Kerckhoff's lines of minima in Teichmüller space

Mathematics Institute, University of Warwick,
Coventry CV4 7AL, UK
email: `C.M.Series@warwick.ac.uk`

Abstract. We survey the known results about lines of minima introduced by Kerckhoff in [18], and also Rafi's results about curves which are short on surfaces along Teichmüller geodesics, in particular their use in proving that lines of minima are Teichmüller quasi-geodesics.

2000 Mathematics Subject Classification: 30F60, 30F35, 32G15, 30F40.

Keywords: Line of minima, Teichmüller space, geodesic lamination, earthquake, Teichmüller geodesic, quasifuchsian group.

Contents

1 Introduction

The usual geometries of Teichmüller space relate largely to analytic properties of Riemann surfaces such as quasiconformal maps and quadratic differentials. Lines of minima were introduced by Kerckhoff [18] as a means of endowing Teichmüller space with a geometry directly related to the hyperbolic structure uniformising the surface.

The fundamental observation is the following. Suppose that μ and ν are measured laminations on a hyperbolic surface S, which fill up S in the sense that all the complementary components are disks or once punctured disks. Then the sum of the hyperbolic lengths $l_\mu + l_\nu$ has a unique minimum $M(\mu,\nu)$ on the Teichmüller space $\mathcal{T}(S)$ of S. This result is a consequence of Thurston's earthquake theorem and the convexity of lamination length along earthquake paths. The proof is sketched in Section 2.

Now consider convex combinations $(1-t)l_\mu + tl_\nu$. The set of minima $M((1-t)\mu, t\nu)$ for $t \in (0,1)$ is a 1-manifold $\mathcal{L}_{\mu,\nu}$ embedded in $\mathcal{T}(S)$, called the *Kerckhoff line of minima* of μ and ν. Kerckhoff showed that the lines $\mathcal{L}_{\mu,\nu}$ mimic many properties of geodesics in the Poincaré disc model of hyperbolic space $\mathbb{H}^2$. For example, two projective laminations in $PML(S)$, the Thurston boundary of $\mathcal{T}(S)$, determine a unique line of minima. The lines $\mathcal{L}_{\mu,\nu}$ for a fixed measured lamination μ foliate $\mathcal{T}(S)$. Any two points in $\mathcal{T}(S)$ lie on at least one line, although whether or not this line is unique, is unknown. Lines of minima are intimately connected with earthquakes, in particular $M(\mu,\nu)$ is the unique point in $\mathcal{T}(S)$ at which $\partial/\partial t_\mu = -\partial/\partial t_\nu$, where $\partial/\partial t_\mu$ is the tangent to the earthquake path along μ. More such results are detailed in Section 2. In Section 3 we discuss various extensions and examples of Kerckhoff's results, most of which are extracted from [8, 9, 35].

Thus far, the main application of lines of minima has to been to problems about small deformations of Fuchsian groups, via a link discovered by Series [35, 36]. One can deform a Fuchsian group by bending along a lamination μ, a bend or quakebend being the complex analogue of an earthquake, see Section 4.2. For small values of the bending parameter this gives a quasifuchsian group the boundary of whose convex core on one side is bent along the lamination μ. It turns out that the bending lamination on the other side of the convex core is ν, if and only if the initial Fuchsian group lies on $\mathcal{L}_{\mu,\nu}$. Subsequently Bonahon [3] used these ideas partially to prove a conjecture of Thurston about the uniqueness of groups with given bending data. These results are explained in Section 4.

It is natural to make a comparison between lines of minima and Teichmüller geodesics. The minimisation property of the length functions is analogous to an important minimisation property along Teichmüller geodesics. A Teichmüller geodesic is also determined by a pair of laminations μ,ν which fill up S, namely the horizontal and vertical foliations of the defining quadratic differential, see Section 2.5. Gardiner and Masur [12] showed that this Teichmüller geodesic $\mathcal{G}_{\mu,\nu}$ can be characterised as the line along which the product of the extremal lengths of μ,ν is minimised. Unlike the sum of lengths along $\mathcal{L}_{\mu,\nu}$, this product remains constant along $\mathcal{G}_{\mu,\nu}$. The limiting behaviour of lines of minima and Teichmüller geodesics is at least partly comparable, see Section 3.4 and [9]. More detailed questions about the relationship between lines of minima and Teichmüller geodesics have been explored by Choi, Rafi and Series. In [6] they gave a combinatorial estimate of the distance between the two paths and in [7], proved that a line of minima is a Teichmüller quasi-geodesic. These results are discussed in Section 5.

The methods of Section 5 are heavily dependent on some techniques originally due to Minsky [24] and developed further by Rafi [31, 32, 33], on curves which are short on surfaces along a Teichmüller geodesic. Since this work contains some powerful and interesting techniques, we take the opportunity to summarize it briefly in Section 6.

I would like to thank Athanase Papadopoulos for giving me the opportunity to present this relatively recent body of work in this volume.

2 Kerckhoff's original paper

2.1 Basic definitions

Let S be a surface of hyperbolic type with genus g and b punctures, and denote its Teichmüller space by $\mathcal{T}(S)$. For simplicity, Kerckhoff restricted his statements in [18] to surfaces without punctures, but remarked that the work easily generalises to the finite area case. Here we state the results for surfaces with punctures, noting some of the points where the original proofs need a little attention.

We denote the spaces of measured and projective measured laminations on S, by $ML(S)$ and $PML(S)$ respectively. Then $\mathcal{T}(S)$ and $ML(S)$ are open balls of real dimension $2d$ with $d = 3g-3+b$. The Thurston compactification of $\mathcal{T}(S)$ adjoins $PML(S)$, homeomorphic to the $2d-1$-sphere, as a boundary [[11] Exposé 8]. For $\mu \in ML(S)$ we denote by $[\mu]$ its projective class and by $|\mu|$ its underlying support.

Let $\mathcal{S}$ denote the set of homotopy classes of simple non-peripheral curves on S. We call a lamination $\mu \in ML(S)$ *rational* if $|\mu|$ is a disjoint union of closed geodesics $\alpha_i \in \mathcal{S}$. We write such laminations $\sum_i a_i\alpha_i$, where $a_i \in \mathbb{R}^+$ and α_i (more properly δ_{α_i}) represents the lamination with support α_i which assigns unit mass to each intersection with α_i.

The hyperbolic length l_μ of a lamination $\mu \in ML(S)$ is the function on $\mathcal{T}(S)$ which associates to each $p \in \mathcal{T}(S)$ the total mass of the measure which is the product of hyperbolic distance along the leaves of μ with the transverse measure μ. (Here and in what follows, the hyperbolic structure on S is the one which uniformises the conformal structure on $p \in \mathcal{T}(S)$.) In particular, if $\mu = \sum_i a_i\alpha_i$ is rational, then $l_\mu = \sum_i a_i l_{\alpha_i}$, where $l_{\alpha_i}(p)$ is the hyperbolic length of the geodesic α_i on the surface $p \in \mathcal{T}(S)$. The length is a continuous function $ML(S) \times \mathcal{T}(S) \to \mathbb{R}_{>0}$. Likewise the geometric intersection number $i(\mu,\nu)$ of $\mu,\nu \in ML$ may be defined as continuous function $ML(S) \times ML(S) \to \mathbb{R}_{\geq 0}$ extending the usual geometric intersection number $i(\alpha,\alpha')$ of two geodesics $\alpha,\alpha' \in \mathcal{S}$, see for example [17].

2.1.1 Filling up a surface

Definition 2.1. Laminations $\mu,\nu \in ML(S)$ *fill up* S if $i(\mu,\eta) + i(\nu,\eta) > 0$ for all $\eta \in ML$.

This condition clearly only depends on the projective classes of μ,ν in PML. An equivalent condition is that every component of $S - |\mu| \cup |\nu|$ contains at most one puncture, and whose closure, after filling in the puncture if needed, is compact and simply connected. For the equivalence of these two conditions see [18] Proposition 1.1 and Lemma 4.4, or [28] Proposition 2. These authors treat the case of closed surfaces, however there is no essential difference for surfaces with punctures provided we allow complemetary components which are punctured disks. For rational laminations it is enough to require that the components of the complement be disks or punctured disks. For general laminations this is not enough, since the complement of a generic lamination is a union of ideal polygons. This is why one needs in addition that the complementary components be compact.

If laminations are replaced by foliations, the condition of filling up a surface is also equivalent to a pair of foliations in the measure equivalence classes being realised transversally, see Section 2.5.

2.1.2 Earthquakes The time t *left earthquake* [17] along a lamination $\mu \in ML$ is a real analytic map $\mathcal{E}_\mu(t) : \mathcal{T}(S) \to \mathcal{T}(S)$ which generalises the classical Fenchel-Nielsen twist. The earthquake shifts complementary components of the lamination $|\mu|$ on the surface $p \in \mathcal{T}(S)$ a hyperbolic distance $t\mu(\kappa)$ relative to one another, where $\mu(\kappa)$ is the μ-measure of a transversal κ joining the two components.

If $p \in \mathcal{T}(S)$, we denote by $\mathcal{E}_\mu = \mathcal{E}_\mu(p)$ the earthquake path $\mathcal{E}_\mu(t)(p), t \in \mathbb{R}$. This path induces a flow and hence a tangent vector field $\partial/\partial t_\mu$ on $\mathcal{T}(S)$. In [17], Kerckhoff shows that if $\nu \in ML$, then the length l_ν is a real analytic function of t along $\mathcal{E}_\mu(p)$. One has [[17] Lemma 3.2, [42] Theorem 3.3]:

$$\partial l_\nu/\partial t_\mu = \int\int \cos\theta d\mu \times d\nu, \tag{2.1}$$

where the integral is over all intersection points of a leaf of $|\mu|$ with one of $|\nu|$, and for each such intersection point, θ is the anticlockwise angle from ν to μ. Following [18], we sometimes write $\mathrm{Cos}(\mu,\nu) = \int\int \cos\theta d\mu \times d\nu$. It follows that l_ν is strictly convex along $\mathcal{E}_\mu(p)$ if $i(\mu,\nu) > 0$ and constant otherwise, in particular, l_ν has a unique minimum on $\mathcal{E}_\mu(p)$. Equation (2.1) also immediately gives Wolpert's antisymmetry formula [[41] Theorem 2.11]

$$\partial l_\nu/\partial t_\mu = -\partial l_\mu/\partial t_\nu, \tag{2.2}$$

from which one deduces easily that the minimum points for l_ν along $\mathcal{E}_\mu$ and l_μ along $\mathcal{E}_\nu$ coincide. Wolpert also showed [[42] Theorem 3.4] that if $i(\mu,\nu) > 0$, then at the unique minimum, $\partial^2 l_\nu/\partial t_\mu^2 > 0$ (see [[10] 3.10] for the same result for laminations). This observation becomes crucial in Bonahon's Theorem 4.7 below.

Thurston's earthquake theorem [17, 40] states that for any pair of points $p, p' \in \mathcal{T}(S)$, there is a unique $\mu \in ML(S)$ such that $\mathcal{E}_\mu(1)(p) = p'$. (The proof of the earthquake theorem in [17] requires a bit of attention when S has punctures. The main point is that in the approximation arguments, one has to use the well known fact that every leaf in the support of a lamination $\mu \in ML$ avoids a definite horocycle neighbourhood of each cusp.)

2.2 Kerckhoff's main results

The foundational result on the existence of minima of length functions is a classic application of the earthquake theorem. In fact the proof gives a rather stronger statement:

Theorem 2.2 ([18] Theorem 1.2). *Let $\mu, \nu \in ML$ be laminations which fill up S. Then $l_\mu + l_\nu$ has a unique critical point on $\mathcal{T}(S)$ which is necessarily a minimum.*

Proof. We first show $f_{\mu,\nu} = l_\mu + l_\nu : \mathcal{T}(S) \to \mathbb{R}^+$ is proper. We have to show that $f_{\mu,\nu}(p_n) \to \infty$ whenever $p_n \to \infty$ in $\mathcal{T}(S)$. Pick a subsequence of (p_n) which converges to $\eta \in PML$. Then there exist $c_n \to \infty$ such that $l_\mu/c_n \to i(\mu,\eta)$ and $l_\nu/c_n \to i(\nu,\eta)$, see [11, 27]. Since μ,ν fill up S, we have $i(\mu,\eta) + i(\nu,\eta) > 0$ so $f_{\mu,\nu} \to \infty$ as claimed. It follows that $f_{\mu,\nu}$ has at least one minimum on $\mathcal{T}(S)$.

Now we show there is a unique critical point, which from the above must be a minimum. Suppose that $p, p' \in \mathcal{T}(S)$ are both critical points. By Thurston's earthquake theorem there is a unique $\eta \in ML$ such that $\mathcal{E}_\eta(1)(p) = p'$. Both l_μ and l_ν are convex along this path. Moreover since μ,ν fill up S, we have $i(\mu,\eta) + i(\nu,\eta) > 0$ so that at least one of l_μ, l_ν must be strictly convex. Thus so is $f_{\mu,\nu}$, hence the critical point along this path is unique. □

The next result shows that the assumption that μ,ν fill up S is necessary.

