BQ-conditions implies PS.*

Proof. Choose a finite sink tree $T_F = T_F(M_0)$ as in Theorem 3.4. Use Proposition 3.8 to enlarge $T_F = T_F(M_0)$ if necessary so that the W- and T-arrows coincide for every edge outside T_F. By further increasing M_0 if necessary we can assume that $|\operatorname{Tr} \rho(u)| > M_0$ implies $\ell(U) > \max\{L_0, L_2\}$ with L_0, L_2 as in Lemma 4.8 and Proposition 4.11 respectively.

Suppose now that $e \notin T_F$. Then at least one of the regions $\mathbf{u}$ adjacent to e has $\ell(U) > \max\{L_0, L_2\}$ and moreover the W- and T-arrows on e coincide. Let $\mathbf{v}$ be the other region adjacent to e and suppose that $u \in \mathbf{u}, v \in \mathbf{v}$ are chosen to be a proper generator pair, so that $||uv|| > ||uv^{-1}||$. Since the W-arrow on e points the same direction as the T-arrow it follows that $|\operatorname{Tr} UV| \geq |\operatorname{Tr} UV^{-1}|$. For the same reason, every region in $\mathcal{W}(\vec{e})$ corresponds to a word which is a product of positive powers of u's and v's. Thus by Proposition 4.11 the collection of all broken geodesics corresponding to regions in $\mathcal{W}(\vec{e})$ is uniformly quasigeodesic.

Since T_F is finite, there are finitely many edges $\{\vec{e}_i, i = 1, \ldots, k\}$ whose heads meet T_F. Moreover every region not adjacent to an edge in T_F is in $\mathcal{W}(\vec{e}_i)$ for some i.

There are only finitely many regions $\mathbf{w}$ adjacent to some edge of T_F. By Lemma 5.2, for each such $\mathbf{w}$ and $w \in \mathbf{w}$, the broken geodesic $\mathbf{br}_\rho(w; (a, b))$ is quasigeodesic with constants depending on $\mathbf{w}$.

It follows that there is a finite set of generator pairs $\mathcal{S}$, such that any $w \in F_2$ can be expressed as a word in some $(s, s') \in \mathcal{S}$ in such a way that $\mathbf{br}_\rho(w; (s, s'))$ is quasigeodesic with constants depending only on (s, s'). For fixed (s, s') each quasigeodesic $\mathbf{br}_\rho(w; (s, s'))$ can be replaced by a broken geodesic $\mathbf{br}_\rho(w; (a, b))$ which is also quasigeodesic with a change of constants depending only on (s, s') and not on w. The total number of replacements required involves only finitely many constants and the result follows. □

6. Palindromicity and the Bounded Intersection Property

It is easy to prove Theorem B, that $\rho \in \mathcal{B}$ implies that ρ has the bounded intersection property, using Theorem 5.3.

Proposition 6.1. *If a representation $\rho\colon F_2 \to SL(2, \mathbb{C})$ is primitive stable then it satisfies BIP.*

Proof. The broken geodesic corresponding to any primitive element by definition passes through the basepoint O. The broken geodesics $\{\mathbf{br}_\rho(u; (a, b))\}, u \in \mathcal{P}$ are by definition uniformly quasigeodesic, so, as explained in Remark 1.2, each is at uniformly bounded distance to its corresponding axis. Hence all the axes are at uniformly bounded distance to O and so in particular axes corresponding to primitive palindromic elements cut the three corresponding special hyperelliptic axes in bounded intervals. □

This result is of course much more interesting once we know that all primitive elements have palindromic representatives. We make a precise statement in Proposition 6.2. In Theorem 6.4 we then give a direct proof that $\rho \in \mathcal{B}$ implies that ρ has the bounded intersection property.

