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QUANTIZATION OF KÄHLER MANIFOLDS. III

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Abstract. We use Berezin's dequantization procedure to define a formal $*$-product on the algebra of smooth functions on the unit disk in $\mathbb{C}$. We prove that this formal $*$-product is convergent on a dense subalgebra of the algebra of smooth functions.

1. Introduction.

In two previous papers [4], [5] we have examined various geometrical methods for quantizing Kähler manifolds M. In particular we showed in [5] that the quantization method of Berezin [3] can be used to construct a formal $*$-product [1], [2] on the smooth functions on coadjoint orbits of compact Lie groups.

The formal $*$-product is constructed by using Berezin's quantization method to make a correspondence between operators and functions (their Berezin symbols) and transferring the operator composition to the symbols (this multiplication of symbols is denoted by $*$). By introducing a parameter k into the quantization we obtain a rule of composition on formal power series in k^{-1} by taking the asymptotic expansion of the $*_k$. Our main result in [5] is that this asymptotic expansion exists for compact M, defines an associative multiplication on formal power series in k^{-1} with coefficients in $C^\infty(M)$ for compact coadjoint orbits, and this formal power series converges on the space of symbols for M a Hermitian symmetric space of compact type.

In the present paper we make the first steps in extending these results to some non-compact Kähler manifolds. Section 2 sets up the notation we will use in this paper. In section 3 we show how to extend our asymptotic expansion of Berezin's integral formula for the composition of symbols, proved in [5] for the compact case, to arbitrary Kähler manifolds for suitably chosen classes of functions on M. In section 4 we show that if M is the unit disk in $\mathbb{C}$ with the Poincaré metric then symbols of polynomial differential operators are a suitable class of functions for the results of section 3 to apply. These symbols have similar nesting properties to those we used in the compact case and allow us to show that the asymptotic expansion of the product of two of these symbols exists. On the other hand we can actually work out the composition of two differential operators explicitly and examine the dependence of the composition on the parameter k. We see at once that the dependence is rational and so the asymptotic expansion is convergent on symbols of differential operators. From this it easily follows that the asymptotic expansion defines an associative product on formal power series with smooth coefficients, and by the method of construction this is invariant under $SU(1,1)$ and also covariant. These results are also related to the calculations of Moreno [7].

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2. Preliminaries.

We denote by (L, ∇, h) a quantization bundle for the Kähler manifold (M, ω, J) and by $\mathcal{H}$ the Hilbert space of square-integrable holomorphic sections of L which we assume to be non-trivial. The coherent states are vectors $e_q \in \mathcal{H}$ such that

$$s(x) = \langle s, e_q \rangle q, \qquad \forall q \in L_x^\times, \quad x \in M, \quad s \in \mathcal{H}$$

where $L^\times$ denotes the complement of the zero-section in L. The function

$$\epsilon(x) = |q|^2 \|e_q\|^2, \qquad q \in L_x^\times$$

is well-defined and real analytic.

We introduce also the 2-point function

$$\psi(x, y) = \frac{|\langle e_{q'}, e_q \rangle|^2}{\|e_{q'}\|^2 \|e_q\|^2}, \qquad q \in L_x^\times, \quad q' \in L_y^\times$$

which is a globally defined real analytic function on $M \times M$ provided ϵ has no zeros. The latter condition will be a consequence of a stronger condition which we shall assume shortly, so we shall assume it is true from now on without further comment. It is a consequence of the Cauchy–Schwartz inequality that $\psi(x, y) \leq 1$ everywhere, with equality where the lines spanned by e_q and $e_{q'}$ coincide ($q \in L_x^\times$, $q' \in L_y^\times$). It is again a consequence of our assumptions below that this only happens for $x = y$. The function ψ vanishes only where the coherent states are orthogonal.

Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator and let

$$\widehat{A}(x) = \frac{\langle A e_q, e_q \rangle}{\langle e_q, e_q \rangle}, \qquad q \in L_x^\times, \quad x \in M$$

be its symbol. The function $\widehat{A}$ has an analytic continuation to an open neighbourhood of the diagonal in $M \times \overline{M}$ given by

$$\widehat{A}(x, y) = \frac{\langle A e_{q'}, e_q \rangle}{\langle e_{q'}, e_q \rangle}, \qquad q \in L_x^\times, \quad q' \in L_y^\times$$

which is holomorphic in x and antiholomorphic in y. It is clear that the only source of singularities in the analytically continued symbol is from zeros in the denominator and these are the zeros of ψ. We denote by $\widehat{E}(L)$ the space of symbols of bounded operators on $\mathcal{H}$. In [5] this space played a crucial role when M was compact. In the present situation where we drop the compactness, one of our problems is to find a suitable substitute for $\widehat{E}(L)$ and this involves the use of symbols of unbounded operators.