Proposition 2.3 ([18] Theorem 2.1 Part II). *Suppose that $\mu,\nu \in ML$ do not fill up S. Then $l_\nu + l_\mu$ has no critical point on $\mathcal{T}(S)$.*

Proof. It suffices to show that if μ and ν fail to fill up S, then $l_\nu + l_\mu$ can always be decreased. Precisely this statement is proved in [[18] Theorem 2.1 Part II, pps. 194-5]. The argument needs minor changes if S has cusps to allow for the possibility that the complementary components of various laminations may be punctured disks. □

Definition 2.4. Suppose that $\mu,\nu \in ML$ fill up S. The *line of minima* $\mathcal{L}_{\mu,\nu}$ of μ,ν is the image of the path $(0,1) \to \mathcal{T}(S), t \mapsto M((1-t)\mu, t\nu)$. Note that this definition depends only on the projective classes of μ,ν, and that equivalently, $\mathcal{L}_{\mu,\nu}$ is the image of the path $(0,\infty) \to \mathcal{T}(S), k \mapsto M(\mu, k\nu)$. Since the notation is slightly lighter, we shall from now on frequently parameterise the line of minima in this way.

At the critical point $p = M(\mu, k\nu)$, we have $\partial/\partial t_\eta (l_\mu + k l_\nu) = 0$, equivalently using (2.1):

$$\mathrm{Cos}(\mu,\eta) + k\,\mathrm{Cos}(\nu,\eta) = 0 \tag{2.3}$$

for all $\eta \in ML$.

Proposition 2.5 ([18] Theorem 2.1 Parts I, III). *The map $k \to M(\mu, k\nu)$ is a continuous injection $(0, \infty) \to \mathcal{T}(S)$.*

Proof. Injectivity is a special (easy) case of [[18] Theorem 2.1 Part III]. To see this, use (2.3) at the point $M(\mu, k\nu)$, together with the easy observation that $\mathrm{Cos}(\nu, \eta)$ certainly does not vanish simultaneously for all η, to show that $M(\mu, k\nu) = M(\mu, k'\nu)$ implies $k = k'$. Continuity is proved in [[18] Theorem 2.1 Part I]. □

One of Kerckhoff's main results is that the lines $\mathcal{L}_{\mu,\nu}$ for varying ν foliate $\mathcal{T}(S)$ in a manner analogous to the foliation of $\mathbb{H}^2$ by geodesics emanating from one point on the boundary $\partial\mathbb{H}^2$. Fixing $[\mu] \in PML(S)$, we therefore need to understand PML_μ, the set of laminations which together with μ fill up S. Clearly, $PML_\mu \subset PML - \{[\mu]\} \sim \mathbb{R}^{6g-7+2b} = \mathbb{R}^{2d-1}$. If μ is uniquely ergodic and maximal, then by definition $i(\mu, \eta) = 0$ implies $\eta \in [\mu]$. It is easy to see in this case that $PML_\mu = PML - \{[\mu]\}$. In general, PML_μ can smaller, nevertheless we have:

Theorem 2.6 ([18] Theorem 4.7, see also [12] Theorem 8). *Suppose $\mu \in ML \setminus \{0\}$. Then PML_μ is homeomorphic to $\mathbb{R}^{2d-1}$.*

Proof. If μ is rational, one can prove this as follows. Extend the support curves of μ to a pants decomposition $\mathcal{A} = \{\alpha_1, \ldots, \alpha_d\}$ of S. It is not hard to see that μ, ν fill up S if and only if $i(\nu, \alpha_i) > 0$ for all $\alpha_i \in \mathcal{A}$. Setting $q_i(\eta) = i(\eta, \alpha_i)$, let $\{(p_i(\eta), q_i(\eta))_{i=1}^d\}$ denote the Dehn-Thurston coordinates of $\eta \in ML$ relative to $\mathcal{A}$, see for example [30]. Then PML_μ is the image $\mathcal{Q}$ in PML of the set $\{\eta \in ML : q_i(\eta) > 0 \text{ for all } i\}$. We can map any point in $\mathcal{Q}$ to the point in $\mathbb{R}^d \times \Sigma^{d-1}$ whose i^{th} coordinates in $\mathbb{R}^d$ and Σ^{d-1} are p_i/q_i and $q_i/\sum q_j$ respectively, where Σ^{d-1} denotes the $d-1$ dimensional simplex $\{(x_1, \ldots, x_d) : \sum x_i = 1, x_i > 0\}$. This shows that $\mathcal{Q}$ is homeomorphic to $\mathbb{R}^d \times \Sigma^{d-1}$. For the general case, see [18]. □

Fix a continuous section $j : PML \to ML$, for example by choosing a fixed point $p_0 \in \mathcal{T}(S)$ and defining $j([\nu]) = \frac{\nu}{l_\nu(p_0)}$.

Theorem 2.7 ([18] Theorem 2.1). *The map $PML_\mu \times (0, \infty) \to \mathcal{T}(S)$ which sends $([\nu], k)$ to $M(\mu, kj([\nu]))$, is a homeomorphism.*

Proof. By Theorem 2.6, the domain and range are both balls of dimension $2d$. The proof consists in showing that the map is continuous, proper and injective. If the map is not proper, one shows that there exists $(\nu, k) \in PML \times (0, \infty)$

such that $l_\mu + kl_\nu$ has a critical point at $p \in \mathcal{T}(S)$, contradicting Proposition 2.3. To prove injectivity, the question is reduced using (2.3) to showing that if $\mathrm{Cos}(\nu, \zeta) = \mathrm{Cos}(\eta, \zeta)$ for all $\zeta \in ML$ at some point $p \in \mathcal{T}(S)$, then $\nu = \eta$. This is resolved using the more technical but useful Proposition 2.8 which follows. □

Proposition 2.8 (Kerckhoff [18] Theorem 4.8). *Suppose that $\mu, \nu \in ML$ and that $i(\mu, \nu) > 0$. Let $p \in \mathcal{T}(S)$. Then there exists $\eta \in ML$ such that at p:*

$$\mathrm{Cos}(\nu, \eta) > \mathrm{Cos}(\mu, \eta), \mathrm{Cos}(\nu, \eta) > \mathrm{Cos}(\nu, \mu) \text{ and } \mathrm{Cos}(\mu, \nu) > \mathrm{Cos}(\mu, \eta).$$

It follows from Theorem 2.7, that for each $p \in \mathcal{T}(S), \mu \in ML$, there is a unique $\nu_\mu = \nu_\mu(p)$ such that $l_\mu + l_{\nu_\mu}$ is minimised at p. Let

$$\widehat{PML} = PML \times [1, \infty) / PML \times \{1\}$$

be the cone over PML with the cone point as $PML \times \{1\}$. Then:

Theorem 2.9 ([18] Theorem 3.6). *Fix $p \in \mathcal{T}(S)$. Then the map $\widehat{PML} \to \mathcal{T}(S)$ which sends (μ, k) to $M(\mu, k\nu_\mu(p))$ is surjective.*

Corollary 2.10 ([18] Corollary 3.7). *There is a line of minima between every pair of points in $\mathcal{T}(S)$.*

It is not known whether or not this line is unique, see [[18] p. 188].

2.3 Lines of minima and earthquakes

There is a close connection between lines of minima and earthquakes. As remarked above, the earthquake flow $\mathcal{E}_\mu(t)$ generates a field of tangent vectors $\partial/\partial t_\mu$ on $\mathcal{T}(S)$. Likewise, the length function l_μ defines a cotangent vector field dl_μ.

Theorem 2.11. ([18] Theorem 3.5) *For all $p \in \mathcal{T}(S)$, the maps $\mu \to \partial/\partial t_\mu(p)$ and $\mu \to dl_\mu(p)$ are homeomorphisms from ML to the tangent space $T_p(\mathcal{T}(S))$ and the cotangent space $T^*_p(\mathcal{T}(S))$ respectively.*

Proof. To prove that $\mu \to \partial/\partial t_\mu(p)$ is a homeomorphism, one uses invariance of domain. The key step is to prove that the map $\mu \to \partial/\partial t_\mu(p)$ is injective, which follows from Proposition 2.8. The second statement can be proved using (2.2). □

Remark 2.12. The fact that $\{dl_\alpha, \alpha \in \mathcal{S}\}$ span $T^*_p(\mathcal{T}(S))$ is classical and goes back to Fricke-Klein, see [42] p.229.

Corollary 2.13 ([18] Theorem 3.4). *Given $\mu, \nu \in ML$ which fill up S, and $k \in (0, \infty)$, there is a unique $p \in \mathcal{T}(S)$ such that $dl_\mu = -kdl_\nu$, or equivalently $\partial/\partial t_\mu = -k\partial/\partial t_\nu$, at p. Moreover $p = M(\mu, k\nu)$.*

Proof. By Theorem 2.2, $dl_\mu = -kdl_\nu$ at $p \in \mathcal{T}(S)$ if and only if $p = M(\mu, k\nu)$. At $M(\mu, k\nu)$, we have $\partial/\partial t_\eta(l_\mu + kl_\nu) = 0$ for every $\eta \in ML$. Using (2.2) this gives $\dfrac{\partial l_\eta}{\partial t_\mu} = -k\dfrac{\partial l_\eta}{\partial t_\nu}$ for all η. We conclude by Theorem 2.11 that $\partial/\partial t_\mu = -k\partial/\partial t_\nu$. Reversing the argument concludes the proof. $\square$

In fact, the conditions on k, μ, ν in Corollary 2.13 are automatic:

Corollary 2.14 ([35] Proposition 4.7). *Suppose that $\mu, \nu \in ML$ and that $\mu \notin [\nu]$. Suppose also that $\partial/\partial t_\mu = -k\partial/\partial t_\nu$ for some $k \in \mathbb{R}$. Then μ and ν fill up S and $k > 0$. In particular, $p = M(\mu, k\nu)$.*

Proof. The main part of the proof is the case $i(\mu, \nu) > 0$. By Proposition 2.8, for any point $p \in \mathcal{T}(S)$, there exists a lamination $\eta \in ML$ such that at p:

$$\mathrm{Cos}(\nu, \eta) > \mathrm{Cos}(\nu, \mu) \quad \text{and} \quad \mathrm{Cos}(\mu, \nu) > \mathrm{Cos}(\mu, \eta).$$

From $\partial/\partial t_\mu = -k\partial/\partial t_\nu$ we deduce $-k\,\mathrm{Cos}(\nu, \mu) = \mathrm{Cos}(\mu, \mu) = 0$ and hence

$$\mathrm{Cos}(\nu, \eta) > 0 > \mathrm{Cos}(\mu, \eta) = -k\,\mathrm{Cos}(\nu, \eta)$$

which forces $k > 0$. By the same argument as Corollary 2.13, we deduce that $dl_\mu = -kdl_\nu$ at p and the conclusion follows from Proposition 2.3. $\square$

2.4 Action of the mapping class group

Lines of minima are natural for the action of the mapping class group:

Theorem 2.15 ([18] Theorems 3.1, 3.2). *A pseudo-Anosov mapping class ϕ fixes the unique line of minima defined by its stable and unstable laminations. More generally, the action of the mapping class group carries lines of minima to lines of minima.*

Proof. Let $\mu^\pm$ be the two fixed laminations of ϕ, so that $\phi(\mu^\pm) = \lambda^{\pm 1}(\mu^\pm)$ for some $\lambda \in \mathbb{R}^+$, see [11, 27]. From this we deduce immediately that $\mathcal{L}_{\phi(\mu^+),\phi(\mu^-)} = \mathcal{L}_{\mu^+,\mu^-}$. The second statement follows in a similar way. $\square$

2.5 The analogy with Teichmüller geodesics

Let q be a quadratic differential on a Riemann surface R. Its horizontal trajectories equipped with the vertical measure $|\Im\sqrt{q(z)}dz|$ form a measured foliation $\mathcal{H}_q$ on R, similarly the vertical trajectories with the horizontal measure

$|\Re\sqrt{q(z)}dz|$ form a measured foliation $\mathcal{V}_q$, see [39]. Hubbard and Masur [13] showed that on every compact Riemann surface, every equivalence class of measured foliations is realised as the horizontal foliation of a unique quadratic differential q.

Suppose $K \in (0, \infty)$. Then q also determines a Teichmüller map from R to another surface R' with dilatation K. If q' is the terminal quadratic differential on R' with the same norm as q, then the horizontal and vertical foliations of q' are $K^{-1/2}\mathcal{H}_q$ and $K^{1/2}\mathcal{V}_q$ respectively. The foliations $\mathcal{H}_q, \mathcal{V}_q$ are clearly transverse. Gardiner and Masur [[12] Theorem 5.1] showed that conversely, for any pair of transversally realisable measured foliations F, F', there is a unique Teichmüller geodesic determined by a quadratic differential q on a Riemann surface R for which F, F' are measure equivalent to $\mathcal{H}_q$ and $\mathcal{V}_q$ respectively. We note that the condition of being transversally realisable is equivalent to the two corresponding pair of measured laminations filling up the surface, see [[12] Lemmas 3.4 and 5.3].