6.1. **Generators and palindromicity.** Let $\mathbb{E} = \{0/1, 1/0, 1/1\}$ and define a map $\beta\colon \hat{\mathbb{Q}} \to \mathbb{E}$ by $\psi(p/q) = \bar{p}/\bar{q}$, where $\bar{p}, \bar{q}$ are the mod 2 representatives of p, q in $\{0, 1\}$. We refer to $\psi(p/q)$ as the mod 2 equivalence class of p/q. Say $p/q \in \hat{\mathbb{Q}}$ is of type $\eta \in \mathbb{E}$ if $\psi(p/q) = \eta$. Say a generator $u \in F_2$ is of type η if $u \in [p/q]$ and p/q is of type η; likewise a generator pair (u, v) is of type (η, η') if u, v are of types η, η' respectively. As in Section 1.3, we fix once and for all a generator pair (a, b) and identify a with $0/1$, b with $1/0$ and ab with $1/1$. The *basic generator pairs* are the three generator pairs (a, b), (a, ab) and (ab, b) corresponding to $(0/1, 1/0)$, $(0/1, 1/1)$ and $(1/0, 1/1)$ respectively. For $\eta, \eta' \in \mathbb{E}$ we say u is palindromic with respect to $(\eta, \eta'), \eta \neq \eta'$ if it is palindromic when rewritten in terms of the basic pair of generators corresponding to (η, η'); equally we say that a generator pair (u, v) is cyclically reduced (respectively palindromic with respect to the pair (η, η')) if each of u, v have the same property. We refer to a generator pair (u, v) which is palindromic with respect to some pair of generators, as a *palindromic pair*. Finally, say a generator pair (u, v) is conjugate to a pair (u', v') if there exists $g \in F_2$ such that $gug^{-1} = u'$ and $gvg^{-1} = v'$.

As previously, we use the term 'Farey word' and 'Farey generator pair' to denote the special words obtained by concatentation from the basic generator pair (a, b) in the anticlockwise order from the Farey diagram as in Figure 2. Notice that the notion of a 'positive' word, which was introduced to distinguish between a generator and its inverse, refers to words written in the generators (a, b), that is, as they appear in their natural arrangement in Figure 2. When describing a word as 'palindromic' in one of the other basic generator pairs, it has to be first rewritten in terms of the new generators and the term 'positive' is not directly applicable.

Proposition 6.2. *If $u \in \mathcal{P}$ is a cyclically reduced positive word of type $\eta \in \mathbb{E}$, then, for each $\eta' \neq \eta$, then there is exactly one cyclically reduced generator u' which is conjugate to u and which is palindromic with respect to (η, η'). If (u, v) is a proper generator pair of type (η, η'), then there is exactly one proper generator pair (u', v') conjugate to (u, v) and palindromic with respect to (η, η').*

Proof. We begin by proving the existence part of the second statement. Observe that the edges of the Farey tree $\mathcal{T}$ may be divided into three classes, depending on the mod two equivalence classes of the generators labelling the neighbouring regions. In this way we may assign colours r, g, b to the pairs $(0/1, 1/0); (0/1, 1/1); (1/0, 1/1)$ respectively and extend to a map *col* from edges to $\{r, g, b\}$, see Figure 2. Note that no two edges of the same colour are adjacent, and that the colours round the boundary of each complementary region alternate. For simplicity in what follows, we label each complementary region by the corresponding Farey word.