The composition of operators on $\mathcal{H}$ gives rise to a product for the corresponding symbols, which is associative and which we shall denote by $*$ following Berezin, [3]. For the basic facts about $*$ quantization see [1], [2]. The $*$-product of symbols is given in terms of the symbols by the integral formula

$$(\widehat{A} * \widehat{B})(x) = \int_M \widehat{A}(x, y) \widehat{B}(y, x) \psi(x, y) \epsilon(y) \frac{\omega^n(y)}{n!}. \tag{2.1}$$

This formula is derived, for example in [5], by use of the adjoint A^* of A.

Remark 1. In order to apply (2.1) to the case where the operators are unbounded we need to be able to use the adjoint of A on coherent states. This is not automatic, but is a condition which needs to be checked. Thus our unbounded operators must satisfy two conditions:

(1) $Ae_q \in \mathcal{H}, \ \forall q$;
(2) the sections $s(x) = \langle e_{q'}, Ae_q \rangle q, \ q \in L_x$ should be holomorphic and square integrable for each q'.

Of course we also need to be able to take the symbol of the composition, so we need to check that the result of applying B to a coherent state is in the domain of A.

Example 1. The identity map has symbol $\widehat{I} = 1$ and so $\widehat{A} * 1 = 1 * \widehat{A} = \widehat{A}$ for any operator A. In particular $1 * 1 = 1$, a fact which we shall use several times later.

Let k be a positive integer. The bundle $(L^k = \otimes^k L, \nabla^k, h^k)$ is then a quantization bundle for $(M, k\omega, J)$ and we denote by $\mathcal{H}_k$ the corresponding space of square-integrable holomorphic sections. We let $\epsilon^{(k)}$ be the corresponding function and say that the quantization is *regular* if $\epsilon^{(k)}$ is a non-zero constant for all nonnegative k and if $\psi(x, y) = 1$ implies $x = y$. The significance of these conditions has been explained in [4] and [5]. In the regular case we have shown that for powers L^k the function ψ in the integral (2.1) gets replaced by powers ψ^k.

If $\widehat{A}$ is a symbol of an operator then its analytic continuation $\widehat{A}(x, y)$ may have singularities where $\psi(x, y) = 0$ but $\widehat{A}\psi$ is always globally defined on $M \times M$. If M is not compact $\widehat{A}\psi$ may not be bounded, so we introduce the class $\mathcal{B} \subset C^\infty(M)$ of functions f which have an analytic continuation off the diagonal in $M \times \overline{M}$ so that $f(x, y)\psi(x, y)^l$ is globally defined, smooth and bounded on $K \times M$ and on $M \times K$ for each compact subset K of M for some positive power l and denote by $\mathcal{B}_l$ those for which the power l suffices. Since ψ is smooth and bounded it is clear that $\mathcal{B}$ is a subalgebra of $C^\infty(M)$. In the case M is compact we obviously have $\widehat{E}(L^l) \subset \mathcal{B}_l$.

Denote by $\widetilde{\mathcal{B}}$ the set of functions f on $M \times M \setminus \psi^{-1}(0)$ such that $f(x, y)\psi(x, y)^l$ has a smooth extension to all of $M \times M$ which is bounded on $K \times M$ for each compact subset $K \subset M$ for some l and denote by $\widetilde{\mathcal{B}}_l$ those for which the power l suffices. If f, g are in $\mathcal{B}$ then $f(x, y)g(y, x)$ is in $\widetilde{\mathcal{B}}$. Note also that if $f \in \widetilde{\mathcal{B}}$ then its restriction to the diagonal $\widehat{f}(x) = f(x, x)$ is smooth.

3. The composition of operators and an asymptotic formula.

In order to localise the integral (2.1) we use a version of the Morse lemma which we adapt from Combet [6]. This is a modification of proposition 2 of §2 of [5] which is valid without the assumption that the Kähler manifold is compact.

Let (M, ω, J) be a Kähler manifold with metric g. We denote by $\exp_x X$ the exponential at x of $X \in T_x M$. If g is not complete the exponential map may not be defined for all x and X, but in any case there is an open subset $V \subset TM$ where it is defined and which contains the zero-section. The differential of the exponential map at 0 is the identity so the map $\alpha : V \to M \times M$ given by $\alpha(X) = (p(X), \exp_{p(X)} X)$ where p is the projection in the tangent bundle $p: TM \to M$ is a diffeomorphism near the zero-section. At any point of the zero-section the differential of α is the identity.

Proposition 1. *Let (M, ω, J) be a Kähler manifold with metric g and $\alpha: V \to M \times M$ be the map defined above. Let (L, ∇, h) be a regular quantization bundle over M and let ψ be* *the corresponding 2-point function on $M \times M$. Then there exists an open neighbourhood $W \subset V$ of the zero-section in TM and a smooth open embedding $\nu : W \to TM$ such that*

$$(-\log \psi \circ \alpha \circ \nu)(X) = \pi g_{p(X)}(X, X), \quad X \in W$$

and the differential of ν at any point of the zero-section is the identity.