The Gardiner-Masur Teichmüller line is found by the following minimisation property. Generalising the usual definition of extremal length of a curve, define the extremal length of a measured foliation $[F]$ as

$$\mathrm{Ext}([F]) = ||q|| = \int_R |q(z)|dxdy,$$

where q is the unique quadratic differential on R for which F is equivalent to the horizontal foliation $\mathcal{H}_q$. It is not hard to see that the product $\mathrm{Ext}([\mathcal{H}_q])\mathrm{Ext}([\mathcal{V}_q])$ is constant along the Teichmüller geodesic $\mathcal{G}$ determined by q. [[12] Theorem 5.1] proves that conversely, $\mathrm{Ext}([\mathcal{H}_q])\mathrm{Ext}([\mathcal{V}_q])$ achieves its infimum along $\mathcal{G}$, and that at any point not on $\mathcal{G}$ the product is strictly larger. In the course of the proof, it is shown that if μ and ν are not transversally realisable, then the product can always be decreased, in analogy with Proposition 2.3.

3 Further straightforward properties

In this section, we describe some relatively straightforward extensions of Kerckhoff's results.

3.1 Lines of minima and fixed length horoplanes

In the very special case $2d = dim_{\mathbb{R}}\mathcal{T}(S) = 2$, the analogy between $\mathbb{H}^2$ and $\mathcal{T}(S)$ suggests viewing lines of minima as geodesics and earthquake paths as horocycles. It is not hard to see, [[14] Lemma 6], that in this case a line of minima meets an earthquake path in exactly one point. One way to generalise this is as follows. Note that along the earthquake path $\mathcal{E}_\mu$, the hyperbolic length of the lamination μ is constant. For $\mu \in ML, c > 0$, define the *horoplane* $\mathcal{H}_{\mu;c} = \{p \in \mathcal{T}(S) : l_\mu(p) = c\}$, so that $\mathcal{E}_\mu \subset \mathcal{H}_{\mu;c}$, with equality in the case $d = 1$. As above, let $ML_\mu = \{\nu \in ML : \mu, \nu \text{ fill up } S\}$, and let j be a fixed section $PML \to ML$. The following result is an improvement on [[35] Theorem 7.2].

Theorem 3.1. *For each $\nu \in ML_\mu$, the restriction of l_ν to $\mathcal{H}_{\mu;c}$ has a unique minimum. The minimum point p_ν is also the unique point in which $\mathcal{L}_{\mu,\nu}$ meets $\mathcal{H}_{\mu;c}$. The map $PML_\mu \to \mathcal{H}_{\mu,c}$ sending to $[\nu]$ to $p_{j([\nu])}$ is a homeomorphism.*

Proof. The proof in [35] can be substantially simplified by using Lagrange multipliers to minimise l_ν subject to the constraint that $l_\mu = c$. This shows that at a critical point, $dl_\nu = \lambda dl_\mu$ for some $\lambda \in \mathbb{R}$, which is exactly the condition in Corollary 2.14. □

By contrast, if we prescribe the lengths of all the curves in a pants decomposition $\mathcal{A} = \{\alpha_1, \dots, \alpha_d\}$, the situation is more subtle. For $\mathbf{c} = (c_1, \dots, c_d), c_i > 0$, define the *shearing plane* $\mathcal{E}_{\mathcal{A},\mathbf{c}} = \{p \in \mathcal{T}(S) : l_{\alpha_i}(p) = c_i, i = 1, \dots, d\}$. Let $ML_{\mathcal{A}}$ be the set of laminations ν such that $\sum_i \alpha_i$ and ν together fill up S. We have:

Theorem 3.2 ([35] Theorem 7.3). *Let $\mathcal{A}$ be a pants decomposition of S and let $\mathbf{c} = (c_1, \dots, c_d)$, $c_i > 0$. Then there is a non-empty open set $U \subset ML_{\mathcal{A}}$ such that for all $\nu \in U$, there are no laminations $\mu = \sum_i a_i \alpha_i$ for which $\mathcal{L}_{\mu,\nu}$ meets $\mathcal{E}_{\mathcal{A},\mathbf{c}}$.*

However [8] gives an example of a pants decomposition $\mathcal{A}$ and a curve $\gamma \in ML_{\mathcal{A}}$ for which $\mathcal{L}_{\mathcal{A},\gamma}$ meets $\mathcal{E}_{\mathcal{A},\mathbf{c}}$ for *any* choice of $\mathbf{c}$.

3.2 The simplex of minima

Series [35] and Díaz and Series [8] studied the special case of lines of minima for families of disjoint curves $\mathcal{A} = \{\alpha_1, \dots, \alpha_N\}$ and $\mathcal{B} = \{\beta_1, \dots, \beta_M\}$ which fill up S. The *simplex of minima* $\Sigma_{\mathcal{A},\mathcal{B}}$ associated to $\mathcal{A}$ and $\mathcal{B}$ is the union of lines of minima $\mathcal{L}_{\mu,\nu}$, where $\mu, \nu \in ML(S)$ are strictly positive linear combinations of $\{\alpha_i\}$ and $\{\beta_i\}$, respectively. We can regard $\Sigma_{\mathcal{A},\mathcal{B}}$ as the image of the affine simplex in $\hat{\Sigma}_{\mathcal{A},\mathcal{B}} \subset \mathbb{R}^{N+M-1}$ spanned by independent points $A_1, \dots, A_N, B_1, \dots, B_M$, under the map $\Phi = \Phi(\mathcal{A}, \mathcal{B})$ which sends the point $(1-s)(\sum_i a_i A_i) + s(\sum_j b_j B_j)$ to $M((1-s)(\sum_i a_i l_{\alpha_i}), s(\sum_j b_j l_{\beta_j}))$, where $0 < s, a_i, b_j < 1, \sum a_i = 1, \sum b_j = 1$. The methods of [18] show that Φ is continuous and proper, extending continuously to the faces of $\Sigma_{\mathcal{A},\mathcal{B}}$ corresponding to the subsets of $\mathcal{A} \times \mathcal{B}$ which still fill up S.

The map Φ may or may not be injective. This may be studied by examining the case in which $\mathcal{A}$ and $\mathcal{B}$ are pants decompositions, so that $M = N = d$. Let $\mathcal{M}(\mathcal{A},\mathcal{B})$ be the $d \times d$ matrix $\left(\frac{\partial l_{\beta_i}}{\partial t_{\alpha_j}}\right)$. If $\mu = \sum_i a_i \alpha_i$, we have $\partial/\partial t_\mu = \sum_i a_i \partial/\partial t_{\alpha_i}$ from which $\partial l_{b_j}/\partial t_\mu = \sum_i a_i \partial l_{\beta_j}/\partial t_{\alpha_i}$ for each i and hence

$$(\partial l_{\beta_1}/\partial t_\mu, \ldots, \partial l_{\beta_d}/\partial t_\mu) = \mathcal{M}(\mathcal{A},\mathcal{B})(a_1, \ldots, a_d)^T.$$

Set $\nu = \sum_j b_j \beta_j$. At the point $M(\mu, k\nu)$ we have $\partial/\partial t_\mu = -k\partial/\partial t_\nu$. Hence $\partial l_{\beta_j}/\partial t_\mu = -k \sum_i b_i \partial l_{\beta_j}/\partial t_{\beta_i} = 0$ for all j so that $\det \mathcal{M}(\mathcal{A},\mathcal{B}) = 0$ at $M(\mu, k\nu)$. The converse statement, that $\det \mathcal{M}(\mathcal{A},\mathcal{B}) = 0$ at $p \in \mathcal{T}(S)$ implies $p = M(\mu, k\nu)$, is false. This is because although the condition implies that $\partial/\partial t_\mu = -k\partial/\partial t_\nu$ for some formal combinations $\mu = \sum_i a_i \alpha_i, \nu = \sum_j b_j \beta_j$, the coefficients a_i, b_j may not all have the same sign. Nevertheless, elaborating these observations, we have:

Proposition 3.3 ([8] Proposition 4.6). *Let $\mathcal{A},\mathcal{B}$ be two pants decompositions which fill up S. Then* $\operatorname{rank} \mathcal{M} < d$ *on* $\operatorname{Im} \Phi$. *Moreover Φ is a homeomorphism onto its image if and only if* $\operatorname{rank} \mathcal{M} = d - 1$ *on* $\operatorname{Im} \Phi$. *If* $\operatorname{rank} \mathcal{M} = d - 1$, *then $p \in$* $\operatorname{Im} \Phi$ *if and only if the adjoint matrix of $\mathcal{M}$ has all its entries of the same sign (with possibly some entries vanishing).*

It can be shown by example, see Section 3.3, that both cases $\operatorname{rank} \mathcal{M} = d-1$ and $\operatorname{rank} \mathcal{M} < d - 1$ occur.

3.3 Concrete examples

Not many explicit computations of lines of minima have been made. In the case of the once punctured torus $S_{1,1}$, there are two special cases, corresponding to the laminations μ, ν being supported on curves α, β which intersect once and twice respectively. It turns out [[15] Section 11] that the lines $\mathcal{L}_{\mu,\nu}$ are the lines in $\mathcal{T}(S)$ which correspond to rectangular and the rhombic tori respectively. One can see this as follows. In the first case, let (V, W) be a generator pair for $\pi_1(S_{1,1})$, so that we may take $V = \alpha, W = \beta$. Then any point on $\mathcal{L}_{\alpha,\beta}$ is fixed by the rectangular symmetry $R : (V, W) \mapsto (V, W^{-1})$, and it follows that $\mathcal{L}_{\alpha,\beta}$ must be the fixed line of R. In the second case, again taking (V, W) to be a generator pair of $\pi_1(S_{1,1})$, the curves $\alpha = VW$ and $\beta = VW^{-1}$ intersect twice. In this case, any point on $\mathcal{L}_{\alpha,\beta}$ is fixed by the rhombic involution $V \to W, W \to V$, so that $\mathcal{L}_{\alpha,\beta}$ must be the fixed line of the rhombic symmetry which is given by the equation $\operatorname{Tr} W = \operatorname{Tr} V$.

If α, β intersect once, one can also determine $\mathcal{L}_{\mu,\nu}$ as follows. The condition $\partial/\partial t_\alpha = -k\partial/\partial t_\beta$ on $\mathcal{L}_{\mu,\nu}$ implies by (2.1) that α and β are orthogonal. Let $t_{\alpha,\beta}$ be the Fenchel-Nielsen twist coordinate of α about β, normalised to be 0 when α and β are orthogonal. Then relative to the Fenchel-Nielsen coordinates

$(l_\alpha, t_{\alpha,\beta})$ for $\mathcal{T}(S_{1,1})$, $\mathcal{L}_{\alpha,\beta}$ is the line $(l,0), 0 < l < \infty$. If α, β are curves on $S_{1,1}$ which meet exactly once, we have [[29] Theorem 2.1]:

$$\cos(t_{\alpha,\beta}/2) = \cosh(l_\alpha/2)\tanh(l_\beta/2). \tag{3.1}$$

It follows that $\mathcal{L}_{\alpha,\beta}$ is also defined by the equation

$$\sinh(l_\alpha/2)\sinh(l_\beta/2) = 1$$

obtained by setting $t_{\alpha,\beta} = 0$ in (3.1).

More interesting examples were computed in [8] for the twice punctured torus $S_{1,2}$. Let $\mathcal{A} = \{\alpha_1, \alpha_2\}$ and $\mathcal{B} = \{\beta_1, \beta_2\}$ be pairs of non-separating, disjoint, simple closed curves on $S_{1,2}$ such that each β_i intersects each α_j exactly once. Also let $\mathcal{D} = \{\beta_1, \delta_1\}$, where δ_1 is a separating curve, disjoint from β_1 and α_2 and intersecting α_1 twice. Using Fenchel-Nielsen coordinates relative to the pants systems $\mathcal{A}$, extensive computations in [8] located the simplices of minima $\Sigma_{\mathcal{A},\mathcal{B}}$ and $\Sigma_{\mathcal{A},\mathcal{D}}$ in $\mathcal{T}(S_{1,2}) = \mathbb{R}^4$. These two examples serve to illustrate the distinction made in Proposition 3.3. We find $\operatorname{rank} \mathcal{M}(\mathcal{A},\mathcal{B}) = d-1 = 3$ on $\Sigma_{\mathcal{A},\mathcal{B}}$ whereas $\operatorname{rank} \mathcal{M}(\mathcal{A},\mathcal{D}) = d-2 = 2$ on $\Sigma_{\mathcal{A},\mathcal{D}}$. It follows that $\Phi(\mathcal{A},\mathcal{B})$ is an embedding of the affine simplex $\hat{\Sigma}_{\mathcal{A},\mathcal{B}}$ and that $\Sigma_{\mathcal{A},\mathcal{B}}$ is a 3-submanifold of $\mathcal{T}(S_{1,2})$, while the image of $\hat{\Sigma}_{\mathcal{A},\mathcal{D}}$ under $\Phi(\mathcal{A},\mathcal{D})$ is not embedded and $\Sigma_{\mathcal{A},\mathcal{D}}$ is a 2-submanifold.

