An equivalent system for studying periodic points of the beta-transformation for a Pisot or a Salem number

by

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Declarations

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work, except where otherwise indicated, cited or commonly known.

Chapter 1 is introductory. Chapters 2 and 3 explain well known results in the area which we will need afterwards. Chapters 4, 5 and 6, and Appendices A and B contain the most important results of my work.

No part of this thesis has been submitted for a degree at another university.

Abstract

We propose an equivalent system $(\widetilde{C}, L)$ for studying the set of eventually periodic points, $Per(T_\beta)$, for the beta-transformation of the unit interval, when β is a *Pisot* or a *Salem* number. This system is defined by a map $\widetilde{C}$, which is closely related to the companion matrix C of the minimal polynomial of β (of degree $d \geq 2$), and by a set of points $L \subset \mathbb{Q}^d$.

The systems $(\widetilde{C}, L)$ and $\big(T_\beta, [0,1) \cap \mathbb{Q}(\beta)\big)$ are semi-conjugate and furthermore the semi-conjugacy is one-to-one. Given that $Per(T_\beta) \subseteq [0,1) \cap \mathbb{Q}(\beta)$, we say that $(\widetilde{C}, L)$ is an equivalent system as far as the study of periodic points is concerned.

We define symbolic dynamics for $(\widetilde{C}, L)$, which is related to the beta-expansions of numbers in the unit interval. We show that $\widetilde{C}$ can be factored to the toral automorphism defined by C and we also study the geometry of $(\widetilde{C}, L)$.

The main motivation for this work is Schmidt's paper [Sch80], and in particular the theorem that $Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$ when β is a *Pisot* number, and the conjecture that the same should be true when β is a *Salem* number. We compare the different dynamical behaviours of $(\widetilde{C}, L)$ when β is *Pisot* and when β is *Salem* , and state some of the implications of Schmidt's theorem and conjecture.

Finally, we use computer simulations and plots for a particular *Salem* case of degree 4, with a view to gaining further insight about the general *Salem* case.

List of notation

β *Pisot* or *Salem* number (except for the definition of T_β, in general)

T_β β-transformation of the unit interval

$[x]$ the greatest integer number less than or equal to x

$\{x\}$ the fractional part of x, that is, $x - [x]$

X_β^+ the set of one-sided sequences which are the β-expansions of some $x \in [0, 1)$

$\widetilde{X}_\beta^+$ the set of one-sided sequences which are the β-expansions of some $x \in [0, 1) \cap \mathbb{Q}(\beta)$

V_β^+ the closure of X_β^+

$\mathbb{N}$ the set of positive integer numbers

$\mathbb{N}_0$ the set of non-negative integer numbers

$\mathbb{Z}$ the set of integer numbers

$\mathbf{x}$ a d-dimensional vector (usually, in $\mathbb{Q}^d$)

$\bar{\beta}$ the vector $(1, \beta, \ldots, \beta^{d-1}) \in \mathbb{R}^d$

$\mathbb{T}^d$ the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$

$\mathbb{Q}^d/\mathbb{Z}^d$ the subgroup of the $\mathbb{T}^d$ consisting of points for whose lifts to the universal cover $\mathbb{R}^d$ have rational coordinates

$p(x)$ a monic polynomial which is the minimal polynomial of the *Pisot* or *Salem* number β

$c_{d-1}, \ldots, c_0$ the integer coefficients which define $p(x)$, and also C

C the companion matrix of the monic polynomial $p(x)$ (which is the minimal polynomial of the *Pisot* or *Salem* number β)

Id the d-dimensional identity matrix

$\widetilde{C}$ a map defined in L, which consists of the linear map defined by C, composed with a translation by an integer multiple of $\overline{1}$, such that the range is L; it is semi-conjugate to the restriction of T_β to $[0,1) \cap \mathbb{Q}(\beta)$

$\overline{C}$ the toral automorphism of $\mathbb{T}^d$ (or its restriction to $\mathbb{Q}^d/\mathbb{Z}^d$) defined by the matrix C (in the *Pisot* case, we require β to be a *Pisot* unit, and therefore $\det C = \pm 1$)

L a subset of $\mathbb{Q}^d$ which is isomorphic to $[0,1) \cap \mathbb{Q}(\beta)$; it consists of all points in $\mathbb{Q}^d$ which lie between two parallel $d-1$-dimensional planes in $\mathbb{R}^d$

f a bijection (field isomorphism) between $\mathbb{Q}^d$ and $\mathbb{Q}(\beta)$, and in particular, between L and $[0,1) \cap \mathbb{Q}(\beta)$