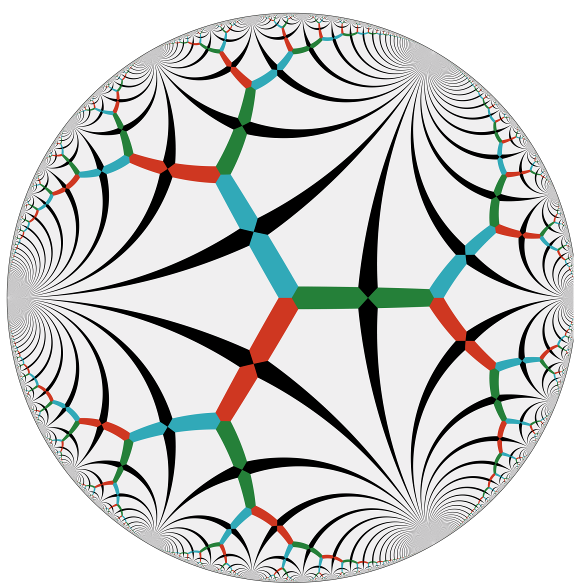

FIGURE 2. The coloured Farey tree. The colours round the boundary of each complementary region alternate. The picture is a conjugated version of the one in Figure 1, arranged so as to highlight the three-fold symmetry between (a, b, ab). Image courtesy of Roice Nelson.

As usual let e_0 be the edge of $\mathcal{T}$ with adjacent regions labelled by (a, b) and let $q^+(e_0)$ and $q^-(e_0)$ denote the vertices at the two ends of e_0, chosen so that the neighbouring regions are labelled (a, b, ab) and (a, b, ab^{-1}) respectively. Removing either of these two vertices disconnects $\mathcal{T}$. We deal first with the subtree $\mathcal{T}^+$ consisting of the connected component of $\mathcal{T} \setminus \{q^-(e_0)\}$ which contains $q^+(e_0)$. Note that the regions adjacent to edges of $\mathcal{T}^+$ correspond to non-negative fractions.

Let e be a given edge of $\mathcal{T}^+$ and let $q^+(e)$ denote the vertex of e furthest from $q^-(e_0)$. Let $\gamma = \gamma(e)$ be the unique shortest edge path joining $q^+(e)$ to $q^-(e_0)$, hence including both e and e_0. The *coloured level* of e, denoted $col.lev(e)$, is the number of edges e' including e itself in $\gamma(e)$ with $col(e') = col(e)$. Note that $\gamma(e)$ necessarily includes e_0, and, provided $e \neq e_0$, one or other of the two edges emanating from $q^+(e_0)$ other than e_0.

Suppose that e is the edge of $\mathcal{T}^+$. The proof will be by induction on $col.lev(e)$.

Suppose first $col.lev(e) = 1$. If $e = e_0$ the result is clearly true, since the pair (a, b) is palindromic with respect to itself. The other two edges emanating from $q^+(e_0)$ also have $col.lev(e) = 1$ and have neighbouring regions corresponding to the base pairs (a, ab) and (ab, b), each of which pair is palindromic with respect to itself.

If any other edge has $col.lev(e) = 1$, then it is connected to $q^-(e_0)$ by a path $\gamma(e)$ which, after the initial edge e, must alternate between the two other colours, because there cannot be two adjacent edges of the same colour. From Figure 3 with labels a, b in place of u, v, one sees that the only such paths must be contained in the boundary of either the region labelled a or that labelled b. Starting from the vertex $q^+(e)$ the corresponding path $\gamma(e)$ goes either in the clockwise direction around the region labelled a or in the anticlockwise direction around the region labelled b. (Note that since every path has to include the initial edge e_0, the only edge with $col.lev(e) = col.lev(e_0) = 1$ is e_0 itself.)

Consider first the edges pointing inwards to the boundary of the region labelled b, around which the edges are alternately solid (red) and dashed (blue). Starting from the vertex labelled by the three regions (ab, ab^2, b), the inward pointing dotted (green) edges have labels (ab, ab^2), (ab^2, ab^3) and so ons, see Figure 3. Rewriting in terms of the basic generators (a, ab) which label the initial dotted edge we have $(ab^n, ab^{n+1}) = ((ab \cdot a^{-1})^{n-1} \cdot ab, (ab \cdot a^{-1})^n \cdot ab)$ which is palindromic in the pair (a, ab) proving the claim.