3.4 Limiting behaviour at infinity

It might be natural to suppose that the ends of $\mathcal{L}_{\mu,\nu}$ would always converge to the points $[\mu], [\nu]$ in Thurston's compactification of $\mathcal{T}(S)$, namely PML. However Kerckhoff [[18] p.192] indicates that this may not always be the case. In fact:

Theorem 3.4 ([9] Theorems 1.1, 1.2). *Suppose that $\mu, \nu \in ML$ fill up S. For $t \in (0,1)$, let $M_t = M((1-t)\mu, t\nu)$. If $\mu = \sum_{i=1}^N a_i \alpha_i \in ML$ is rational with $a_i > 0$ for all i then*

$$lim_{t\to 0} M_t = [\alpha_1 + \cdots + \alpha_N] \in PML\,.$$

If μ is uniquely ergodic and maximal, then

$$lim_{t\to 0} M_t = [\mu] \in PML\,.$$

This should be compared with the analogous results of Masur [22] for Teichmüller geodesics. A geodesic ray is determined by a base surface $p \in \mathcal{T}(S)$ and a quadratic differential q on p. Masur shows that if q is a Jenkins-Strebel differential, that is, if the horizontal foliation $\mathcal{H}$ is supported (apart from saddle connections) on closed leaves, then the associated ray converges in PML to the barycentre of the leaves (the foliation with the same closed leaves all of whose cylinders have unit height). At the other extreme, if $\mathcal{H}$ is uniquely ergodic and every leaf apart from saddle connections is dense in S, the ray converges to the boundary point defined by $\mathcal{H}$.

Recently Anya Lenzhen [19] has shown the existence of Teichmüller geodesics which do not converge to *any* point in PML. One can presumably apply [6] to show the same is true of lines of minima, although to the author's knowledge this has not been written down.

3.4.1 Proof of Theorem 3.4 The condition of being uniquely ergodic and maximal, is equivalent to the condition that $i(\mu, \nu) = 0$ for all $\nu \in ML$ implies that $\nu \in [\mu]$, [[9] Lemma 2.1]. The second statement of Theorem 3.4 is thus easily proved by showing that as $t \to 0$, any subsequence of (M_t) necessarily has a convergent subsequence which limits on $[\mu] \in PML$.

The proof of the first part of Theorem 3.4 is easiest when $\{\alpha_i\}$ form a pants decomposition $\mathcal{A}$ of S. Let $(l_{\alpha_i}, t_{\alpha_i})$ be the Fenchel-Nielsen coordinates of S relative to $\mathcal{A}$, where the twist t_{α_i} is the signed hyperbolic distance twisted round α_i measured from some suitable base point. It is not hard to show by constructing surfaces in which the lengths l_{α_i} are specified, that $l_{\alpha_i}(M_t)/t$ is uniformly bounded away from $0, \infty$ as $t \to 0$ for each i.

An earthquake about a curve α fixes l_α, while the length l_γ of any transverse curve γ is a proper and convex function of the earthquake parameter. It is therefore reasonable to suppose that for fixed lengths l_{α_i}, the sum $l_\mu + l_\nu$ attains its minimum within bounded distance of the point where $t_{\alpha_i} = 0$. To make this more precise, we replace γ by a homotopic piecewise geodesic curve $\hat{\gamma}$ which runs alternately along the pants curves α_i, and across pairs of pants in such a way that it meets the boundary components orthogonally, so that adjacent segments of $\hat{\gamma}$ always meet orthogonally. Now the hypotenuse of a right angled hyperbolic triangle is, up to bounded additive constant, equal to the sum of the lengths of the other two sides. It follows that if the lengths l_{α_i} are bounded above (so that each curve α_i is surrounded by a collar of definite width), the length l_γ is coarsely approximated by the length $l_{\hat{\gamma}}$, with error comparable in magnitude to the intersection number of γ with $\mathcal{A}$. By the hyperbolic collar lemma, the width of the collar round a short curve of length l is approximately $\log 1/l$, see [21, 26]. Using the observation that the twist parameter remains bounded at M_t, gives the estimate

$$l_\gamma(M_t) = 2\sum_{j=1}^{N} i(\alpha_j, \gamma) \log \frac{1}{l_{\alpha_j}(M_t)} + O(1) \tag{3.2}$$

for a curve γ transverse to the pants curves $\mathcal{A}$. Since we have already observed, $l_{\alpha_j}(M_t) \sim t$ for all i, the result follows from the definition of convergence to a point in PML, see [11]. The argument in the case that $\mathcal{A}$ is not a full pants decomposition is considerably more subtle, see [[9] Section 6].

4 Lines of minima and quasifuchsian groups which are almost Fuchsian

As discovered by Series [35, 36], lines of minima are closely related to small deformations of Fuchsian into quasifuchsian groups. To understand the connection, recall first some basic facts about quasifuchsian groups and their convex cores; for more details see [20]. A Kleinian group (that is, a discrete subgroup of $SL(2,\mathbb{C})$) is called *quasifuchsian* if it is isomorphic to $\pi_1(S)$ for some surface S of hyperbolic type and if its limit set is a Jordan curve. Equivalently, a Kleinian group G is quasifuchsian if the associated hyperbolic 3-manifold $\mathbb{H}^3/G$ is homeomorphic to $S \times (0,1)$. Let $\mathcal{C}$ be the convex hull of the limit set of G in $\mathbb{H}^3$, see [10]. Then $\mathcal{C}/G$ is the smallest closed convex set containing all closed geodesics in $\mathbb{H}^3/G$ and is homeomorphic to $S \times [0,1]$. Moreover $\mathcal{C}$ has two simply connected boundary components $\partial^{\pm}\mathcal{C}$ whose quotients $\partial^{\pm}\mathcal{C}/G$ are pleated surfaces [10], each themselves homeomorphic to S. Each component is bent along a geodesic lamination on S, the amount of bending being measured by the *bending measures* $pl^{\pm} = pl^{\pm}(G) \in ML(S)$. The following is a special case of a fundamental result of Bonahon and Otal:

Theorem 4.1 ([4] Theorème 1). *Let S be a hyperbolisable surface and suppose that $\mu,\nu \in ML(S)$. Suppose also that every closed leaf of μ and of ν has weight strictly less than π. Then there is a quasifuchsian group $G(\mu,\nu)$ for which $pl^+(G) = \mu$ and $pl^-(G) = \nu$, if and only if μ,ν fill up S. If μ,ν are rational, then $G(\mu,\nu)$ is unique.*

The following conjecture, known as the *bending measure conjecture*, was originally made by Thurston.

Conjecture 4.2. *For any $\mu,\nu \in ML$ satisfying the conditions of Theorem 4.1, the quasifuchsian group $G(\mu,\nu)$ for which $pl^+(G) = \mu$ and $pl^-(G) = \nu$ is unique. Moreover groups for which both bending laminations are rational are dense in the space of all quasifuchsian groups.*

This was proved by Series [37] for the very special case in which S is a once punctured torus. We shall return to this conjecture in Section 4.3.

From now on, we denote by $G(\mu,\nu)$ any quasifuchsian group for which $pl^+(G) = \mu$ and $pl^-(G) = \nu$. It seems reasonable to assume that as $\theta,\phi \to 0$, the convex cores of the groups $G(\theta\mu,\phi\nu)$ flatten out approaching a Fuchsian limit. In fact we have:

Theorem 4.3 ([36] Theorems 1.4, 1.6). *Let μ,ν be two measured laminations which fill up S, and suppose that $\mu_n \to \mu$, $\nu_n \to \nu$ in $ML(S)$ and that $\theta_n \to 0$. Then the sequence of groups $G(\theta_n\mu_n,\theta_n\nu_n)$ converges to $M(\mu,\nu)$ as $n \to \infty$.*

For a partial proof, see Proposition 4.6 below. It is however essential that the bending measures stay in bounded proportion:

Theorem 4.4 ([36] Theorem 1.5). *Let μ, ν be two measured laminations which fill up S. Then any sequence of groups $G(\theta_n \mu, \phi_n \nu)$ with $\theta_n, \phi_n \to 0$ diverges (that is, no subsequence has an algebraic limit) unless θ_n/ϕ_n is uniformly bounded away from 0 and ∞.*

Let $\mathcal{Q}(S)$ denote the space of quasifuchsian groups with the algebraic topology (inherited from the space of representations $\pi_1(S) \to SL(2,\mathbb{C})$). Then $\mathcal{Q}(S)$ is a $2d$-dimensional complex manifold, as may be seen using complex Fenchel-Nielsen coordinates [38]. Via the uniformisation theorem, $\mathcal{T}(S)$ may be identified with the space of Fuchsian groups $\mathcal{F}(S)$, naturally embedded as a $2d$-dimensional totally real submanifold in $\mathcal{Q}(S)$. The following close connection between lines of minima and bending measures of groups in $\mathcal{Q}(S)$ was originally noted by Keen and Series [15] in the context of the once punctured torus. Given two non-zero measured laminations μ and ν, the *pleating variety* $\mathcal{P}_{\mu,\nu}$ consists of all quasifuchsian groups $G(\lambda\mu, \kappa\nu), \lambda, \kappa \in \mathbb{R}^+$.

Theorem 4.5 ([36] Theorems 1.7). *Let μ, ν be two measured laminations which fill up S. Then the closure of $\mathcal{P}_{\mu,\nu}$ meets $\mathcal{F}(S)$ precisely in $\mathcal{L}_{\mu,\nu}$.*

To prove Theorem 4.5 one needs to invoke Theorem 4.1 for the existence of the groups $G(\theta_n\mu, \tau_n\nu)$. Since for rational laminations this in turn is based on the Hodgson-Kerckhoff theory of deformations of cone manifolds, Theorem 4.5 rests ultimately on the same thing.

We deduce from Theorem 4.5 that given any point $p \in \mathcal{F}(S)$ and lamination $\mu \in ML(S)$, it is possible to move away from p into $\mathcal{Q}(S)$ along a path in $\mathcal{P}_{\mu,\nu}$, if and only if $p \in \mathcal{L}_{\mu,\nu}$. Using Theorem 3.1, we deduce that for any $\mu \in ML, c > 0$ and any $\nu \in ML_\mu$ one can bend (see Section 4.2) along μ to produce groups in $\mathcal{P}_{\mu,\nu}$ for which $l_\mu = c$. One the other hand, Theorem 3.2 implies the rather surprising negative result that if $\mathcal{A} = \{\alpha_1, \dots, \alpha_d\}$ is a pants decomposition of S, then for any choice of fixed lengths $l_{\alpha_i} = c_i, i = 1, \dots, d$, there are laminations $\nu \in ML_{\mathcal{A}}$ with the property that it is *impossible* to bend on the lines α_i obtaining groups with $pl^- \in [\nu]$.

4.1 Applications and examples

The examples of simplices of minima described in Section 3.3 were expanded in [8] into direct computations of pleating varieties in $\mathcal{Q}(S_{1,2})$. With notation as in that section, complex Fenchel-Nielsen coordinates were used to locate the pleating varieties $\mathcal{P}_{\mathcal{A},\mathcal{B}}$ and $\mathcal{P}_{\mathcal{A},\mathcal{D}}$ (those groups $G \in \mathcal{Q}(S)$ for which $pl^{\pm}(G)$ are supported on the curves $\mathcal{A}, \mathcal{B}$ or $\mathcal{A}, \mathcal{D}$ respectively) in $\mathcal{QF}(S_{1,2}) = \mathbb{C}^4$. The computations confirm Theorem 4.5, showing by explicit computation that the closures of $\mathcal{P}_{\mathcal{A},\mathcal{B}}$ and $\mathcal{P}_{\mathcal{A},\mathcal{D}}$ meet $\mathcal{F}(S)$ in the simplices of minima $\Sigma_{\mathcal{A},\mathcal{B}}$ and $\Sigma_{\mathcal{A},\mathcal{D}}$. As expected (see Proposition 4.9) both pleating varieties are smooth submanifolds of real dimension 4.

4.2 Quakebends and the Fuchsian limit

The main part of the proof of Theorem 4.3 involves geometry in $\mathbb{H}^3$ and is beyond the scope of this article. However the proof that if the limit of the groups $G(\theta_n\mu_n, \theta_n\nu_n)$ exists, it must be $M(\mu,\nu)$, is a nice illustration of the results in Section 2.2.

The proof uses *quakebends* [[10] Section 3.5], an extension of earthquakes into the complex domain. Starting from a Fuchsian group G_0, representing a point $p \in \mathcal{F}(S)$, the quakebend construction produces a quasifuchsian group G by bending, more generally quakebending, G_0 along a lamination $\mu \in ML$. Let $|\tilde{\mu}|$ denote the union of the lifts of the (geodesic) leaves supporting μ in the hyperbolic structure on p to $\mathbb{H}^2$. This gives a decomposition of $\mathbb{H}^2$ into plaques, each plaque being a connected component of $\mathbb{H}^2 \setminus |\tilde{\mu}|$. Let $\tau = t + i\theta \in \mathbb{C}$. The time τ *left quakebend* $\mathcal{Q}_\mu(\tau)$ reglues these plaques, in such a way that if two plaques are connected by a transversal κ to $|\tilde{\mu}|$, they are reglued at an angle $\theta\mu(\kappa)$ after being shifted a relative distance $t\mu(\kappa)$, resulting in a pleated surface $\mathcal{Q}_\mu(\tau)(\mathbb{H}^2)$ in $\mathbb{H}^3$. This surface is invariant under the action of a group $G = \mathcal{Q}_\mu(\tau)(G_0) \subset SL(2,\mathbb{C})$. Keen and Series [14] showed that for sufficiently small $|\tau|$, $\mathcal{Q}_\mu(\tau)(\mathbb{H}^2)$ is embedded in $\mathbb{H}^3$ and hence that $G = \mathcal{Q}_\mu(\tau)(G_0)$ is quasifuchsian. Moreover $\mathcal{Q}_\mu(\tau)(\mathbb{H}^2)$ is one of the two boundary components of the hyperbolic convex hull $\mathcal{C}$ of the limit set of G in $\mathbb{H}^3$, and $pl^+(G) = \theta\mu$.