Proof. α is a diffeomorphism of some neighbourhood V_1 of the zero-section of TM onto an open neighbourhood U of the diagonal Δ in $M \times M$ and we can further suppose that $\overline{U} \cap \psi^{-1}(0) = \emptyset$.

Denote by $f : V_1 \to \mathbb{R}$ the smooth function $f = -\log \psi \circ \alpha$. Observe that $f(0_x) = -\log \psi(x, x) = 0$. If we denote by a subscript 2 differentiation in the vertical direction in TM one has

$$(D_2 f)_{0_x} = \frac{1}{\psi(x, x)}(D_2 \psi)_{(x,x)} = 0$$

since all points of the diagonal are critical points of ψ. Finally, using Proposition 4 of §1 of [5] (corrected to include a missing factor of 2) we get

$$(\mathrm{Hess}_2\, f)_{0_x} = 2\pi g_x.$$

For $v \in V_1$, define the function $\tilde{g}_v \colon [0, 1] \to \mathbb{R}$ by $\tilde{g}_v(t) = f(tv)$. Clearly

$$\tilde{g}_v(0) = 0, \qquad \tilde{g}_v'(0) = (D_2 f)_0 v = 0$$

and

$$\tilde{g}_v''(0) = (\mathrm{Hess}_2\, f)_0(v, v) = 2\pi g_x(v, v) > 0$$

whatever v we choose. Taylor's formula with remainder gives us

$$f(v) = \int_0^1 (1 - s)\tilde{g}_v''(s) ds$$

and one sees that

$$\tilde{g}_v'(s) = (D_2 f)_{sv} v, \qquad \tilde{g}_v''(s) = (\mathrm{Hess}_2\, f)_{sv}(v, v).$$

We can thus introduce on each tangent space M_x a family of symmetric bilinear forms, indexed by an element $v \in V_1 \cap M_x$

$$B_v(u, u') = \int_0^1 (1 - s)(\mathrm{Hess}_2\, f)_{sv}(u, u') ds.$$

Clearly $B_0(u, u') = \frac{1}{2}(\mathrm{Hess}_2\, f)_0(u, u') = \pi g_x(u, u')$ is positive definite.

Let V_2 denote the set of points v of V_1 where B_v is positive definite. V_2 will be an open neighbourhood of the zero-section in TM. Recall that $f(v) = B_v(v, v)$. There exists a unique non-singular element C_v of $GL(M_x)$ which is symmetric relative to B_0 such that

$$B_v(u, u') = B_0(C_v u, u')$$

and all eigenvalues of C_v are strictly positive. Furthermore the map $V_2 \subset TM \to \mathrm{End}(TM)$, $v \mapsto C_v$ is smooth. Finally the endomorphism C_v admits a unique symmetric, positive definite square root $C_v^{1/2}$ and

$$B_v(u, u') = B_0(C_v^{1/2}u, C_v^{1/2}u').$$

Also $C_0^{1/2} = I$ and the map $v \mapsto C_v^{1/2}$ is smooth.

Define the map $\beta\colon V_2 \to TM$ by $\beta(v) = C_v^{1/2}v$. This maps the zero-section onto the zero-section and one can find a neighbourhood $\widetilde{W} \subset V_2$ of the zero-section such that $\beta|_{\widetilde{W}} : \widetilde{W} \to TM$ is an embedding. Clearly one may choose $W = \beta(\widetilde{W})$, $\nu = \beta|_W{}^{-1}$. It is trivial to check that β has differential at the zero-section equal to the identity and the proposition is proven.

Remark 1. In [5] we used the compactness of M in several places to get the existence of an asymptotic expansion for the $*$-product. In particular the Liouville volume had a uniform bound. We can replace that here by observing that, from example 1 of §2, we have

$$1 = 1 *_1 1 = \int_M \psi(x,y)\epsilon^{(1)}(y)\frac{\omega^n(y)}{n!}.$$

Since ψ is everywhere non-negative, it follows that, if $\epsilon^{(1)}$ is constant, then on every subset U of M we have

$$\int_U \psi(x,y)\frac{\omega^n(y)}{n!} \le \epsilon^{(1)\,-1}.$$

Proposition 2. *Let (M, ω, J) be a Kähler manifold, (L, ∇, h) be a regular quantization bundle for M and ψ the corresponding 2-point function. Then, for any f belonging to $\widetilde{\mathcal{B}}_l$, the integral*

$$F_k(x) = \int_M f(x,y)\psi(x,y)^k k^n \frac{\omega^n(y)}{n!}, \quad \text{for } k \ge l+1 \tag{3.1}$$

admits an asymptotic expansion

$$F_k(x) \sim \sum_{r\ge 0} k^{-r} C_r(\widehat{f}\,)(x)$$

where C_r is a smooth differential operator of order $2r$ depending only on the geometry of M. The leading term is given by $C_0(\widehat{f}\,)(x) = \widehat{f}$.