Likewise the edges round the region labelled a are alternately solid (red) and dotted (green). Reading in clockwise order starting from the vertex labelled (a, a^2b, ab), the inward pointing dashed (blue) arrows have adjacent labels $(a^2b, ab), (a^3b, a^2b)$ and more generally $(a^{n+1}b, a^nb)$. Rewriting in terms of the generator pair (ab, b) associated to the initial dashed edge, this becomes $((ab{\cdot}b^{-1})^n ab, (ab{\cdot}b^{-1})^{n-1} ab)$ which is palindromic in (ab, b) as required. This completes the first step of the induction.

Suppose now the result is proved for all edges of coloured level $k \geq 1$. Let e be an edge of type (η, η'). Suppose that $col(e) = c$ and let e' be the next edge of γ with $col(e') = c$ along the path $\gamma(e)$ from $q^+(e)$ to $q^-(e_0)$. By the induction hypothesis the Farey generator pair (u, v) adjacent to e' is conjugate to a proper positive pair (u', v') which is palindromic of the same type (η, η').

Let $q^+(e')$ be the vertex of e' closest to e, so that the subpath path γ' of γ from $q^+(e')$ to $q^-(e)$ contains no other edges of colour c, where $q^-(e)$ is the vertex of e other than $q^+(e)$. As above, the edges of γ' must alternate between the two other colours. This implies (see Figure 2) that γ' forms part of the boundary of a complementary region R of $\mathcal{T}^+$. Moreover the third edge at each vertex along ∂R (that is, the one which is not contained in ∂R), is coloured c.

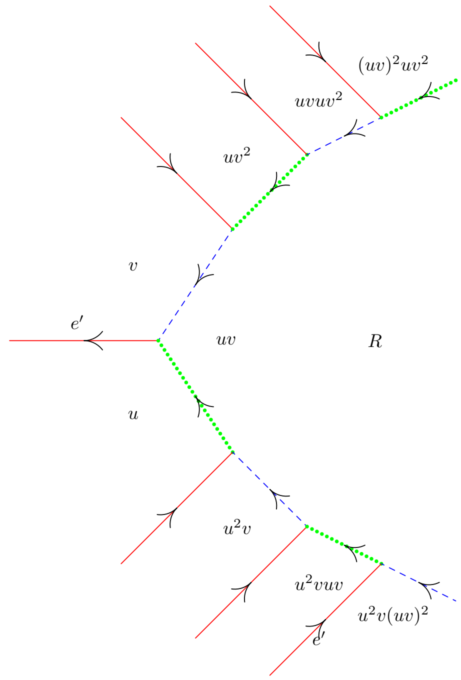

FIGURE 3. Labels of regions round ∂R showing the W-arrows. Note that labels are concatenated in anticlockwise order round the boundary circle; thus for example u is before v and concatenates to uv. 'Red' edges are shown solid, 'green' edges are dashed and 'blue' edges are dotted. In the initial configuration, $u = a$ and $v = b$.

Without loss of generality, suppose that u is before v in the anti-clockwise order round $\partial\mathbb{D}$. Then the generator associated to R is uv. Since (u, v) is a Farey pair, moving in anticlockwise order around ∂R starting from v, successive regions have labels

$$v, u, u^2v, u^2vuv, \ldots, u^2v(uv)^n, \ldots,$$

see Figure 3. Any successive pair, in particular the pair adjacent to e, can be simultaneously conjugated to the form $(uv(uv)^k u, (uv(uv)^{k+1}u)$ for some $k \geq 0$. Since by hypothesis the pair (u, v) is conjugate to a pair (u', v') palindromic with respect to (η, η'), so is $(uv(uv)^k u, (uv(uv)^{k+1}u)$.

Similarly, the regions moving clockwise around ∂R starting from u have labels

$$u, v, uv^2, uvuv^2, \ldots, (uv)^n uv^2, \ldots.$$

Thus any successive pair can be simultaneously conjugated into the form $(v(uv)^k uv, v(uv)^{k+1}uv)$ for some $k \geq 0$ which is likewise conjugate to a pair palindromic with respect to (η, η').