Proposition 4.6 ([36] Proposition 3.1, see also [3] Proposition 6). *Let μ,ν be measured laminations which fill up S. Suppose that as $\theta_n \to 0$, the groups $G(\theta_n\mu, \theta_n\nu)$ converge to $p \in \mathcal{F}(S)$. Then $p = M(\mu,\nu)$.*

Proof. By Theorem 2.11 and Corollary 2.13 it is enough to show that

$$\frac{\partial l_\eta}{\partial t_\mu}(p) + \frac{\partial l_\eta}{\partial t_\nu}(p) = 0 \quad \text{for all} \quad \eta \in ML\,.$$

Now for any $\mu \in ML$ and $p \in \mathcal{F}(S)$, the quakebend construction with $\tau = i\theta$ gives a one parameter family of quasifuchsian groups $G(\theta)$ for which $pl^+ = \theta\mu$ for all small $\theta > 0$. Throughout this deformation, the Fuchsian structure $p^+(\theta)$ on $\partial\mathcal{C}^+(G(\theta))$ (see [10]) remains fixed. Thus we can reach $q(\theta) = G(\theta\mu, \theta\nu) \in \mathcal{Q}(S)$ either by starting at $p^+(\theta)$ and making the pure bend $\mathcal{Q}_\mu(i\theta)$, or by starting at $p^-(\theta)$ and making the pure bend $\mathcal{Q}_\nu(-i\theta)$.

One can extend the definition of the complex length of a curve $\gamma \in \mathcal{S}$ to that of an arbitrary lamination $\eta \in ML$, either by taking limits of real lengths and extending along a suitable branch from $\mathcal{F}(S)$ into $\mathcal{QF}(S)$ [[15] Theorem 3], or using Bonahon's shearing coordinates [2]. The resulting complex length function $\lambda_\eta(\mathcal{Q}_\mu(\tau))$ is a holomorphic function of τ. Hence we can expand λ_η as a Taylor series about $p^\pm(\theta)$ (checking the second derivatives are uniformly bounded) and compare the results:

$$\lambda_\eta(q(\theta)) = l_\eta(p^+(\theta)) + i\theta\frac{\partial l_\eta}{\partial t_\mu}(p^+(\theta)) + O(\theta^2)$$

and

$$\lambda_\eta(q(\theta)) = l_\eta(p^-(\theta)) - i\theta\frac{\partial l_\xi}{\partial t_\nu}(p^-(\theta)) + O(\theta^2).$$

Equating imaginary parts gives:

$$\frac{\partial l_\eta}{\partial t_\mu}(p^+(\theta_n)) + \frac{\partial l_\eta}{\partial t_\nu}(p^-(\theta_n)) = O(\theta_n). \tag{4.1}$$

It is proved in [[36] Proposition 1.8] that up to subsequences, the groups $G(\theta_n\mu, \theta_n\nu)$ necessarily converge in such a way that the real analytic structures on $p^\pm(\theta_n)$ are close. Viewing l_η as a real analytic function on $\mathcal{F}(S)$, this gives

$$\frac{\partial l_\eta}{\partial t_\nu}(p^+(\theta_n)) - \frac{\partial l_\eta}{\partial t_\nu}(p^-(\theta_n)) = O(\theta_n). \tag{4.2}$$

Combining (4.1) and (4.2) and taking limits completes the proof. □

4.3 The bending measure conjecture

We return to Conjecture 4.2. Although the conjecture in general remains open, Bonahon proved:

Theorem 4.7 ([3] Theorem 1). *There exists an open neighbourhood U of the Fuchsian submanifold $\mathcal{F}(S)$ of $\mathcal{QF}(S)$ such that the bending map $\beta : \mathcal{Q}(S) \to ML(S) \times ML(S), \beta(q) = (pl^+(q), pl^-(q))$, is a homeomorphism from U onto its image.*

This provides an alternative proof of Theorem 4.5. One can modify the proof to show that the length map $\ell : \mathcal{Q}(S) \to \mathbb{R}^+ \times \mathbb{R}^+$ which sends $q \in \mathcal{Q}(S)$ to $(l_{j([pl^+(q)])}, l_{j([pl^-(q)])})$ (where j is a section $PML \to ML$), is also a homeomorphism from U onto its image [Series, unpublished].

The key steps in the proof of Theorem 4.7 are summarized below. The essential idea is to translate the non-degeneracy of the Hessian of the function $l_\mu + l_\nu$ at the minimum $M(\mu, \nu)$ into a suitable transversality statement about the pleating varieties $\mathcal{P}^\pm(\mu) = \{q \in \mathcal{Q}(S) : pl^\pm(q) \in [\mu] \cup \{0\}\}$.

Proposition 4.8 ([3] Proposition 3). *Let $\mu, \nu \in ML$ fill up S. Then the sections $p \mapsto \partial/\partial t_\mu(p), p \mapsto -\partial/\partial t_\nu(p)$ from $\mathcal{F}(S)$ to $T(\mathcal{F}(S))$ are transverse.*

Proof. By [42], the Weil-Petersson symplectic form induces the isomorphism $\partial/\partial t_\mu \to dl_\mu$ between the tangent bundle $T(\mathcal{F}(S))$ and the cotangent bundle $T^*(\mathcal{F}(S))$, see also Theorem 2.11. Thus the statement can be converted into the transversality of the sections $dl_\mu, -dl_\nu : \mathcal{F}(S) \to T^*(\mathcal{F}(S))$. The intersection of the tangent spaces to these sections at $p = M(\mu,\nu)$ consists of all vectors which can be written in the form $T_p(dl_\mu)(w) = T_p(-dl_\nu)(w)$, where $w \in T_p(\mathcal{F}(S))$ and $T_p(dl_\mu), T_p(-dl_\nu) : T_p(\mathcal{F}(S)) \to T_p(T^*(\mathcal{F}(S)))$ are the tangent maps of the two sections. One shows by calculation in local coordinates that $T_p(dl_\mu)(w) - T_p(-dl_\nu)(w)$ is the image of w under the natural isomorphism $T_p(\mathcal{F}(S)) \to T^*_p(\mathcal{F}(S))$ induced by the Hessian of $l_\mu + l_\nu$. Since by [41] (see also Section 2.1.2) the Hessian is non-degenerate, the result follows. □

Proposition 4.9 ([3] Lemma 7). *Let $\mu \in ML$. The pleating varieties $\mathcal{P}^\pm(\mu)$ are submanifolds of $\mathcal{Q}(S)$ of dimension $2d+1$ and with boundary $\mathcal{F}(S)$.*

Proof. If $\mu = \sum_i a_i \alpha_i$ is a rational lamination with weight $a_i > 0$ on all the curves in a pants decomposition $\mathcal{A}$, this follows using complex Fenchel-Neilsen coordinates $(\lambda_{\alpha_i}, \tau_{\alpha_i})$ relative to $\mathcal{A}$ [[38], see also [14]]. Here λ_{α_i} is the complex length given by $2\cosh \lambda_{\alpha_i}/2 = \operatorname{Tr} \alpha_i$ and τ_{α_i} is the complex twist as in the quakebend construction 4.2. A necessary condition for a curve α to be contained in the bending locus is that its complex length be real [[15] Proposition 22], giving d conditions $\lambda_{\alpha_i} \in \mathbb{R}$. The bending angle on α_i is the imaginary part of τ_{α_i}, giving a further $d-1$ conditions to ensure the bending angles are in the proportion $[a_1 : a_2 : \ldots : a_d]$ specified by $[\mu]$.

To prove the result in general is substantially harder. One needs to use Bonahon's shear-bend coordinates [2] which provide a substitute for Fenchel-Nielsen coordinates for laminations which are not rational. □

The final key step is to blow up $\mathcal{Q}(S)$ along its boundary $\mathcal{F}(S)$. Define $\check{\mathcal{Q}}(S)$ to be the union of $\mathcal{Q}(S) \setminus \mathcal{F}(S)$ with the unit normal bundle $\mathcal{N}^1(\mathcal{F}(S))$ of $\mathcal{F}(S) \subset \mathcal{Q}(S)$ with suitable topology. Then $\check{\mathcal{Q}}(S)$ is a manifold with boundary $\mathcal{N}^1(\mathcal{F}(S))$. The complex structure gives a natural identification of $\mathcal{N}(\mathcal{F}(S))$ with $i\mathcal{T}(\mathcal{F}(S))$.

The inclusion $\mathcal{P}^+(\mu) \to \mathcal{Q}(S)$ extends uniquely to an embedding $\mathcal{P}^+(\mu) \cup \mathcal{F}(S) \to \check{\mathcal{Q}}(S)$ by sending $p \in \mathcal{F}(S)$ to $\mathbf{n}_\mu(p)$, where $\mathbf{n}_\mu(p)$ is the unit normal vector in the direction $i\partial/\partial t_\mu$ at p. We define $\check{\mathcal{P}}^+(\mu)$ to be the image of $\mathcal{P}^+(\mu) \cup \mathcal{F}(S)$ under this map. Likewise we embed $\mathcal{P}^-(\nu)$ by the map $p \mapsto -\mathbf{n}_\nu(p)$, and define $\check{\mathcal{P}}^-(\nu)$ in a similar way. The following result is then a translation of Proposition 4.8.

Proposition 4.10 ([3] Proposition 9). *The boundaries of $\check{\mathcal{P}}^+(\mu), \check{\mathcal{P}}^-(\nu)$ in $\mathcal{N}^1(\mathcal{F}(S))$ have non-empty intersection if and only if μ, ν fill up S. If they fill up, the intersection is transverse and is equal to the image of $\mathcal{L}_{\mu,\nu}$ under the map $p \mapsto \mathbf{n}_\mu(p) = -\mathbf{n}_\nu(p) \in \check{\mathcal{Q}}(S)$.*

The injectivity of the map β in Theorem 4.7 follows from the transversality of $\check{\mathcal{P}}^+(\mu), \check{\mathcal{P}}^-(\nu)$ in a neighbourhood of $\mathcal{N}^1(\mathcal{F}(S))$ in $\check{\mathcal{Q}}(S)$. Using invariance of domain, one then shows that β is a homeomorphism in a neighbourhood U of $\mathcal{F}(S)$ in $\mathcal{Q}(S) \setminus \mathcal{F}(S)$. The density of groups with rational pleating loci (see Conjecture 4.2) is immediate from the density of rational laminations in $ML \times ML$.

5 Relationship to Teichmüller geodesics

In two papers [6, 7], Choi, Rafi and Series investigated the relationship between Teichmüller geodesics and lines of minima. The first paper derives a combinatorial formula for the Teichmüller distance between the time t surfaces on these two paths, and the second proves that a line of minima is a Teichmüller quasi-geodesic.

The first step is to parameterise both paths in a comparable way. Given laminations $\mu, \nu \in ML$ which fill up S, define

$$\mu_t = e^t \mu, \quad \nu_t = e^{-t} \nu.$$

We define the line of minima $t \mapsto \mathcal{L}_t$ to be the path which takes t to the minimising surface $M(\mu_t, \nu_t)$. As explained in Section 2.5, for each t, there is a unique Riemann surface $\mathcal{G}_t$ and quadratic differential q_t such that μ_t, ν_t are the horizontal and vertical foliations of q_t respectively. The path $t \mapsto \mathcal{G}_t$ is a Teichmüller geodesic.

For $\alpha \in \mathcal{S}$ let $l_\alpha(\mathcal{G}_t), l_\alpha(\mathcal{L}_t)$ denote the lengths of α in the hyperbolic metrics on $\mathcal{G}_t, \mathcal{L}_t$ respectively. We say a curve is *extremely short* if its hyperbolic length is less than some fixed constant $\epsilon_0 > 0$ determined in the course of the proofs. For functions f, g we write $f \asymp g$ and $f \overset{*}{\asymp} g$ to mean respectively, that there are constants $C, c > 1$, depending only on the topology of S such that

$$\frac{1}{c} g(x) - C \le f(x) \le c g(x) + C \quad \text{and} \quad \frac{1}{c} g(x) \le f(x) \le c g(x).$$

Theorem 5.1 ([6] Theorems A and D). *The extremely short curves in the hyperbolic metrics on $\mathcal{L}_t$ and $\mathcal{G}_t$ coincide. The Teichmüller distance $d_{\mathcal{T}}$ between $\mathcal{L}_t$ and $\mathcal{G}_t$ is given by*

$$d_{\mathcal{T}}(\mathcal{L}_t, \mathcal{G}_t) = \frac{1}{2} \log \max_\alpha \frac{l_\alpha(\mathcal{G}_t)}{l_\alpha(\mathcal{L}_t)} + O(1),$$

where the maximum is taken over all simple closed curves α that are extremely short in $\mathcal{G}_t$. In particular, the distance between corresponding thick parts of $\mathcal{L}_t$ and $\mathcal{G}_t$ is bounded.