Note: We are not claiming that for this very general class of functions $f(x,y)$ the integral (3.1) depends smoothly on x, only that the coefficients of the asymptotic expansion do.

Proof of proposition 2. Use Proposition 1 to construct a neighbourhood U_1 of the diagonal Δ in $M \times M$ and a neighbourhood V_1 of the zero-section in TM such that the following hold:

(i) $\alpha : V_1 \to U_1$, $X \mapsto (x, \exp_x X)$ is a smooth diffeomorphism;

(ii) $\exists\, \nu^{-1} : V_1 \to \nu^{-1}(V_1) \subset TM$ a smooth embedding such that $-\log\psi\circ\alpha\circ\nu = \pi g$ on $\nu^{-1}(V_1)$;

(iii) $\overline{U}_1 \cap \psi^{-1}(0) = \emptyset$.

Shrinking U_1, V_1, if necessary, we can assume that $V_1 \cap T_xM$ is a bounded subset of T_xM. Going back to the proof of Proposition 2, one observes that $\alpha \circ \nu : \nu^{-1}(V_1) \cap T_xM \to M = \{x\} \times M$ is an embedding and hence one may define a non-zero smooth function θ by

$$((\alpha \circ \nu)^* \frac{\omega^n}{n!})(x, v) = \theta(x, v) dv$$

where dv denotes the linear Lebesgue measure on M_x. Shrinking V_1, if necessary, one may assume that θ is defined on $\overline{V}_1$ and hence is bounded as well as all its derivatives for x in a compact subset of M.

Choose an open neighbourhood U_2 of Δ in $M \times M$, with $\overline{U}_2 \subset U_1$ and define $V_2 = \alpha^{-1}(U_2)$. Let $\chi : M \times M \to [0, 1]$ be a smooth function such that $\chi|_{U_2} = 1$ and $\operatorname{supp} \chi \subset U_1$. Set $\eta = \max_{x,y \notin U_2} \psi(x, y)$. Clearly $\eta < 1$ and $\psi(x, y) \leq \eta$ on $M \setminus U_2$. Let $U_{1,x} = \{y \in M \mid (x, y) \in U_1\}$, $U_{2,x} = \{y \in M \mid (x, y) \in U_2\}$ and $\chi_x(y) = \chi(x, y)$. The function χ_x is equal to 1 on $U_{2,x}$ and has compact support in $U_{1,x}$.

The function f appearing in the statement of the proposition is a smooth function on $(M \times M) \setminus \psi^{-1}(0)$. In particular it is smooth in a neighbourhood of the diagonal. One has

$$F_k(x) = \int_{U_{1,x}} f(x, y)\psi^k(x, y)k^n \frac{\omega^n(y)}{n!} + \int_{M \setminus U_{1,x}} f(x, y)\psi^k(x, y)k^n \frac{\omega^n(y)}{n!}.$$

By assumption there exists a positive constant C_1 such that $|f\psi^l| \leq C_1$ on $\{x\} \times M$. Thus

$$\left| \int_{M \setminus U_{1,x}} f(x, y)\psi^k(x, y)k^n \frac{\omega^n(y)}{n!} \right| \leq C_1 \epsilon^{(1)^{-1}} \eta^{k-l-1} k^n.$$

Also

$$\begin{aligned} &\left| \int_{U_{1,x}} f(x, y)(1 - \chi_x(y))\psi^k(x, y)k^n \frac{\omega^n(y)}{n!} \right| \\ &\qquad = \left| \int_{U_{1,x} \setminus U_{2,x}} f(x, y)(1 - \chi_x(y))\psi^k(x, y)k^n \frac{\omega^n(y)}{n!} \right| \\ &\qquad \leq C_1 \epsilon^{(1)^{-1}} \eta^{k-l-1} k^n. \end{aligned}$$

Grouping the terms we get

$$\left| F_k(x) - \int_{U_{1,x}} \chi(x, y) f(x, y)\psi^k(x, y)k^n \frac{\omega^n(y)}{n!} \right| \leq C\eta^k k^n, \quad \forall k \geq l + 1$$

where $C = 2C_1 \epsilon^{(1)^{-1}} \eta^{-l-1}$. Thus this difference is exponentially small for each x. The remaining integral may be computed in the tangent space T_xM as

$$\begin{aligned} &\int_{U_{1,x}} \chi(x, y) f(x, y)\psi^k(x, y)k^n \frac{\omega^n(y)}{n!} \\ &\qquad = \int_{V_{1,x}} \chi(\alpha \circ \nu)(x, v) f(\alpha \circ \nu)(x, v) e^{-k\pi g(v,v)} k^n \theta(x, v) dv \end{aligned}$$

where $V_{1,x} = (\alpha\circ\nu)^{-1}\, U_{1,x}$.