By the same argument for the tree $\mathcal{T}^-$ consisting of the connected component of $\mathcal{T} \setminus \{q^+(e_0)\}$ which contains $q^-(e_0)$ we arrive at the statement that the generators associated to each edge of $\mathcal{T}^-$ can be written in a form which is palindromic with respect to one of the three generator pairs associated to the edges emanating from $q^-(e_0)$, that is, (a, b^{-1}), $(a, b^{-1}a)$ or $(b^{-1}a, b^{-1})$. The first pair is obviously palindromic with respect to (a, b^{-1}). Noting that $b^{-1}a = (b^{-1}a^{-1})a^2$ which is conjugate to the word $a(b^{-1}a^{-1})a$ palindromic with respect to (a, ab), and that $b^{-1}a = b^{-1}(ab)b^{-1}$ which is palindromic with respect to (b, ab), the result follows.

Now we prove the existence part of the first claim. Suppose that $u \in \mathcal{P}$ is of type $\eta \in \mathbb{E}$ and that $\eta' \neq \eta$. Choose a generator v of type η' so that (u, v) is a proper generator pair. By the above there is a conjugate pair (u', v') palindromic with respect to (η, η') and u' is a generator as required.

To see that u' is unique, suppose that cyclically reduced positive primitive elements u and u' are in the same conjugacy class and are both palindromic with respect to the same pair of generators, which we may as well take to be $\{0/1, 1/0\}$. Notice that u necessarily has odd length, for otherwise the exponents of a and b are both even.

Let $u = e_r \dots e_1 f e_1 \dots e_r$ and suppose that $f' = e_k$ is the centre point about which u' is palindromic for some $1 \le k \le r$. Then $\dots uu \dots$ is periodic with minimal period of length $2r + 1$ and contains the subword

$$e_r \dots e_1 f e_1 \dots e_{k-1} f' e_{k-1} \dots e_1 f e_1 \dots e_r$$

so after $f e_1 \dots e_{k-1} f' e_{k-1} \dots e_1$ the sequence repeats. Since this subword has length $2k < 2r+1$ this contradiction proves the result.

The claimed uniqueness of generator pairs follows immediately. □

As a corollary to this result we obtain the independence of BIP property from the generators (a, b), see also [11] Propsition 6.11.

Proposition 6.3. *The definition of the bounded intersection property is independent of the choice of generating set.*

Proof. Denote the hyperellipitic axis corresponding to (η, η') by $\mathcal{E}(\eta, \eta')$, so that $\mathcal{E}(\eta, \eta')$ is one of $\mathcal{E}(A, B), \mathcal{E}(A, AB), \mathcal{E}(B, AB)$. Suppose ρ satisfies BIP with respect to the generator pair (a, b). Then by definition every primitive axes palindromic with respect to (η, η') intersects $\mathcal{E}(\eta, \eta')$ orthogonally within a bounded interval $J(\eta, \eta')$.

Now suppose (u, v) is a proper generator pair of type (η, η'). It is enough to show that every primitive element palindromic with respect to the generators (u, v) intersects the common perpendicular $\mathcal{E}(U, V)$ of $U = \rho(u), V = \rho(v)$ in a bounded interval $J(U, V)$.

By Proposition 6.2 there is a conjugate generator pair $(u', v') = (gug^{-1}, gvg^{-1})$ which is palindromic with respect to (η, η'), indeed gug^{-1}, gvg^{-1} are cyclic permutations of u, v. Hence the axes of both $U' = \rho(u')$ and $V' = \rho(v')$ both intersect $\mathcal{E}(\eta, \eta')$ orthogonally, so that the common perpendicular $\mathcal{E}(U', V')$ of U' and V' coincides with $\mathcal{E}(\eta, \eta')$. Moreover any pair palindromic with respect to (u', v') is also palindromic with respect to (η, η'), and by Proposition 6.2 applied to the initial pair (u', v') and writing the generators (η, η') in terms of (u', v'), one sees that conversely every pair palindromic with respect to (η, η') is also palindromic with respect to (u', v'). Hence the bounded intervals $J(U, V)$ and $J(\eta, \eta')$ coincide.