Theorem 5.2 ([7] Theorem A). *The line of minima $t \mapsto \mathcal{L}_t$, $t \in \mathbb{R}$, is a quasi-geodesic with respect to the Teichmüller metric $d_{\mathcal{T}}$. In other words,*

$$d_{\mathcal{T}}(\mathcal{L}_a, \mathcal{L}_b) \asymp |b - a|$$

for any $a, b \in \mathbb{R}$.

The constants involved in these two theorems depend only on the genus and number of punctures of S.

The method is to compare the curves which are short on the time t surfaces $\mathcal{G}_t, \mathcal{L}_t$. Perhaps surprisingly, although the same curves are short on the two surfaces, their lengths are not necessarily in bounded proportion. Thus one can construct examples of surfaces and pairs of laminations for which the two paths $t \mapsto \mathcal{G}_t$ and $t \mapsto \mathcal{L}_t$ are unboundedly far apart.

Theorem 5.1 is proved using estimates for short curves in $\mathcal{L}_t$ and $\mathcal{G}_t$ which involve two quantities $D_t(\alpha), K_t(\alpha)$ defined in Section 5.0.1 below. Both $D_t(\alpha)$ and $K_t(\alpha)$ have a combinatorial interpretation in terms of the topological relationship between α, μ and ν: $D_t(\alpha)$ is large iff the relative twisting of μ and ν about α is large, while $K_t(\alpha)$ is large iff μ and ν have large relative complexity in $S \setminus \alpha$ (the completion of S minus α), in the sense that every essential arc or closed curve in $S \setminus \alpha$ must have large intersection with both μ and ν.

Theorem 5.3 ([31], see also [6] Theorem B). *Suppose that $\alpha \in \mathcal{S}$ is extremely short on $\mathcal{G}_t$. Then*

$$\frac{1}{l_\alpha(\mathcal{G}_t)} \asymp \max\{D_t(\alpha), \log K_t(\alpha)\}.$$

Theorem 5.4 ([6] Theorem C). *Suppose that $\alpha \in \mathcal{S}$ is extremely short on $\mathcal{L}_t$. Then*

$$\frac{1}{l_\alpha(\mathcal{L}_t)} \asymp \max\{D_t(\alpha), \sqrt{K_t(\alpha)}\}.$$

The motivation for this approach stems in part from a central ingredient of the proof of the ending lamination theorem. Suppose that N is a hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$. The ending lamination theorem states that N is completely determined by the asymptotic invariants of its two ends. A key step is to show that if these end invariants are given by the laminations μ, ν, then the curves on S which have short geodesic representatives in N can be characterized in terms of their combinatorial relationship to μ and ν. (The relationship is expressed using the the complex of curves of S. Roughly speaking, a curve is short in N if and only if the distance between the projections of μ and ν to some subsurface $Y \subset S$ is large in the curve complex of Y, see [5].)

Since the proofs of Theorems 5.1–5.4 involve some interesting techniques, the remainder of Section 5 will be spent outlining their proofs.

5.0.1 Definitions of the combinatorial parameters With the terminology introduced in Section 6, $D_t(\alpha)$ is approximately the modulus of the maximal flat annulus round α while $\log K_t(\alpha)$ is approximately the modulus of the maximal expanding annulus. The following definitions are independent of these notions.

The term $K_t(\alpha)$ depends on the (possibly coincident) hyperbolic thick components Y_1, Y_2 of the surface $\mathcal{G}_t$ adjacent to α. Let q_t be the area 1 quadratic differential on $\mathcal{G}_t$ whose horizontal and vertical foliations are respectively μ_t and ν_t. Associated to q_t is a singular Euclidean metric; we denote the geodesic length of a curve γ in this metric by $l_\gamma(q_t)$, see Section 5.1.1. By definition

$$K_t(\alpha) = \max\left\{\frac{\lambda_{Y_1}}{l_\gamma(q_t)}, \frac{\lambda_{Y_2}}{l_\gamma(q_t)}\right\}$$

where λ_{Y_i} is the length of the shortest non-trivial non-peripheral simple closed curve on Y_i with respect to the q_t-metric. If Y_i is a pair of pants, there is a slightly different definition, see [6].

To define $D_t(\alpha)$, we need the notion of the *relative twist* $d_\alpha(\mu, \nu)$ of μ and ν around α. Following [26], for $p \in \mathcal{T}(S)$ and $\eta \in ML$, define

$$tw_\alpha(\eta, p) = \inf s/l_\alpha(p),$$

where $l_\alpha(p)$ is the hyperbolic length of α in the surface p, s is the signed hyperbolic distance between the perpendicular projections of the endpoints of a lift of a geodesic in $|\eta|$ at infinity onto a lift of α, and the infimum is over all lifts of leaves of $|\eta|$ which intersect α. Now define

$$d_\alpha(\mu, \nu) = \inf_p |tw_\alpha(\mu, p) - tw_\alpha(\nu, p)|,$$

where the infimum is over all points $p \in \mathcal{T}(S)$. Then $tw_\alpha(\mu, p) - tw_\alpha(\nu, p)$ is independent of p up to a universal additive constant, see [26]. Note that $d_\alpha(\mu, \nu)$ agrees up to an additive constant with the definition of subsurface distance between the projections of $|\mu|$ and $|\nu|$ to the annular cover of S with core α, as defined in [23] Section 2.4 and used throughout [31, 32].

The definition of $D_t(\alpha)$ also involves the *balance time* of α, namely the unique time $t = t_\alpha$ for which $i(\alpha, \mu_{t_\alpha}) = i(\alpha, \nu_{t_\alpha})$. (If $i(\alpha, \nu) = 0$ define $t_\alpha = -\infty$, if $i(\alpha, \mu) = 0$ define $t_\alpha = \infty$.) Finally, define

$$D_t(\alpha) = e^{-2|t - t_\alpha|} d_\alpha(\mu, \nu).$$

By [[32] Theorem 3.1], the length $l_\alpha(\mathcal{G}_t)$ is approximately convex along $\mathcal{G}$ and is close to its minimum at t_α.

For later use we note that the twist is closely related to the normalised Fenchel-Nielsen twist coordinate $s_\alpha(p)$ defined in Section 5.2.1:

Lemma 5.5 (Minsky [26] Lemma 3.5). *For any lamination $\eta \in ML$ and any two hyperbolic metrics $p, p' \in \mathcal{T}(S)$,*

$$|(tw_\alpha(\eta, p) - tw_\alpha(\eta, p')) - (s_\alpha(p) - s_\alpha(p'))| \leq 4.$$

5.1 Comparison in the thick part of Teichmüller space

It is relatively easy to compare the two surfaces $\mathcal{G}_t$ and $\mathcal{L}_t$ when both are contained in the thick part of $\mathcal{T}_{\text{thick}(\epsilon)}$ of $\mathcal{T}(S)$, consisting of all surfaces on which the hyperbolic injectivity radius has some fixed lower bound $\epsilon > 0$. We have:

Theorem 5.6 ([6] Theorem 3.8). *If $\mathcal{G}_t, \mathcal{L}_t \in \mathcal{T}_{\text{thick}(\epsilon)}$ then $d_{\mathcal{T}}(\mathcal{G}_t, \mathcal{L}_t) = O(1)$.*

To gain some insight into how the two surfaces may be compared, we outline the proof.

5.1.1 Quadratic differential metrics A finite area holomorphic quadratic differential q on a Riemann surface $p \in \mathcal{T}(S)$ defines a singular Euclidean metric, which away from the singularities is just the Euclidean metric defined by the horizontal and vertical foliations $\mathcal{H}_q, \mathcal{V}_q$, see Section 2.5 and [39, 24]. On the surface $\mathcal{G}_t$, by definition $\mathcal{H}_q$ and $\mathcal{V}_q$ are equivalent to the laminations μ_t, ν_t respectively. Every simple closed curve γ in (S, q) either has a unique q-geodesic representative, or is contained in a family of closed Euclidean geodesics foliating an annulus whose interior contains no singularities, see [39] and Section 6.1. Thus on $\mathcal{G}_t$, the horizontal and vertical lengths of γ are $i(\gamma, \mu_t)$ and $i(\gamma, \nu_t)$ respectively, from which it follows that its q-geodesic length $l_\gamma(q_t)$ satisfies

$$l_\gamma(q_t) \stackrel{*}{\asymp} i(\mu_t, \gamma) + i(\nu_t, \gamma). \tag{5.1}$$

5.1.2 Short markings Define a *marking* of a surface S to be a collection of pants curves $\mathcal{A}$, together with a dual set of curves which intersect each curve in $\mathcal{A}$ either once or twice, depending on the topology [[6] Section 3]. The marking is *short* with respect to a hyperbolic metric on S, if there is a uniform upper bound to the lengths of all curves in $\mathcal{A}$, and if in addition the dual curves are as short as possible among all choices which have the same intersections with $\mathcal{A}$. By a result of Bers [1], one can always choose $\mathcal{A}$ so that $\sum_{\alpha \in \mathcal{A}} l_\alpha$ is less than some universal upper bound depending only on the genus of S.

Since curves in $\mathcal{A}$ may be arbitrarily short, there is in general no upper bound to the length of the dual curves. However in $\mathcal{T}_{\text{thick}}$, the lengths of the dual curves are uniformly bounded above. It follows, see e.g.[[25] Lemma 4.7], that for any hyperbolic metric $h \in \mathcal{T}_{\text{thick}}$ and short marking M_h, we have

$$l_\gamma(h) \overset{*}{\asymp} i(\gamma, M_h) \text{ and } l_{M_h}(h) \overset{*}{\prec} 1. \tag{5.2}$$

for any $\gamma \in \mathcal{S}$, where $l_{M_h}(h), i(\gamma, M_h)$ are the sums over curves in M_h of the lengths and intersections numbers with γ respectively.

5.1.3 Proof of Theorem 5.6 Let $M_{\mathcal{G}_t}$ and $M_{\mathcal{L}_t}$ be short markings for $\mathcal{G}_t, \mathcal{L}_t$ respectively. In $\mathcal{T}_{\text{thick}}$, any two metrics, in particular the hyperbolic metric and the quadratic differential metric, are comparable. Hence using (5.1) and (5.2):

$$l_{M_{\mathcal{G}_t}}(\mathcal{G}_t) \overset{*}{\asymp} l_{M_{\mathcal{G}_t}}(q_t) \overset{*}{\asymp} i(M_{\mathcal{G}_t}, \mu_t) + i(M_{\mathcal{G}_t}, \nu_t) \overset{*}{\asymp} l_{\mu_t}(\mathcal{G}_t) + l_{\nu_t}(\mathcal{G}_t). \tag{5.3}$$

In a similar way

$$l_{M_{\mathcal{L}_t}}(\mathcal{G}_t) \overset{*}{\asymp} l_{M_{\mathcal{L}_t}}(q_t) \overset{*}{\asymp} i(M_{\mathcal{L}_t}, \mu_t) + i(M_{\mathcal{L}_t}, \nu_t) \overset{*}{\asymp} l_{\mu_t}(\mathcal{L}_t) + l_{\nu_t}(\mathcal{L}_t).$$

Since $\mathcal{L}_t$ minimizes $l_{\mu_t}(h) + l_{\nu_t}(h)$ over all hyperbolic metrics $h \in \mathcal{T}(S)$:

$$l_{\mu_t}(\mathcal{G}_t) + l_{\nu_t}(\mathcal{G}_t) \geq l_{\mu_t}(\mathcal{L}_t) + l_{\nu_t}(\mathcal{L}_t).$$

Putting together the preceding three equations, we have

$$l_{M_{\mathcal{G}_t}}(\mathcal{G}_t) \overset{*}{\succ} l_{M_{\mathcal{L}_t}}(\mathcal{G}_t).$$

We deduce from (5.2) that

$$l_{M_{\mathcal{L}_t}}(\mathcal{G}_t) \overset{*}{\prec} 1 \tag{5.4}$$

It is not hard to see that the set of surfaces for which a given marking M has a diameter bounded by $B > 0$, has bounded diameter with respect to the Teichmüller distance, with a bound which depends only on B. The result follows.

5.2 Comparison in the thin part of Teichmüller space

Let us assume for a moment the results of Theorems 5.3 and 5.4. If $\alpha \in \mathcal{S}$ is short in the hyperbolic metrics on two surfaces $p, p' \in \mathcal{T}(S)$, then Kerckhoff's formula [16]

$$d_{\mathcal{T}}(p, p') = \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{\mathrm{Ext}_\gamma(p)}{\mathrm{Ext}_\gamma(p')} \tag{5.5}$$

(where $\mathrm{Ext}(\gamma)$ is the extremal length of γ) together with Theorem 6.3 below, shows that the expression

$$\frac{1}{2}\log\max_{\alpha}\frac{l_\alpha(\mathcal{G}_t)}{l_\alpha(\mathcal{L}_t)}$$

in Theorem 5.1 is an appoximate lower bound for $d_{\mathcal{T}}(\mathcal{G}_t,\mathcal{L}_t)$. To turn this into a precise estimate, one uses Minsky's product region theorem. This is worth explaining in its own right.