$\otimes$ the product operation in the field $\mathbb{Q}^d$, induced by f^{-1}

F a bijection (field isomorphism) between $\mathbb{Q}(\beta)$ and $\mathbb{Q}(C)$

π_β the map from $\Sigma^+_{[\beta]}$ (or the restriction to any subset, such as $\widetilde{X}^+_\beta$) which maps a one-sided sequence symbols $\overline{\varepsilon}$ to $x = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}$

Π_β the bijection between $\widetilde{X}^+_\beta$ and L which associates each sequence $\overline{\varepsilon} \in \widetilde{X}^+_\beta$, and which is the β-expansion of some $x \in [0,1) \cap \mathbb{Q}(\beta)$, to the unique $\mathbf{x} \in L$ which represents such x (that is, $f(\mathbf{x}) = x$)

$d_\beta(x)$ the map from $[0,1)$ to X_β^+ which maps each x to the one-sided sequence which is its β-expansion (as obtained by the *greedy* algorithm); it is exceptionally defined for $x = 1$ as well

$\overline{\omega}$ the lexicographic supremum of the β-expansions of every $x \in [0,1)$; this is also called the modified β-expansion of 1

$Per(T_\beta)$ the subset of $[0,1)$ which is the set of all eventually periodic points (points which have finite orbits) for the map T_β

$Per(\widetilde{C})$ the subset of L which is the set of all eventually periodic points (points which have finite orbits) for the map $\widetilde{C}$ ($f : Per(\widetilde{C}) \to Per(T_\beta)$ is a bijection)

E the subset of $Per(\widetilde{C})$ which consists of all eventually periodic points for $\widetilde{C}$, which are not strictly periodic (their pre-period is greater than 0)

P the subset of $Per(\widetilde{C})$ which consists of all strictly periodic points for $\widetilde{C}$ (their pre-period is 0)

N the subset of $Per(\widetilde{C})$ which consists of all non-periodic points for $\widetilde{C}$ (those having infinite orbits)

^{t} we use column vector notation for right multiplication with a matrix (or row vector), and row vector notation for left multiplication with a matrix (or column vector)

$\overline{1}$ the integer vector $(1,0,\ldots,0)^t \in \mathbb{Z}^d$, which is also $f^{-1}(1)$ and therefore the unity element of the field $(\mathbb{Q}, +, \otimes)$

Chapter 1

Introduction

The representation of real numbers in decimal base is a rather familiar notion to most people. In order to determine the decimal digits which represent any non-negative real number in base 10, we choose the highest possible integers between 0 and 9, which are associated to each power of 10, starting from the highest power to the lowest. For example if $x = 100\sqrt{2} \approx 141.4213\ldots$, then:

$$x = 1\times 10^2 + 4\times 10^1 + 1\times 10^0 + 4\times 10^{-1} + 2\times 10^{-2} + 1\times 10^{-3} + 3\times 10^{-4} + \ldots \quad (1.1)$$

This way of generating the integer coefficients is called the *greedy* algorithm, since in each iterative step it chooses the highest integer possible. The generalization to any other integer base $\beta > 1$ is straightforward. If we want to define this algorithm explicitly, it helps to normalize any non-negative real number to the unit interval $[0, 1)$, dividing it by a suitable power of the base.

For any integer $\beta > 1$ we define the *greedy* expansion in base β of $x \in [0, 1)$ as

$$x := \frac{a_1(x)}{\beta} + \frac{a_2(x)}{\beta^2} + \frac{a_3(x)}{\beta^3} + \ldots = \sum_{k=1}^{\infty} \frac{a_k(x)}{\beta^k}, \quad (1.2)$$

where $a_k(x) \in \{0, 1, \ldots, \beta - 1\}$ are determined as $a_k(x) := [\beta\, T_\beta^{k-1}(x)]$, with:

$$\begin{array}{rccc} T_\beta : & [0,1) & \longrightarrow & [0,1) \\ & x & \longmapsto & \beta x - [\beta x] \end{array} \quad (1.3)$$

and $[x] := \max\{n \in \mathbb{N}_0 \mid n \leq x\}$.

The symbolic dynamics associated to the map T_β is defined in terms of the one-sided shift on β-symbols. This one-sided space of sequences is trivially defined, and it is known that the sequences which have periodic tails (that is, which are eventually periodic under the shift map) correspond to the rational numbers in the unit interval.