Denote by $G(x,v)$ the function on TM defined by

$$G(x,v) = \begin{cases} \chi((\alpha\circ\nu)(x,v))f((\alpha\circ\nu)(x,v))\theta(x,v), & \text{if } (x,v)\in V_1; \\ 0, & \text{if } (x,v)\notin V_1. \end{cases}$$

It is smooth and compactly supported for x in a compact set and

$$\begin{aligned}\int_{U_{1,x}} \chi(x,y)f(x,y)\psi(x,y)k^n\frac{\omega^n(y)}{n!} &= \int_{M_x} G(x,v)e^{-k\pi g(v,v)}k^n dv \\ &= \int_0^\infty dr\int_{S_xM} G(x,rv)e^{-k\pi r^2}r^{2n-1}k^n dv\end{aligned}$$

where $r(v) = g(v,v)^{1/2}$ and S_xM is the unit sphere in T_xM. Now use Taylor's formula with integral remainder for $G(x,rv)$

$$G(x,rv) = \sum_{p=0}^{2N}\frac{r^p}{p!}(D_v^pG)(x,0) + r^{2N+1}\int_0^1\frac{(1-s)^{2N}}{(2N)!}(D_v^{2N+1}G)(x,rsv)ds.$$

The integral of the remainder term is easily bounded since G is compactly supported.

$$\begin{aligned}&\left|\int_0^\infty dr\int_{S_xM} dv\, k^n r^{2n+2N}\int_0^1 ds\frac{(1-s)^{2N}}{(2N)!}(D_v^{2N+1}G)(x,rsv)e^{-k\pi r^2}\right| \\ &\quad = \left|k^{-N}\int_0^\infty dt\int_{S_xM} dv\int_0^1 ds\frac{t^{n+N-\frac12}(1-s)^{2N}}{(2N)!\pi^{n+N+\frac12}}(D_v^{2N+1}G)(x,\sqrt{\frac{t}{k\pi}}sv)\frac{e^{-t}}{2\sqrt{k}}\right| \\ &\quad \le k^{-N}\frac{C_3}{\sqrt{k}}.\end{aligned}$$

Observe finally that if p is odd

$$\int_{S_xM}(D_v^pG)(x,0)dv = 0$$

since this is the integral of the restriction to the sphere of a homogeneous polynomial of odd degree. Putting these facts together we get

$$k^N\left|F_k(x) - \sum_{p=0}^{N}\frac{(p+n-1)!}{(2p)!}\frac{k^{-p}}{2\pi^{p+n}}\int_{S_xM}(D_v^{2p}G)(x,0)dv\right| \le C\eta^k k^{n+N} + \frac{C_3}{\sqrt{k}}.$$

The derivatives of the function G in the vertical direction for $v=0$ do not depend on the choice of the cut-off function χ, but depend only on f and θ (which is related to the geometry alone). The smoothness of the coefficients follows since they are determined by the integral of a compact supported function in U_1 which allows us to differentiate freely under the integral sign even though we could not necessarily do that when evaluating the original integral over the whole of M.

Finally, the leading term is given by

$$\frac{(n-1)!}{2\pi^n}\int_{S_xM} G(x,0)dv = \frac{(n-1)!\,\mathrm{vol}(S^{2n-1})}{2\pi^n}\theta(x,0)\widehat{f}(x,x).$$

Since the differentials of both α and ν at the zero section are equal to the identity, and on a unitary frame of the tangent bundle the Liouville volume $\frac{\omega^n}{n!}$ is equal to 1 it follows that $\theta(x,0)=1$. Moreover $\mathrm{vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$ so $C_0(\widehat{f}\,) = \hat{f}$ as desired.

Corollary 1. *In the regular case $\epsilon^{(k)}$ has an asymptotic expansion $\sum_{r\geq 0}\epsilon_r/k^r$ as k tends to infinity with $\epsilon_0 = 1$.*

Proof. Apply the formula for the product to the identity operator using the fact that the symbol of the identity is always the constant function 1:

$$1 = 1 *_k 1 = \epsilon^{(k)} \int_M \psi(x,y)^k k^n \frac{\omega^n(y)}{n!}.$$

The integral has an asymptotic expansion in k^{-1} with leading term 1 by the previous proposition. We can then invert the asymptotic expansion to obtain one for $\epsilon^{(k)}$.