5.2.1 Minsky's product region theorem This theorem, proved in [26], is an approximate formula for the Teichmüller metric $d_{\mathcal{T}}$ in the part of $\mathcal{T}(S)$ in which a collection $\mathcal{A}=\{\alpha_1,\dots,\alpha_n\}$ of disjoint, homotopically distinct, simple closed curves on S are short. Extending $\mathcal{A}$ to a pants decomposition $\hat{\mathcal{A}}$ if necessary, we fix Fenchel-Nielsen coordinates $(l_{\alpha_i},s_{\alpha_i})$ on $\hat{\mathcal{A}}$, where s_α is the *normalised* twist, that is the hyperbolic signed distance twisted away form some base point divided by the length l_{α_i}.

Let $\mathcal{T}_{thin}(\mathcal{A},\epsilon_0)\subset\mathcal{T}(S)$ be the subset on which all curves $\alpha_i\in\mathcal{A}$ have hyperbolic length at most ϵ_0. Let $S_{\mathcal{A}}$ denote the surface obtained from S by removing all the curves in $\mathcal{A}$ and replacing the resulting boundary components by punctures. For $\alpha\in\mathcal{A}$, let $\mathbb{H}_\alpha$ be the hyperbolic plane and let $d_{\mathbb{H}_\alpha}$ be half the usual hyperbolic metric on $\mathbb{H}_\alpha$. Define $\Pi_\alpha\colon\mathcal{T}(S)\to\mathbb{H}_\alpha$ by

$$\Pi_\alpha(p)=s_\alpha(p)+i/l_\alpha(p)\in\mathbb{H}_\alpha.$$

Also define $\Pi_0:\mathcal{T}(S)\to\mathcal{T}(S_{\mathcal{A}})$ by forgetting the coordinates of the curves in $\mathcal{A}$ and keeping the same Fenchel-Nielsen coordinates for the remaining surface.

Theorem 5.7 (Minsky [26]). *Let $p,p'\in\mathcal{T}_{thin}(\mathcal{A},\epsilon_0)$. Then*

$$d_{\mathcal{T}}(p,p')=\max_i\{d_{\mathcal{T}(S_{\mathcal{A}})}(\Pi_0(p),\Pi_0(p')),d_{\mathbb{H}_{\alpha_i}}(\Pi_i(p),\Pi_i(p')\}\pm O(1).$$

A consequence of this formula is that unless the difference between the twist coordinates $s_\alpha(p),s_\alpha(p')$ is extremely large in comparison to $l_\alpha(p),l_\alpha(p')$, their contribution to $d_{\mathcal{T}}(p,p')$ can be neglected. Lemma 5.5 shows that we can replace $|s_\alpha(p)-s_\alpha(p')|$ with $|tw_\alpha(\eta,p)-tw_\alpha(,\eta,p')|$, for any $\eta\in ML$. In fact:

Corollary 5.8 ([6] Corollary 4.7). *Suppose that $p,p'\in\mathcal{T}_{thin}(\alpha,\epsilon_0)$ and that there exists $\eta\in ML$ such that $|tw_\alpha(\eta,r)|\,l_r(\alpha)=O(1)$ for $r=p,p'$. Then*

$$d_{\mathbb{H}_\alpha}(\Pi_\alpha(p),\Pi_\alpha(p'))=\left|\log\frac{l_\alpha(p)}{l_\alpha(p')}\right|\pm O(1).$$

5.2.2 Proof of Theorem 5.1 Theorems 5.3 and 5.4 show that, with suitable choices of constants, the extremely short curves on $\mathcal{L}_t$ and G_t coincide. To use Minsky's product theorem to deduce Theorem 5.1 from Theorems 5.3

and 5.4, we need to estimate the Teichmüller distance between the hyperbolic thick components of $\mathcal{G}_t$ and $\mathcal{L}_t$, and also the difference between the Fenchel-Nielsen twist coordinates corresponding to the short curves in the two surfaces.

The first part is done by an elaboration of the method in Theorem 5.6. One needs a substitute for the fact that the surface $\mathcal{L}_t$ minimises $l_{\mu_t} + l_{\nu_t}$ over $\mathcal{T}(S)$. Let Q be a thick part of the hyperbolic surface $\mathcal{L}_t$. One shows [[6] Proposition 7.4] that the contribution to $l_{\mu_t} + l_{\nu_t}$ coming from the intersection with Q is less, up to multiplicative constants, than the similar sum for the surface $\mathcal{G}_t$.

For the second part, by Corollary 5.8, it is enough to show that

$$|tw_\alpha(\eta, p)| l_\alpha(p) = O(1)$$

for $p = \mathcal{L}_t, \mathcal{G}_t$ and some $\eta \in ML$, where as usual $l_\alpha(p)$ means the hyperbolic length of α in the hyperbolic metric uniformising p. On the surface $\mathcal{L}_t$, the result follows from equation (2.3) with $\eta = \mu$ if $t > t_\alpha$ and $\eta = \nu$ if $t < t_\alpha$, see [[6] Theorems 6.2, 6.9]. For the surface $\mathcal{G}_t$, we use the analogous notion of the twist $tw_\alpha(\nu, q)$ of ν about α with respect to a quadratic differential metric q, introduced by Rafi [32]. We have:

Proposition 5.9 ([32] Theorem 4.3, [6] Proposition 5.7). *Suppose that $p \in \mathcal{T}(S)$ and that q is a compatible quadratic differential. For any geodesic lamination η intersecting α, we have*

$$l_\alpha(p)|tw_\alpha(\eta, p) - tw_\alpha(\eta, q)| = O(1).$$

As explained in Section 6, a hyperbolically short curve α on $\mathcal{G}_t$ is surrounded by an annulus of large modulus which in the associated quadratic differential metric q_t is either flat (that is, isometric to a Euclidean cylinder) or expanding. It follows immediately from the Gauss-Bonnet theorem (6.1) that the contribution to $tw_\alpha(\mu, q)$ and $tw_\alpha(\nu, q)$ from an expanding annulus is bounded. In a flat annulus F, at the balance time t_α the leaves of the horizontal and vertical foliations make angles of $\pm\pi/4$ with ∂F. It is an exercise in Euclidean geometry to determine the angle with the boundary at time $t - t_\alpha$, from which one deduces the required bounds on twists with the aid of Proposition 5.9.

5.2.3 Proof of Theorem 5.2 It follows from Theorem 5.1, that along intervals on which either there are no short curves, or on which $D_t(\alpha)$ dominates $K_t(\alpha)$ for all short curves α, the surfaces $\mathcal{L}_t$ and $\mathcal{G}_t$ remain a bounded distance apart. However the path $\mathcal{L}_t$ may deviate arbitrarily far from $\mathcal{G}_t$ along time intervals on which $K_t(\alpha)$ is large and dominates $D_t(\alpha)$. To prove Theorem 5.2, in addition to Theorems 5.3 and 5.4, we therefore need to control distance along intervals along which $K_t(\alpha)$ is large. Let S_α be the surface obtained by cutting S along a short curve α and replacing the two resulting boundary components by punctures. The following somewhat surprising result shows that in this situation, the distance in $\mathcal{T}(S)$ is dominated by the distance in the thick part $\mathcal{T}(S_\alpha)$.

Proposition 5.10 ([7] Theorem D). *If $K_t(\alpha)$ is sufficiently large for all $t \in [a,b]$, the distance in $\mathcal{T}(S_\alpha)$ between the restrictions of $\mathcal{G}_a$ and $\mathcal{G}_b$ to S_α is equal to $b-a$, up to an additive error that is bounded by a constant depending only on the topology of S.*

The proof of Theorem 5.2 also requires a detailed comparison of the rates of change of $D_t(\alpha)$ and $K_t(\alpha)$ with t. The situation is further complicated because the family of curves which are short on $\mathcal{L}_t$ varies with t, so that the intervals along which different curves α are short may overlap. One needs a somewhat involved induction to complete the proof.

5.2.4 Proof of Theorem 5.4 The main tool in the proof of Theorem 5.4 is the formula (2.1) for the variation of length with respect to Fenchel-Nielsen twist, together with the extension proved by Series [34] for variation with respect to the *lengths* of pants curves. To explain, if $\gamma \in \mathcal{S}$, its hyperbolic length l_γ is a real analytic function of the Fenchel-Nielsen coordinates (l_α, t_α) for $\mathcal{T}(S)$ relative to a set $\mathcal{A}$ of pants curves of S. Series' formula is an expression for $\partial l_\gamma/\partial l_\alpha$, analogous to the formula (2.1) for $\partial l_\gamma/\partial t_\alpha$. As in Section 3.4.1, homotope γ to run alternately along and perpendicular to the boundaries of the pairs of pants in the decomposition defined by $\mathcal{A}$. The formula is a sum similar to one in (2.1), but involving additionally both the distance twisted by γ round the pants curves and the complex distance between a given segment of γ across a pair of pants P and the common perpendicular joining the corresponding components of ∂P. Imposing the condition that $\partial(l_{\mu_t} + l_{\nu_t})/\partial l_\alpha = 0$ for short curves α gives additional constraints which need to be satisified at the minimum of the function $l_{\mu_t} + l_{\nu_t}$. It requires considerable work to bring these results into the form in the statement of Theorem 5.4.

The proof of Theorem 5.3 is explained in Section 6.4 below.

6 Short curves on Teichmüller geodesics

In [24], Minsky analyses the curves which are hyperbolically short in a surface on which one has a metric defined by a quadratic differential q as in Section 5.1.1. In a series of papers [31, 32, 33], Rafi has developed this into a technique for estimating the lengths of curves which are short on a surface on a Teichmüller geodesic, in particular implying a proof of Theorem 5.3. Since this work contains some important ideas, we conclude this article with a brief summary of their results.

6.1 Flat and expanding annuli

Let q be a finite area quadratic differential on a Riemann surface $p \in \mathcal{T}(S)$ and let $A \subset S$ be an annulus with piecewise smooth boundary. We say A is *regular* if its boundary components ∂_0, ∂_1 are are q-equidistant and monotonically curved in the sense that their acceleration vectors always points into A, with suitable definition at singular points, namely the zeros of q. The total curvature of ∂_i is

$$\kappa(\partial_i) = \int_{\partial_i} \kappa(p) + \sum [\pi - \theta(P)],$$

where the sum is over all singular points $P \in \partial_i$ and $\theta(P)$ is the interior angle at P. The annulus A is called *flat* if $\kappa(\partial_i) = 0$ for each i and *expanding* if $\kappa(\partial_i) \neq 0$ for $i = 0, 1$.

By the Gauss-Bonnet theorem,

$$\kappa(\partial_0) + \kappa(\partial_1) = \pi \sum \mathrm{ord} P \tag{6.1}$$

where the sum is over the singularities of q in the interior of A and $\mathrm{ord}P$ is the order of the zero at P. A regular annulus is *primitive* if it contains no singularities of q in its interior. It follows from (6.1) that a primitive annulus is either flat or expanding. A flat annulus is necessarily primitive, and is foliated by Euclidean geodesics homotopic to the boundaries. Thus a flat annulus is isometric to a cylinder obtained as the quotient of a Euclidean rectangle in $\mathbb{R}^2$. Expanding annuli are exemplified by an annulus bounded by a pair of concentric circles in $\mathbb{R}^2$. In this case, with a suitable choice of sign convention, $\kappa(\partial_0) = -2\pi, \kappa(\partial_1) = 2\pi$. Any expanding annulus is coarsely isometric to this example [24].

Theorem 6.1 ([24] Theorem 4.5, [31]). *Let $A \subset S$ be an annulus that is primitive with respect to q and with boundaries ∂_0 and ∂_1. Let d be the q-distance between ∂_0 and ∂_1. Then either*

(i) *A is flat and* $\mathrm{mod}\, A = d/l_{\partial_0}(q)$ *or*

(ii) *A is expanding and* $\mathrm{mod}\, A \asymp \log[d/l_{\partial_0}(q)]$.

Theorem 6.2 ([24] Theorem 4.6). *Let $p \in \mathcal{T}(S)$ be a Riemann surface and let q be a quadratic differential on p. Let A be any homotopically non-trivial annulus whose modulus on p is sufficiently large. Then A contains an annulus B that is primitive with respect to q and such that* $\mathrm{mod}\, A \asymp \mathrm{mod}\, B$.

(The statement of Theorem 4.6 in [24] should read $\mathrm{mod}\, A \geq m_0$ not $\mathrm{mod}\, A \leq m_0$.)