In [Ren57], Rényi suggested generalizing the expansion of a number to any non-integer base $\beta > 1$. In this case, the corresponding expanding map of the unit interval is called the beta-transformation. The symbolic dynamics associated to this map are related to a subshift of the full shift on $[\beta]$ symbols, but it is not immediate to give a description of that subshift. In [Par60], Parry characterized this subshift, describing the sequences which belong to it.

In [Ber77] and [Sch80], an important result concerning the eventually periodic points of the β-transformation was proved. It brought about a connection with the theory of algebraic numbers, with crucial results related to two special classes of algebraic numbers: *Pisot* and *Salem* numbers. The *Pisot* case is completely understood, but the *Salem* case presents additional difficulties, and therefore there is still an open conjecture concerning the *Salem* case. As far as we know, the only published works directly addressing the conjecture stated in Schmidt's paper, were Boyd's articles [Boy89], [Boy96] and [Boy97]. Despite the interest and the partial results obtained by Boyd, Schmidt's conjecture still remains unanswered.

We attempted to deal with this difficult open problem, going back to the original paper by Schmidt. We relied on some of the original ideas used by Schmidt, and when $\beta > 1$ is *Pisot* or *Salem* , we were able to explicitly define a dynamical system which is semi-conjugate to a restriction of the β-transformation to a subset of the unit interval which is suitable for studying eventually periodic points. This system is defined in a subset of $\mathbb{Q}^d$, and its definition is related to the linear map defined by the companion matrix of β.

This system can be factored to the toral automorphism defined by the companion matrix. The way in which the periodic orbits are mapped into the torus follow a different pattern, according to whether β is *Pisot* or *Salem* . We also show how the main result by Schmidt is translated into this new setting. Finally, our system also allows us to explicitly define the points in $\mathbb{Q}^d$ which represent an eventually periodic point, defined by its sequence.

Chapter 2

The beta-transformation and symbolic dynamics

2.1 Introduction

In this chapter, we summarize some classical results concerning a particular way of representing real numbers in an arbitrary non-integer base $\beta > 1$. Following the analogy with the representation of real numbers in an integer base, we will apply the *greedy* algorithm in order to obtain sequences of symbols representing real numbers $x \in [0, 1)$, and we will call these sequences β-expansions.

The dynamical way of defining the *greedy* algorithm is connected to a map of the unit interval which is called the β-transformation. The dynamical properties of this map are related to the allowed sequences generated by the *greedy* algorithm. The study of β-transformations was first carried out by Rényi and Parry in [Ren57] and [Par60]. In [Par60], Parry described the sequences generated by the *greedy* algorithm applied to numbers $x \in [0, 1)$. This description is related to the expansion in base β of the number 1 and uses the notion of *lexicographic inequality* between sequences. We will introduce some of the classical concepts and results from [Par60], while using the most recent terminology which in the mean time has become standard. We introduce a sequence of steps which prove these results.

The study of periodic orbits for the β-transformation (or equivalently, the study of periodic sequences generated by the *greedy* algorithm) is one of the non-trivial problems in which we will be interested in the following chapters.

2.2 The beta-expansion

Let $\beta > 1$ be a non-integer real number. Following the motivation of the representation of real numbers in integer base, we would like to express any $x \in [0,1)$ as a sum of terms consisting of negative powers of β weighted by non-negative integer coefficients:

$$x = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \frac{\varepsilon_3}{\beta^3} + \ldots = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k}. \tag{2.1}$$

We say that (2.1) is an expansion of x in base β and that the sequence of integers $\overline{\varepsilon} := (\varepsilon_1, \varepsilon_2, \ldots)$ represents x in base β. There exist many ways of representing x with sequences such that (2.1) holds, but we shall be concerned with a particular one which is called the β-expansion.

The example of the integer base expansion of real numbers suggests that the integers ε_k should satisfy $0 \leq \varepsilon_k < \beta$. If we define the notation for the integer part of a real number x as:

$$[x] := \max\{n \in \mathbb{Z} \mid n \leq x\}, \tag{2.2}$$

then $\varepsilon_k \in \{0, 1, \ldots, [\beta]\}$, and we write $\overline{\varepsilon} \in \Sigma^+_{[\beta]} := \{0, 1, \ldots, [\beta]\}^{\mathbb{N}}$.

We also define the fractional part of x as: $\{x\} := x - [x]$.