Corollary 2. *In the regular case $\widehat{A} *_k \widehat{B}$ has an asymptotic expansion in k^{-1} as k tends to infinity for $\widehat{A}$ and $\widehat{B}$ in $\mathcal{B}_l$ for some l with coefficients which are bidifferential operators.*

Proof. We have already observed that if $\widehat{A}$ and $\widehat{B}$ are in $\mathcal{B}_l$ for some l then $\widehat{A}(x,y)\widehat{B}(y,x)$ is in $\widetilde{\mathcal{B}}$ and so proposition 2 applies to the integral

$$\int_M \widehat{A}(x,y)\widehat{B}(y,x)\psi(x,y)^k k^n \frac{\omega^n(y)}{n!}$$

to show that it has an asymptotic expansion with coefficients which are bidifferential operators. By (2.1), since $\epsilon^{(k)}$ is constant, $\widehat{A} *_k \widehat{B}$ is a product of two functions of k each of which has an asymptotic expansion. Hence the result.

We compute the first two terms of the asymptotic expansion of $\widehat{A} *_k \widehat{B}$ for $\widehat{A}$ and $\widehat{B}$ in $\mathcal{B}_l$ and $k \geq l$.

$$(\widehat{A} *_k \widehat{B})(x) \sim \left(1 + \frac{\epsilon_1}{k}\right)\widehat{A}(x,x)\widehat{B}(x,x) + \frac{n!}{4\pi^{n+1}k}\int_{S_x M} D_v^2 G(x,0)dv + \cdots$$

where

$$G(x,v) = \widehat{A}(x, \exp_x \nu(v))\widehat{B}(\exp_x \nu(v), x)\theta(x,v).$$

Taking the antisymmetric part we have

$$\begin{aligned}(\widehat{A} *_k \widehat{B} - \widehat{B} *_k \widehat{A})(x) \\ \sim \frac{n!}{4\pi^{n+1}k}\int_{S_x M}\left[D_v^2 G_{(A,B)}(x,0) - (D_v^2 G)_{(B,A)}(x,0)\right]dv + \cdots\end{aligned}$$

Using the identities $1 *_k B = B *_k 1 = B$, $1 *_k 1 = 1$ when expanding the derivatives the above integrand reduces to

$$\begin{aligned}(\widehat{A} *_k \widehat{B} - \widehat{B} *_k \widehat{A})(x) \\ \sim \frac{n!}{2\pi^{n+1}k}\int_{S_x M}\big[D_{v,2}\widehat{A}(x,x)D_{v,1}\widehat{B}(x,x) \\ - D_{v,2}\widehat{B}(x,x)D_{v,1}\widehat{A}(x,x)\big]\theta(x,0)dv + \cdots\end{aligned} \tag{3.2}$$

where the indices $1,2$ refer to the first (second) variable in a function of the form $\widehat{A}(x,y)$.

If we take an orthonormal basis e_i of tangent vectors at x and v has components v_i then

$$\int_{S_x M} v_i v_j dv = \mathrm{vol}(S^{2n-1})/n\ \delta_{ij}$$

so that, if $D_{i,1}$ denotes differentiation in the direction e_i in the first variable, and so on, then

$$\begin{aligned}(\widehat{A} *_k \widehat{B} - &\widehat{B} *_k \widehat{A})(x) \\ &\sim \frac{1}{\pi k}\sum_{j=1}^{2n} D_{j,2}\widehat{A}(x,x)D_{j,1}\widehat{B}(x,x) - D_{j,2}\widehat{B}(x,x)D_{j,1}\widehat{A}(x,x) + \cdots\end{aligned}$$

The function $\widehat{A}(x,y)$ is holomorphic in x and antiholomorphic in y. Thus, taking an orthonormal basis such that $e_{j+n} = J_x e_j$, $j \leq n$ and setting $u_j = e_j - ie_{j+n}$ we have $\omega_x(u_j, \overline{u}_k) = 2i\delta_{jk}$ and, for $j \leq n$, $D_{j,2}\widehat{A} = \frac{1}{2}\overline{u}_j(\widehat{A})$, $D_{j+n,2}\widehat{A} = -\frac{i}{2}\overline{u}_j(\widehat{A})$, $D_{j,1}\widehat{A} = \frac{1}{2}u_j(\widehat{A})$, $D_{j+n,1}\widehat{A} = \frac{i}{2}u_j(\widehat{A})$, etc, so that

$$\begin{aligned}\sum_{j=1}^{2n} D_{j,2}\widehat{A}D_{j,1}\widehat{B} - D_{j,2}\widehat{B}D_{j,1}\widehat{A} &= \sum_{j=1}^{n} D_{j,2}\widehat{A}D_{j,1}\widehat{B} - D_{j,2}\widehat{B}D_{j,1}\widehat{A} \\ &\qquad + D_{j+n,2}\widehat{A}D_{j+n,1}\widehat{B} - D_{j+n,2}\widehat{B}D_{j+n,1}\widehat{A} \\ &= \frac{1}{2}\sum_{j=1}^{n}\overline{u}_j(\widehat{A})u_j(\widehat{B}) - u_j(\widehat{A})\overline{u}_j(\widehat{B}) \\ &= i\{\widehat{A},\widehat{B}\}\end{aligned}$$

so that

$$(\widehat{A} *_k \widehat{B} - \widehat{B} *_k \widehat{A})(x) \sim \frac{i}{\pi k}\{\widehat{A},\widehat{B}\} + \cdots$$