6.2 Moduli of annuli and hyperbolically short curves

One can link the hyperbolic and quadratic differential metrics on a surface using annuli of large modulus. Let h be a hyperbolic metric on S. If α is short in h, Maskit [21] showed that the extremal length $\mathrm{Ext}(\alpha)$ and hyperbolic length $l_\alpha(h)$ are comparable, up to multiplicative constants. Moreover, there is an embedded collar $C(\alpha)$ around α whose modulus is comparable to $1/l_\alpha(h)$ (see [26] for an explicit calculation), and therefore also to $1/\mathrm{Ext}(\alpha)$. Combining with the results in Section 6.1, this gives

Theorem 6.3 ([6] Theorem 5.2). *If α is a simple closed curve which is sufficiently short in a hyperbolic metric h on S, then for any compatible quadratic differential q, there is an annulus A that is primitive with respect to q with core homotopic to α such that*

$$\frac{1}{l_\alpha(h)} \asymp \mathrm{mod}(A).$$

6.3 Rafi's thick-thin decomposition for the q-metric

Rafi [32] used the above results to analyse the relationship between the q-metric on a Riemann surface $p \in \mathcal{T}(S)$ and the uniformizing hyperbolic metric h in the thick components of the standard thick-thin decomposition of h. The main result is that on the hyperbolic thick parts of (S, h), the two metrics are comparable, up to a factor which depends only on the moduli of the expanding annuli around the short curves in the boundary of the thick components.

For a subsurface Y of S, let $\hat{Y}$ be the unique subsurface of (S, q) with q-geodesic boundary in the homotopy class of Y that is disjoint from all the maximal flat annuli containing boundary components of ∂Y.

Theorem 6.4 (Rafi [32]). *Let $p \in \mathcal{T}(S)$ be a Riemann surface, let h be the hyperbolic metric that uniformizes p and let Y be a thick component of the hyperbolic thick-thin decomposition of (S, h). Then there exists $\lambda_Y > 0$ such that*

(i) *$diam_q \hat{Y} \stackrel{*}{\asymp} \lambda_Y$,*

(ii) *For any non-peripheral simple closed curve $\gamma \subset Y$, we have*

$$l_\gamma(q) \stackrel{*}{\asymp} \lambda_Y l_\gamma(h).$$

In fact λ_Y is the length of the q-shortest non-peripheral simple closed curve contained in $\hat{Y}$ unless Y is a pair of pants, in which case $\lambda_Y = \max\{l_q(\gamma_i)\}$ where $\gamma_i, i = 1, 2, 3$ are the boundary curves of $\hat{Y}$.

6.4 Proof of Theorem 5.3

This is an application of Theorem 6.1. Let R be a Riemann surface and let q be a quadratic differential on R. It follows as above from Equation (6.1) that every simple closed curve γ on (R, q) either has a unique q-geodesic representative, or is contained in a family of closed Euclidean geodesics which foliate a flat annulus. Denote by $F(\gamma)$ the maximal flat annulus, which necessarily contains all q-geodesic representatives of γ. (If the geodesic representative of γ is unique, then $F(\gamma)$ is taken to be the degenerate annulus containing this geodesic alone.) Denote the (possibly coincident) boundary curves of $F(\gamma)$ by ∂_0, ∂_1 and consider the q-equidistant curves from ∂_i outside $F(\gamma)$. Let $\hat{\partial}_i$ denote the first such curve which is not embedded. If $\hat{\partial}_i \neq \partial_i$, then the pair ∂_i, $\hat{\partial}_i$ bounds a region $E_i(\gamma)$ whose interior is an annulus with core homotopic to γ, and which by its construction is regular and expanding. Let h be the uniformising hyperbolic metric on R. Combining Theorems 6.1, 6.2 and 6.3 we have:

Corollary 6.5. *If α is an extremely short curve on (R, h), then*

$$\frac{1}{l_\alpha(h)} \asymp \max\left\{\operatorname{mod} F(\alpha), \operatorname{mod} E_0(\alpha), \operatorname{mod} E_1(\alpha)\right\}.$$

The proof of Theorem 5.3 then follows from the following two propositions.

Proposition 6.6 ([6] Proposition 5.6). *Let $q = q_t$ denote the quadratic differential on the surface $\mathcal{G}_t$ whose horizontal and vertical foliations are μ_t, ν_t respectively. Let $\alpha \in \mathcal{S}$ be a curve that is neither vertical nor horizontal. Then*

$$\operatorname{mod} F_t(\alpha) \asymp D_t(\alpha).$$

Proposition 6.7 ([6] Proposition 5.7). *Let $p \in \mathcal{T}(S)$ be a Riemann surface with compatible quadratic differential q. Suppose that α is extremely short in the uniformising hyperbolic metric h and let Y be a thick component of the hyperbolic thick-thin decomposition of (S, h), one of whose boundary components is α. Let $\hat{\alpha}$ be the q-geodesic representative of α on the boundary of $\hat{Y}$ and let $E(\alpha)$ be a maximal expanding annulus on the same side of $\hat{\alpha}$ as $\hat{Y}$. Then*

$$\operatorname{mod} E(\alpha) \asymp \log \frac{\lambda_Y}{l_\alpha(q)}.$$

Proposition 6.6 is an exercise in Euclidean geometry and Proposition 6.7 follows from Theorems 6.1 and 6.4.

6.5 Rafi's combinatorial formula for Teichmüller distance

Rafi [33] has developed the above ideas further into a combinatorial expression for the Teichmüller distance between any two surfaces $p, p' \in \mathcal{T}(S)$ in which different families of curves may be short. Let h, h' be the hyperbolic metrics uniformising p, p' respectively, and let $M_p, M_{p'}$ be short markings for h, h' (see 5.1.2). Then:

Theorem 6.8 ([33] Theorem 6.1).

$$d_{\mathcal{T}(S)}(p,p') \asymp \sum_Y \left[d_Y(M_p, M_{p'})\right]_k + \sum_{\alpha \notin \Gamma} \log \left[d_\alpha(M_p, M_{p'})\right]_k + \\ + \max_{\beta \in \Gamma_p} \log \frac{1}{l_\beta(h)} + \max_{\beta \in \Gamma_{p'}} \log \frac{1}{l_\beta(h')} + \max_{\alpha \in \Gamma} d_{\mathbb{H}^2_\alpha}(p,p'). \tag{6.2}$$

where $\Gamma_p, \Gamma_{p'}$, *and* Γ *are defined as follows:*

$$\begin{aligned} \Gamma_p &= \{\alpha \in \mathcal{S} : l_\alpha(h) < \epsilon,\ l_\alpha(h') > \epsilon\}, \\ \Gamma_{p'} &= \{\alpha \in \mathcal{S} : l_\alpha(h') < \epsilon,\ l_\alpha(h) > \epsilon\}, \\ \Gamma &= \{\alpha \in \mathcal{S} : l_\alpha(h) < \epsilon,\ l_\alpha(h') < \epsilon\}. \end{aligned} \tag{6.3}$$

Here $d_Y(M_p, M_{p'})$ is the distance between the projections of the union of the curves in the markings $M_p, M_{p'}$ in the curve complex of Y and $d_\alpha(M_p, M_{p'})$ is the relative twist of the curves in $M_p, M_{p'}$ round α. For $X \geq 0$, the function $[X]_k$ takes the value 0 when $X < k$ and X when $X \geq k$.

This formula is not needed in the proofs of Theorems 5.1 and 5.2.

References

[1] L. Bers, An inequality for Riemann surfaces, in *Differential Geometry and Complex Analysis*, I. Chavel and H. Farkas eds., Springer Verlag, (1985), 87 – 93.

[2] F. Bonahon, Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form, *Ann. Fac. Sci de Toulouse Math.* 5 (1996), 233 – 297.

[3] F. Bonahon, Kleinian groups which are almost Fuchsian, *Journal fur die reine und angewandte Mathematik* 10 (2005), 1–15.

[4] F. Bonahon and J-P. Otal, Laminations mesurées de plissage des variétés hyperboliques de dimension 3, *Annals of Math.* 160 (2004), 1013–1055.

[5] J. Brock, R. Canary and Y. Minsky. The classification of Kleinian surface groups, II: The Ending Lamination Conjecture, `arXiv: math.GT/0412006`.

[6] Y. Choi, K. Rafi, and C. Series, Lines of minima and Teichmüller geodesics, *Geom. Funct. Anal.*, GAFA online first 2008, also `arXiv: math.GT/0605135`.

[7] Y. Choi, K. Rafi, and C. Series, Lines of minima are uniformly quasi-geodesic, *Pacific J. Math.* 237 (2008), 21–44.

[8] R. Díaz and C. Series, Examples of pleating varieties for twice punctured tori, *Trans. Amer. Math. Soc.* 356 (2003), 621- 658.

[9] R. Díaz and C. Series, Limit points of lines of minima in Thurston's boundary of Teichmüller space, *Algebraic and Geometric Topology* 3 (2003), 207–234.

[10] D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Analytical and Geometric Aspects of Hyperbolic Space*, D. B. A. Epstein ed, LMS Lecture Notes 111,Cambridge University Press (1987), 112–253.

[11] A. Fahti, P. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque 66-67, Société Mathématique de France, 1979.

[12] F. Gardiner and H. Masur, Extremal length geometry of Teichmüller space, *Complex Variables*, 16 (1991), 209–237.

[13] J. Hubbard and H. Masur, Quadratic differentials and foliations, *Acta Math.* 142 (1979), 221–274.

[14] L. Keen and C. Series, How to bend pairs of punctured tori, in *Lipa's Legacy*, J. Dodziuk and L. Keen eds, Contemp. Math. 211, 359–388. AMS, 1997.

[15] L. Keen and C.Series, Pleating invariants for punctured torus groups, *Topology*, 43 (2004), 447 – 491.

[16] S. Kerckhoff, The asymptotic geometry of Teichmüller space, *Topology*, 19 (1980) 2 –41.

[17] S. Kerckhoff, The Nielsen realization problem, *Ann. of Math.* 117 (1983), 235–265.

[18] S. Kerckhoff, Lines of Minima in Teichmüller space, *Duke Math J.* 65 (1992), 187–213.

[19] A. Lenzhen, Teichmüller geodesics that do not have a limit in PMF, *Geom. and Topology* 12 (2008), 177–197.

[20] A. Marden, *Outer Circles: An introduction to hyperbolic* 3*-manifolds,* Cambridge University Press, 2007.

[21] B. Maskit, Comparison of extremal and hyperbolic lengths, *Ann. Acad. Sci. Fenn.* 10 (1985), 381 – 386.

[22] H. Masur, Two boundaries of Teichmüller space, *Duke Math. J* 49 (1982), 183–190.

[23] H. Masur and Y. Minsky, Geometry of the complex of curves II: Hierarchical structure, *Geom. Funct. Anal.* **10** (2000), 902–974.

[24] Y. Minsky, Harmonic maps, length, and energy in Teichmüller space, *J. Differential Geometry* 35 (1992), 151–217.

[25] Y. Minsky, Teichmüller geodesics and ends of hyperbolic 3-manifolds, *Topology* 32 (1993), 625–647.

[26] Y. Minsky, Extremal length estimates and product regions in Teichmüller space, *Duke Math. J.* 83 (1996), 249–286.

[27] J-P. Otal, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension* 3, Astérisque 235, Société Mathématique de France, 1996.

[28] A. Papadopoulos, Geometric intersection functions and Hamiltonian flows on the space of Measured Foliations on a surface, *Pacific J. Math.* 124 (1986), 375–402.

[29] J. Parker and C. Series, Bending formulae for convex hull boundaries, *J. d'Analyse Math.* 67 (1995), 165–198.

[30] R. C. Penner with J. Harer, *Combinatorics of Train Tracks.* Annals of Math. Studies 125, Princeton University Press, 1992.

[31] K. Rafi, A characterization of short curves of a Teichmüller geodesic, *Geometry and Topology* 9 (2005), 179–202.

[32] K. Rafi, Thick-thin decomposition of quadratic differentials, *Math. Res. Lett.* 14 (2007), 333–342.

[33] K. Rafi, A combinatorial model for the Teichmüller metric, *Geom. Funct. Anal.* 17 (2007), 936–659.

[34] C. Series, An extension of Wolpert's derivative formula, *Pacific J. Math.* 197 (2001), 223–239.

[35] C. Series, On Kerckhoff minima and pleating varieties for quasifuchsian groups, *Geometriae Dedicata* 88 (2001), 211–237.

[36] C. Series, Limits of quasifuchsian groups with small bending, *Duke Mathematical J.*128 (2005), 285–329.

[37] C. Series, Thurston's bending measure conjecture for once punctured torus groups, in *Spaces of Kleinian Groups*, Y. Minsky, M. Sakuma and C. Series eds., LMS Lecture Notes 329, 75–90, Cambridge Univ. Press 2006.

[38] S. P. Tan, Complex Fenchel-Nielsen coordinates for quasifuchsian structures, *Int'l. J. Math* 5 (1994), 239–251.

[39] K. Strebel, *Quadratic Differentials,* Springer Verlag (1980).

[40] W. Thurston, Earthquakes in two dimensional hyperbolic geometry, in *Low dimensional geometry and Kleinian groups, Warwick and Durham 1984*, D. B. A. Epstein, ed., LMS Lecture Notes 112, 91–112, Cambridge Univ. Press, 1986.

[41] S. Wolpert, The Fenchel-Nielsen deformation, *Ann. Math.* 115 (1982), 50–528.

[42] S. Wolpert, On the symplectic geometry of deformations of a hyperbolic surface, *Ann. Math.* 117 (1983), 207–234.