We shall now describe an algorithm that for each $x \in [0,1)$ uniquely determines coefficients ε_k which satisfy (2.1).

Given any $x \in [0,1)$, multiply (2.1) by β:

$$\beta x = \varepsilon_1 + \Big(\frac{\varepsilon_2}{\beta} + \frac{\varepsilon_3}{\beta^2} + \ldots\Big), \tag{2.3}$$

and compare it with:

$$\beta x = [\beta x] + \{\beta x\}. \tag{2.4}$$

Comparing (2.3) and (2.4) suggests that:

$$\varepsilon_1 := [\beta x], \qquad \{\beta x\} = \frac{\varepsilon_2}{\beta} + \frac{\varepsilon_3}{\beta^2} + \dots . \tag{2.5}$$

This defines $\varepsilon_1 \in \{0, 1, \dots, [\beta]\}$, and the coefficients $(\varepsilon_2, \varepsilon_3, \dots)$ still need to be determined. But we can apply the same method to the number $\{\beta x\} \in [0, 1)$ in order to determine ε_2, and this will define the coefficients $\{\varepsilon_k\}_{k\in\mathbb{N}}$ by induction. In order to simplify the description of this algorithm, we introduce the following map:

Definition 2.2.1. *For any non-integer $\beta > 1$, we define the β-transformation of the unit interval:*

$$\begin{aligned} T_\beta : \quad [0,1) \quad &\longrightarrow \qquad [0,1) \\ x \quad &\longmapsto \quad \beta x - [\beta x]. \end{aligned} \tag{2.6}$$

The β-transformation relates to the algorithm that we're describing in the following way:

$$T_\beta^k(x) = \frac{\varepsilon_{k+1}}{\beta} + \frac{\varepsilon_{k+2}}{\beta^2} + \frac{\varepsilon_{k+3}}{\beta^3} + \dots . \tag{2.7}$$

With this new notation, we can define:

$$\varepsilon_k := [\beta\, T_\beta^{k-1}(x)] \quad \forall k \in \mathbb{N}. \tag{2.8}$$

This algorithm is known as the *greedy* algorithm, because the choice of each term ε_k corresponds to be the greatest possible integer in each step of the iterative process. Therefore, an equivalent way of defining the algorithm is the following: if the integers $\{\varepsilon_i\}_{0<i<k}$ have already been determined, then define ε_k as the greatest possible integer such that the sum of the first k terms in the expansion in base β doesn't exceed x:

$$\varepsilon_k := \max\Big\{\varepsilon \in \{0, 1, \dots, [\beta]\} \mid \sum_{0<i<k} \frac{\varepsilon_i}{\beta^i} + \frac{\varepsilon}{\beta^k} \le x\Big\}. \tag{2.9}$$

For each $x \in [0, 1)$, the expansion (2.1) determined by the *greedy* algorithm is called the β-expansion of x. Very often in the literature, it became a common practice to call the sequence $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ generated by the *greedy* algorithm the β-expansion of x as well. This is a slight abuse of language (with respect to the original definition by Rényi and Parry, [Par60]), but given that it is usually clear from the context whether we are referring to the sequence of symbols determined by the *greedy* algorithm, or to the actual sum (2.1) with those symbols replaced, we will follow the same practice.

2.3 Symbolic dynamics

Define $\pi_\beta : \Sigma^+_{[\beta]} \to \mathbb{R}$ as:

$$\pi_\beta(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) := \sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k}. \tag{2.10}$$

We also define the following notation: $X^+_\beta \subset \Sigma^+_{[\beta]}$ is the set of all possible sequences which are β-expansions of numbers $x \in [0, 1)$.

Definition 2.3.1. *We call d_β the function which maps each $x \in [0,1)$ to the sequence $\overline{\varepsilon}$ which is the β-expansion of x:*

$$\begin{array}{rccc} d_\beta : & [0,1) & \longrightarrow & X^+_\beta \\ & x & \longmapsto & \overline{\varepsilon} := d_\beta(x). \end{array} \tag{2.11}$$

This means that $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ is determined by the *greedy* algorithm, as defined in (2.8), or equivalently in (2.9).

Note that $\pi_\beta \circ d_\beta(x) = x$, that is to say, the restriction of π_β to X^+_β is the inverse of d_β.