Now since $(u', v') = (gug^{-1}, gvg^{-1})$, if w is a primitive element palindromic with respect to (u', v') then $g^{-1}wg$ is primitive and palindromic with respect to (u, v) and conversely. In this case $\mathrm{Ax}\, w$ cuts $\mathcal{E}(U', V')$ orthogonally and $g^{-1}(\mathrm{Ax}\, w) = \mathrm{Ax}\, gwg^{-1}$ cuts $\mathcal{E}(U, V)$ orthogonally, and g^{-1} carries $J(\eta, \eta')$ to a bounded interval on $\mathcal{E}(U, V)$. The result follows. □

6.2. **Direct proof of Theorem B.** It may also be of interest to give a direct proof that $\rho \in \mathcal{B}$ implies that ρ has the bounded intersection property. Theorem 6.4 below is a simplified version of the proof of this result from [15]. It is based on estimating the distance between pairs of palindromic axes along their common perpendicular. We use the estimate (7) derived from the invariance of the amplitude (up to sign) under change of generators to improve the corresponding estimate in Proposition 4.6 in [15].

Theorem 6.4. *(Direct proof of Theorem B.) If $\rho \in \mathcal{B}$ then ρ has the bounded intersection property.*

Proof. Assume that $\rho \in \mathcal{B}$ and choose $M_0 \geq 2$ and a finite connected non-empty subtree tree T_F of $\mathcal{T}$ as in Theorem 3.4. Let $\Omega(T_F)$ be the set of regions $\mathbf{u} \in \Omega$ such that $\mathbf{u}$ is adjacent to an edge of T_F. By enlarging T_F if necessary, we can ensure that every region in $\Omega(2)$ is adjacent to some edge of T_F. In addition, since there are only finitely many possible pairs of elements of $\Omega(2)$, we may yet further enlarge T_F so that no edge outside T_F is adjacent to a region in $\Omega(2)$ on both sides.

Suppose the generator $u = u_1$ is palindromic with respect (η, η') and that $\eta' \neq \eta$. Without loss of generality, we may take u positive. Let $\mathcal{E} = \mathcal{E}_{\eta,\eta'}$ be the corresponding special hyperelliptic axis. Let Ξ denote the set of axes corresponding to palindromic representatives of $\mathbf{v} \in \Omega(T_F)$ which are of types either η or η'. It is sufficient to see that $\operatorname{Ax} U$ meets $\mathcal{E}$ at a uniformly bounded distance to one of the finitely many axes in Ξ.

If $\mathbf{u}_1 \in \Omega(T_F)$ there is nothing to prove, so suppose that $\mathbf{u}_1 \notin \Omega(T_F)$. Choose an oriented edge $\vec{e}_1$ in $\partial \mathbf{u}_1$. Then there is a strictly descending path β of T-arrows $\vec{e}_1, \dots, \vec{e}_n$ so that the head of $\vec{e}_n$ meets an edge in T_F, and this is the first edge in β with this properly. We claim that there is a sequence of positive cyclically reduced generators $u_1 = u, u_2, \dots, u_k \in \mathcal{P}$ such that for $i = 1, \dots, k-1$:

(1) $(\mathbf{u}_i, \mathbf{u}_{i+1})$ are neighbours adjacent to an edge of β.
(2) $u_i \in \mathbf{u}_i, u_{i+1} \in \mathbf{u}_{i+1}$ and (u_i, u_{i+1}) is a proper generator pair palindromic with respect to (η, η').
(3) $\mathbf{u}_k \in \Omega(T_F)$ but $\mathbf{u}_i \notin \Omega(T_F), 1 \leq i < k$.