Theorem 1. *Let (M, ω, J) be a Kähler manifold and (L, ∇, h) be a regular quantization bundle over M. Let $\widehat{A}, \widehat{B}$ be in $\mathcal{B}$. Then*

$$(\widehat{A} *_k \widehat{B})(x) = \int_M \widehat{A}(x,y)\widehat{B}(y,x)\psi^k(x,y)\epsilon^{(k)}k^n\frac{\omega^n(y)}{n!}(y),$$

defined for k sufficiently large, admits an asymptotic expansion in k^{-1} as $k \to \infty$

$$(\widehat{A} *_k \widehat{B})(x) \sim \sum_{r\geq 0} k^{-r}C_r(\widehat{A},\widehat{B})(x)$$

where the cochains C_r are smooth bidifferential operators, invariant under the automorphisms of the quantization and determined by the geometry alone. Furthermore

$$C_0(\widehat{A},\widehat{B}) = \widehat{A}\widehat{B},$$

and

$$C_1(\widehat{A},\widehat{B}) - C_1(\widehat{B},\widehat{A}) = \frac{i}{\pi}\{\widehat{A},\widehat{B}\}.$$

4. The Poincaré disk.

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$ with the Poincaré metric. The Kähler form is given by

$$\omega = \frac{-i\lambda dz \wedge d\overline{z}}{2\pi(1-|z|^2)^2} = d\left(\frac{i\lambda \overline{z} dz}{2\pi(1-|z|^2)}\right).$$

$\mathbb{D}$ can be written as the homogeneous domain $SU(1,1)/U(1)$ and the action of $SU(1,1)$ is Hamiltonian.

If (L, ∇, h) is a homogenous quantization for the simply-connected group $\widetilde{SU(1,1)}$ then L can be trivialised on all of $\mathbb{D}$ by a section s_0 with

$$|s_0|^2 = (1-|z|^2)^\lambda.$$

The norm on holomorphic sections is then

$$\|fs_0\|^2 = \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\lambda \frac{\lambda d^2z}{\pi(1-|z|^2)^2}$$

where d^2z denotes the usual Lebesgue measure $dxdy$ for $z = x+iy$. s_0 will have finite norm for $\lambda > 1$ which we assume.

Since $\mathbb{D}$ is homogeneous, ϵ will be constant and so the coherent states for the Hilbert space $\mathcal{H}$ of square-integrable holomorphic sections can be written down immediately as

$$e_{s_0(w)}(z) = \epsilon(1-\overline{w}z)^{-\lambda}s_0(z).$$

Note that this makes sense since $2\Re e(1-\overline{w}z) = 1-|w|^2+1-|z|^2+|w-z|^2 > 0$ and so $1-\overline{w}z$ takes values in a half space and all real powers can be defined as single-valued holomorphic functions.

We calculate ϵ by observing that $s_0(0) = \langle s_0, e_{s_0(0)}\rangle s_0(0)$ and thus

$$\begin{aligned} 1 &= \epsilon \int_{\mathbb{D}} (1-|z|^2)^{\lambda-2}\frac{\lambda}{\pi} d^2z \\ &= \epsilon\frac{\lambda}{\lambda-1}. \end{aligned}$$

Hence

$$\epsilon = 1-\lambda^{-1}.$$

The 2-point function ψ is given by

$$\psi(z,w) = \left(\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2}\right)^\lambda = \left(\frac{(1-|z|^2)(1-|w|^2)}{(1-|z|^2)(1-|w|^2)+|z-w|^2}\right)^\lambda$$

and so has no zeros on $\mathbb{D}\times\mathbb{D}$ and equals 1 only on the diagonal so we have a regular quantization.