Definition 2.3.2. *The one-sided full shift on $\Sigma^+_{[\beta]}$ is the pair $(\Sigma^+_{[\beta]}, \sigma)$, where σ is:*

$$\begin{array}{rccc} \sigma : & \Sigma^+_{[\beta]} & \longrightarrow & \Sigma^+_{[\beta]} \\ & (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) & \longmapsto & (\varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots). \end{array} \tag{2.12}$$

The restriction of σ to X^+_β is well defined, because X^+_β is σ-invariant: for any $\overline{\varepsilon} \in X^+_\beta$, there exists $x \in [0,1)$ such that $\overline{\varepsilon} = d_\beta(x)$ and $\sigma(\overline{\varepsilon}) = d_\beta(y)$, with $y := T_\beta(x)$. Therefore $\sigma(\overline{\varepsilon}) \in X^+_\beta$.

Proposition 2.3.3. *The dynamical system* (σ, X_β^+) *is semi-conjugate to* $(T_\beta, [0,1))$*:*

$$\begin{array}{ccc} X_\beta^+ & \xrightarrow{\ \sigma\ } & X_\beta^+ \\ \Big\downarrow \pi_\beta & & \Big\downarrow \pi_\beta \\ [0,1) & \xrightarrow[T_\beta]{} & [0,1) \end{array} \tag{2.13}$$

Proof. $\pi_\beta : X_\beta^+ \to [0,1)$ is a continuous surjective map.

Let us check that the diagram commutes.

$$\begin{aligned} \pi_\beta \circ \sigma(\varepsilon_1, \varepsilon_2, \dots) &= \pi_\beta(\varepsilon_2, \varepsilon_3, \dots) \\ &= \frac{\varepsilon_2}{\beta} + \frac{\varepsilon_3}{\beta^2} + \dots. \end{aligned} \tag{2.14}$$

On the other hand,

$$\begin{aligned} T_\beta \circ \pi_\beta(\varepsilon_1, \varepsilon_2, \dots) &= T_\beta\Big(\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \dots\Big) \\ &= \frac{\varepsilon_2}{\beta} + \frac{\varepsilon_3}{\beta^2} + \dots, \end{aligned} \tag{2.15}$$

therefore the diagram is commutative. □

This means that the orbits of points $x \in [0,1)$ under T_β are in one-to-one correspondence with the orbits of sequences $\overline{\varepsilon} \in X_\beta^+$ under the shift map σ. An eventually periodic point $x \in [0,1)$ corresponds to a sequence $\overline{\varepsilon} \in X_\beta^+$ which has a periodic tail of symbols.

2.4 Lexicographic order

The order of real numbers in $[0,1)$ is carried by d_β to an order between sequences in X_β^+, which is called the *lexicographic order*.

Definition 2.4.1. *We say that a sequence of integers* $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots)$ *is* lexicographically less *than a different sequence* $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots)$*, and we write*

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) <_{lex} (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots), \tag{2.16}$$

if and only if:

$$\varepsilon_m < \varepsilon'_m, \quad \textit{where} \quad m := \min\{k \in \mathbb{N} \mid \varepsilon_k \neq \varepsilon'_k\}.$$

Proposition 2.4.2. *Consider β-expansions of $x, y \in [0,1)$:*

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) := d_\beta(x) \quad \textit{and} \quad (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots) := d_\beta(y). \tag{2.17}$$

Then:

$$x < y \quad \Leftrightarrow \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) <_{lex} (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots). \tag{2.18}$$

Proof. Choose $x, y \in [0,1)$, such that $x < y$ and let $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots)$ and $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots)$ be defined as in (2.17). Since $x \neq y$, then $d_\beta(x) \neq d_\beta(y)$ and we can define a positive integer $m := \min\{k \in \mathbb{N} \mid \varepsilon_k \neq \varepsilon'_k\}$.

$$x = \sum_{0<k<m} \frac{\varepsilon_k}{\beta^k} + \sum_{k=m}^{\infty} \frac{\varepsilon_k}{\beta^k}. \tag{2.19}$$

$$y = \sum_{0<k<m} \frac{\varepsilon'_k}{\beta^k} + \sum_{k=m}^{\infty} \frac{\varepsilon'_k}{\beta^k}. \tag{2.20}$$

Since $\forall_{0<k<m} \;\; \varepsilon_k = \varepsilon'_k$, then $\sum_{0<k<m} \frac{\varepsilon_k - \varepsilon'_k}{\beta^k} = 0$. Subtracting (2.20) from (2.19) we obtain:

$$x - y = \sum_{k=m}^{\infty} \frac{\varepsilon_k}{\beta^k} - \sum_{k=m}^{\infty} \frac{\varepsilon'_k}{\beta^k} \tag{2.21}$$