Suppose that $u_1, \dots, u_i$ have been constructed with properties (1) and (2) with $i \geq 1$ and that $\mathbf{u}_i \notin \Omega(T_F)$. The path β travels round $\partial \mathbf{u}_i$, eventually leaving it along an arrow $\vec{e}$ which points out of $\partial \mathbf{u}_i$. If $\mathbf{u}_i$ is of type η (respectively η') then of the two regions adjacent to $\vec{e}$, one, $\mathbf{u}'$ say, is of type η' (respectively η). Set $\mathbf{u}_{i+1} = \mathbf{u}'$ and choose $u_{i+1} \in \mathbf{u}_{i+1}$ so that (u_i, u_{i+1}) is positive and palindromic with respect to (η, η'). (Notice that we are using the uniqueness of the palindromic form for u_i, in other words if (u_{i-1}, u_i) is the positive palindromic pair associated to the regions $(\mathbf{u}_{i-1}, \mathbf{u}_i)$ then (u_i, u_{i+1}) is the positive palindromic pair associated to the regions $(\mathbf{u}_i, \mathbf{u}_{i+1})$.) If $\mathbf{u}_{i+1} \in \Omega(T_F)$ we are done, otherwise continue as before. Since β eventually lands on an edge of T_F, the process terminates. This proves the claim.

Since (u_i, u_{i+1}) are palindromic with respect to (η, η'), the axes $\operatorname{Ax} U_i, \operatorname{Ax} U_{i+1}$ are orthogonal to the hyperelliptic axis $\mathcal{E}_{\eta,\eta'}$ and hence Inequality (7) gives $d(\operatorname{Ax} U_i, \operatorname{Ax} U_{i+1}) \leq O(e^{-\ell(U_i)}), 1 \leq i < k$.

Now let $\vec{e}$ be the oriented edge between $\mathbf{u}_{k-1}, \mathbf{u}_k$ and let $\mathcal{W}(\vec{e})$ be its wake. Then since the edge between $\mathbf{u}_i, \mathbf{u}_{i+1}$ is always oriented towards $\vec{e}$, we see that $\mathbf{u}_i \in \mathcal{W}(\vec{e}), 0 \leq i \leq k$. Let $F_{\vec{e}}$ be the Fibonacci function on $\mathcal{W}(\vec{e})$ defined immediately above Lemma 3.7. It is not hard to see that for $0 \leq i \leq k$ we have $F_{\vec{e}}(\mathbf{u}_i) \geq k - i$. By construction, $\mathbf{u}_{k-1} \notin \Omega(T_F)$ so that, by our assumption on T_F, we have $\mathbf{u}_{k-1} \notin \Omega(2)$. Moreover by connectivity of T_F, no edge in $\mathcal{W}_{\mathcal{E}}(\vec{e})$ is in T_F and hence none of these edges is adjacent on both sides to regions in $\Omega(2)$. Thus by Lemma 3.7, there exist $c > 0, n_0 \in \mathbb{N}$ depending only on ρ and not on $\vec{e}$ such that $\log^+ |\operatorname{Tr} U_i| \geq c(k-i)$ for all but at most n_0 of the regions $\mathbf{u}_i$.

Hence for all except some uniformly bounded number of the regions $\mathbf{u}_i$, $\ell(U_i) \geq c(k-i) - \log 2$. Since all axes $\operatorname{Ax} U_i$ intersect $\mathcal{E}$ orthogonally in points P_i say, it follows that

$d(\mathrm{Ax}\, U_1, \mathrm{Ax}\, U_k)$ is bounded above by the sum $\sum_1^{k-1} d(\mathrm{Ax}\, U_i, \mathrm{Ax}\, U_{i+1})$ of the distances between the points P_i, P_{i+1}. Since $d(\mathrm{Ax}\, U_i, \mathrm{Ax}\, U_{i+1}) \leq O(e^{-\ell(U_i)}), 1 \leq i < k$, the distance from $\mathrm{Ax}\, U_1$ to one of the finitely many axes in Ξ is uniformly bounded above, and we are done. $\square$

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