The class of symbols which we shall use are the symbols of differential operators on L^k. Denote by $D(p,q,k)$ the operator

$$D(p,q,k)(fs_0^k)(z) = \left\{z^p\left(\frac{\partial}{\partial z}\right)^q f(z)\right\} s_0^k(z).$$

then applied to a coherent state we have

$$D(p,q,k)e^{(k)}_{s_0(w)}(z) = \epsilon^{(k)} P_q(k\lambda) z^p \left(\frac{\overline{w}}{1-\overline{w}z}\right)^q (1-\overline{w}z)^{-k\lambda} s_0^k(z)$$

where P_q is the polynomial of degree q given by $P_q(x) = x(x+1)\dots(x+q-1)$. The coefficient of s_0^k is bounded and so both conditions of Remark 1 of §2 hold. The symbol of $D(p,q,k)$ is given by

$$\widehat{D(p,q,k)}(z) = P_q(k\lambda) z^p \left(\frac{\overline{z}}{1-|z|^2}\right)^q$$

and its analytic continuation by

$$\widehat{D(p,q,k)}(z,w) = P_q(k\lambda) z^p \left(\frac{\overline{w}}{1-\overline{w}z}\right)^q.$$

Since

$$\begin{aligned} |1-\overline{w}z|^2 &= 1 - \overline{w}z - \overline{z}w + |z|^2|w|^2 \\ &\geq (1-|z||w|)^2 \\ &\geq (1-|z|)^2 \end{aligned}$$

as a function of $w \in \mathbb{D}$ for $z \in \mathbb{D}$ fixed, $\widehat{D(p,q,k)}$ is in $\mathcal{B}_0$.

It follows that $z^p \left(\frac{\overline{z}}{1-|z|^2}\right)^q$ is the symbol of the densely defined operator $\frac{D(p,q,k)}{P_q(k\lambda)}$ on $\mathcal{H}_k$. We can clearly compose such operators since the result of applying the first to a coherent state is a coherent state for a different parameter and these are in the domain of the second as one easily checks and so the $*$-operation is well-defined and is the symbol of the composition. An easy calculation yields

$$\begin{aligned} &\left\{ z^p \left(\frac{\overline{z}}{1-|z|^2}\right)^q \right\} *_k \left\{ z^r \left(\frac{\overline{z}}{1-|z|^2}\right)^s \right\} \\ &\qquad = \{P_q(k\lambda)P_s(k\lambda)\}^{-1} \widehat{D(p,q,k) \circ D}(r,s,k) \\ &\qquad = \sum_{m=0}^{\min(q,r)} \binom{q}{m} \frac{r!}{(r-m)!} \frac{P_{s+q-m}(k\lambda)}{P_q(k\lambda)P_s(k\lambda)} z^{p+r-m} \left(\frac{\overline{z}}{1-|z|^2}\right)^{s+q-m}. \end{aligned}$$

From this we deduce that $\left\{ z^p \left(\frac{\overline{z}}{1-|z|^2}\right)^q \right\} *_k \left\{ z^r \left(\frac{\overline{z}}{1-|z|^2}\right)^s \right\}$ is a rational function of k, and hence that the asymptotic expansion given by proposition 2 of §3 is convergent on symbols of polynomial differential operators. Secondly, if we consider symbols of this type with $p < N$, $q < N$ then all the functions on the right hand side are of this form with $p < 2N$, $q < 2N$ and so lie in a fixed finite dimensional space.

Similarly, if we consider the $*_k$-product of three symbols in order to check associativity, the result depends rationally on k with coefficients which are symbols with $p < 3N$, $q < 3N$. Since we know that $*_k$ is associative, we can use the same argument as in [5] to conclude associativity of the asymptotic expansion, provided we show that symbols of this kind are enough to determine the differential operators involved.

To see this we observe that

$$\frac{1}{1-|z|^2} = 1 + z\left(\frac{\overline{z}}{1-|z|^2}\right)$$

is in the space of symbols, and hence so are all the functions of the form $\frac{z^p\overline{z}^q}{(1-|z|^2)^r}$ for p and q less than or equal to r. Suppose that D_1 and D_2 are two differential operators of order r with $D_1 f = D_2 f$ for all symbols f of this form. Then we can conjugate D_1 and D_2 by $(1-|z|^2)^r$ to yield differential operators $\widetilde{D}_1$ and $\widetilde{D}_2$ of order r which satisfy $\widetilde{D}_1 z^p\overline{z}^q = \widetilde{D}_2 z^p\overline{z}^q$ for all p and q less than or equal to r. Hence $\widetilde{D}_1 = \widetilde{D}_2$ and so $D_1 = D_2$.

Thus associativity holds for the formal $*$-product defined on smooth functions given by the asymptotic expansion of $*_k$, and the coefficients are differential operators determined by the geometry, and independent of any choices of symbols, etc, made to determine them. In particular it is invariant under $SU(1,1)$. We have thus proven the following theorem.

Theorem 1. *There is a formal $*$-product on smooth functions on the disk $\mathbb{D}$ which is determined by asymptotic expansion of the Berezin $*_k$. The coefficients are bidifferential operators invariant under $SU(1,1)$. The formal series converges for pairs of functions which are symbols of polynomial differential operators.*

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