But $x - y < 0$, therefore:

$$\sum_{k=m}^{\infty} \frac{\varepsilon_k}{\beta^k} < \sum_{k=m}^{\infty} \frac{\varepsilon'_k}{\beta^k}, \quad \text{and so} \quad \sum_{i=1}^{\infty} \frac{\varepsilon_{m-1+i}}{\beta^i} < \sum_{i=1}^{\infty} \frac{\varepsilon'_{m-1+i}}{\beta^i}, \tag{2.22}$$

where $(\varepsilon_m, \varepsilon_{m+1}, \dots) = d_\beta(T_\beta^{m-1}(x))$ and $(\varepsilon'_m, \varepsilon'_{m+1}, \dots) = d_\beta(T_\beta^{m-1}(y))$. According to (2.22), $T_\beta^{m-1}(x) < T_\beta^{m-1}(y)$, and the *greedy* algorithm implies that

$$\varepsilon_m := [\beta T_\beta^{m-1}(x)] \;\leq\; \varepsilon'_m := [\beta T_\beta^{m-1}(y)]. \tag{2.23}$$

But $\varepsilon_m \neq \varepsilon'_m$ and so $\varepsilon_m < \varepsilon'_m$, which proves that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) <_{lex} (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots)$.

Finally, assume that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) := d_\beta(x) <_{lex} (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots) := d_\beta(y)$. Given that $(\varepsilon_1, \varepsilon_2, \dots) \neq (\varepsilon'_1, \varepsilon'_2, \dots)$ and that these sequences are obtained using the *greedy* algorithm for some $x, y \in [0,1)$, then $x \neq y$. Therefore either $x < y$ or $y < x$. But if $y < x$, then $d_\beta(y) <_{lex} d_\beta(x)$, which is a contradiction. This proves that $x < y$. □

2.5 Lexicographic supremum

The set $\Sigma^+_{[\beta]}$ is bounded in the *lexicographic* sense:

$$\forall\, \overline{\varepsilon} := (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) \in \Sigma^+_{[\beta]}, \quad \overline{\varepsilon} \leq_{lex} ([\beta], [\beta], \ldots). \tag{2.24}$$

Given that $X^+_\beta \subset \Sigma^+_{[\beta]}$, it makes sense to investigate the lexicographic supremum of X^+_β. We shall see that the sequence which is the lexicographic supremum of X^+_β always defines an expansion of 1 in base β (although not necessarily the one that would be obtained if we applied the *greedy* algorithm to 1). Furthermore, this sequence plays an important role in the description of the possible sequences in X^+_β through a countable number of *lexicographic* inequalities.

Definition 2.5.1. *The lexicographic supremum of the set X^+_β is defined as:*

$$\sup_{lex} X^+_\beta := \overline{\omega} = (\omega_1, \omega_2, \omega_3, \ldots), \tag{2.25}$$

where $\{\omega_k\}_{k\in\mathbb{N}}$ is defined inductively as:

$$\omega_k := \max\Big\{\omega \in \{0, 1, \ldots, [\beta]\} \mid \sum_{0<i<k} \frac{\omega_i}{\beta^i} + \frac{\omega}{\beta^k} < 1.\Big\} \tag{2.26}$$

Furthermore,

$$\pi_\beta(\overline{\omega}) = \sum_{k=1}^{\infty} \frac{\omega_k}{\beta^k} = 1. \tag{2.27}$$

Proof. We will start by proving (2.27). Consider the non-decreasing sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset [0,1)$:

$$x_k := \pi_\beta(\omega_1, \ldots, \omega_k, 0, 0, \ldots) = \sum_{i=1}^{k} \frac{\omega_i}{\beta^i}. \tag{2.28}$$

Note that (2.28) is the β-expansion of x_k, or equivalently:

$$d_\beta(x_k) = (\omega_1, \dots, \omega_k, 0, 0, \dots), \tag{2.29}$$

because each integer coefficient ω_i up to order k is the greatest possible according to the definition (2.26), and this corresponds to the *greedy* algorithm. This means that in step k, we choose ω_k to be as big as possible. Therefore if we added β^{-k} in the expansion, we would obtain:

$$\sum_{i=1}^{k} \frac{\omega_i}{\beta^i} + \frac{1}{\beta^k} \geq 1, \quad \text{and so} \quad \Big|1 - \sum_{i=1}^{k} \frac{\omega_i}{\beta^i}\Big| \leq \frac{1}{\beta^k}. \tag{2.30}$$

This implies:

$$\lim_{k\to\infty} |1 - x_k| = 0, \quad \text{and so} \quad \sum_{k=1}^{\infty} \frac{\omega_k}{\beta^k} = 1, \tag{2.31}$$

which proves (2.27).

Let us now prove that $\forall_{x\in[0,1)} \quad d_\beta(x) <_{lex} \overline{\omega}$. If $x \in [0,1)$, then $x \neq 1$ therefore $d_\beta(x) := (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) \neq \overline{\omega}$. Let $m := \min\{k \in \mathbb{N} \mid \varepsilon_k \neq \omega_k\}$. By definition, ω_k is the maximum possible integer which implies that $\varepsilon_m < \omega_m$. This proves that $d_\beta(x) <_{lex} \overline{\omega}$, therefore $\overline{\omega}$ is a *lexicographic* upper bound for X_β^+.

It remains to prove that $\overline{\omega}$ is indeed the least *lexicographic* upper bound. Suppose that $\overline{\omega}' \in \Sigma_{[\beta]}^+$, $\overline{\omega}' <_{lex} \overline{\omega}$ and $\overline{\omega}'$ is a *lexicographic* upper bound for X_β^+. Let $m := \min\{k \in \mathbb{N} \mid \omega_k' \neq \omega_k\}$. Since $\overline{\omega}' <_{lex} \overline{\omega}$, then $\omega_m' < \omega_m$. But

$$(\omega_1, \dots, \omega_m, 0, 0, \dots) \in X_\beta^+, \tag{2.32}$$

and therefore $\overline{\omega}' <_{lex} (\omega_1, \dots, \omega_m, 0, 0, \dots)$, which means that $\overline{\omega}'$ isn't a *lexicographic* upper bound for X_β^+, which is a contradiction. This proves that $\overline{\omega} \in \Sigma_{[\beta]}^+$ is the least *lexicographic* upper bound for X_β^+. □

We will now extend the definition of β-expansion to the number 1, according to the *greedy* algorithm. If we extend the definition of T_β to the point 1 as $T_\beta(1) := \{\beta\}$, then we obtain the β expansion of 1 in a similar way as for any other $x \in [0,1)$:

$$d_\beta(1) = (\delta_1, \delta_2, \delta_3, \dots), \quad \text{where} \quad \delta_k := [\beta T_\beta^{k-1}(1)], \tag{2.33}$$

or equivalently:

$$\delta_k := \max\Big\{\delta \in \{0,1,\ldots,[\beta]\} \mid \sum_{0<i<k} \frac{\delta_i}{\beta^i} + \frac{\delta}{\beta^k} \le 1\Big\}. \tag{2.34}$$

We note that the definitions of $\overline{\omega}$ and $d_\beta(1)$ in (2.26) and (2.34) only differ in the inequality sign: in (2.34), it is possible that a finite number of terms might add up to 1, whereas in (2.26) the inequality is strict, and therefore that isn't allowed. We should distinguish two cases: if $d_\beta(1)$ is an *infinite expansion*, that is:

$$d_\beta(1) = (\delta_1, \delta_2, \delta_3, \ldots) \quad \text{and} \quad \forall_{N\in\mathbb{N}} \quad \exists_{k>N} \quad : \; \delta_k \neq 0,$$

then no finite number of terms in the β-expansion ever adds up to 1, and therefore $\overline{\omega}$=$d_\beta(1)$ because (2.26) and (2.34) define the same sequence.

However, if $d_\beta(1)$ is a *finite expansion*, that is:

$$d_\beta(1) = (\delta_1, \ldots, \delta_m, 0, 0, \ldots), \quad \text{with} \quad \delta_m \neq 0,$$

then

$$\sum_{k=1}^{m} \frac{\delta_k}{\beta^k} = 1. \tag{2.35}$$

According to (2.26) and (2.34), $\forall_{0\le k<m}\ \omega_k = \delta_k$, but $\omega_m < \delta_m$, because otherwise $\sum_{k=1}^{m} \frac{\omega_k}{\beta^k} = 1$, which is not allowed. The greatest integer ω_m such that $\sum_{k=1}^{m} \frac{\omega_k}{\beta^k} < 1$ is $\omega_m = \delta_m - 1$. This means that:

$$\sum_{k=1}^{\infty} \frac{\omega_k}{\beta^k} = \left(\sum_{0<k<m} \frac{\delta_k}{\beta^k} + \frac{(\delta_m - 1)}{\beta^m} \right) + \sum_{k=m+1}^{\infty} \frac{\omega_k}{\beta^k} = 1 \Leftrightarrow \tag{2.36}$$

$$\left(1 - \frac{1}{\beta^m}\right) + \sum_{k=m+1}^{\infty} \frac{\omega_k}{\beta^k} = 1 \Leftrightarrow \sum_{i=1}^{\infty} \frac{\omega_{m+i}}{\beta^i} = 1. \tag{2.37}$$

But $(\omega_{m+1}, \omega_{m+2}, \ldots)$ satisfies (2.26), therefore it is the lexicographic supremum of X_β^+. This proves that $\overline{\omega}$ is related to $d_\beta(1) = (\delta_1, \ldots, \delta_m, 0, 0, \ldots)$ in the following way: $\overline{\omega}$ is the infinite repetition of a sequence of m integers, which we denote as $(\,\overline{\delta_1, \ldots, \delta_{m-1}, (\delta_m - 1)}\,)$.

We can sum up these results as follows:

Theorem 2.5.2. *Let $d_\beta(1) := (\delta_1, \delta_2, \delta_3, \ldots)$. The lexicographic supremum of X_β^+ is:*

$$\overline{\omega} := \begin{cases} (\delta_1, \delta_2, \delta_3, \ldots) & \text{if } d_\beta(1) \text{ is infinite} \\ \\ (\,\overline{\delta_1, \ldots, \delta_{m-1}, (\delta_m - 1)}\,) & \text{if } d_\beta(1) \text{ is finite.} \end{cases} \tag{2.38}$$

The lexicographic supremum $\overline{\omega}$ is some times called the modified β-expansion of 1, and often in the literature the following notation would be used to refer to it: $d_\beta^*(1)$.

2.6 The β-shift

We will now show how the *lexicographic supremum* $\overline{\omega}$ can be used to describe the sequences in X_β^+. This problem was addressed in [Par60].

Lemma 2.6.1. *If $\overline{\varepsilon} \in \Sigma_{[\beta]}^+$ is a finite sequence, then:*

$$\forall_{k\in\mathbb{N}} \;\; \sigma^{k-1}(\overline{\varepsilon}) <_{lex} \overline{\omega} \quad \Rightarrow \quad 0 \leq \pi_\beta(\overline{\varepsilon}) < 1. \tag{2.39}$$

Proof. We only need to prove that $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} < 1$, given that this is a sum of non-negative terms and therefore always non-negative.

Let $\overline{\varepsilon} \in \Sigma_{[\beta]}^+$ be a finite sequence and $\forall_{k\in\mathbb{N}} \;\; (\varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}, \ldots) <_{lex} \overline{\omega}$. Suppose that $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} \geq 1$.

Let $m_1 \in \mathbb{N}$ be $m_1 := \min\{k \in \mathbb{N} \mid \varepsilon_k < \omega_k\}$.

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} = \sum_{0<k<m_1} \frac{\varepsilon_k}{\beta^k} + \frac{\varepsilon_{m_1}}{\beta^{m_1}} + \frac{1}{\beta^{m_1}}\Big(\frac{\varepsilon_{m_1+1}}{\beta} + \frac{\varepsilon_{m_1+2}}{\beta^2} + \ldots\Big). \tag{2.40}$$

Since $\varepsilon_{m_1} < \omega_{m_1}$ and $\sum_{k=1}^{m_1} \frac{\omega_k}{\beta^k} < 1$, then:

$$\sum_{k=1}^{m_1} \frac{\varepsilon_k}{\beta^k} \leq \sum_{k=1}^{m_1} \frac{\omega_k}{\beta^k} - \frac{1}{\beta^{m_1}} < 1 - \frac{1}{\beta^{m_1}}, \tag{2.41}$$

and therefore

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} < \Big(1 - \frac{1}{\beta^{m_1}}\Big) + \frac{1}{\beta^{m_1}}\Big(\frac{\varepsilon_{m_1+1}}{\beta} + \frac{\varepsilon_{m_1+2}}{\beta^2} + \ldots\Big). \tag{2.42}$$

By hypothesis $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} \geq 1$, and together with (2.42) this implies that

$$\frac{\varepsilon_{m_1+1}}{\beta} + \frac{\varepsilon_{m_1+2}}{\beta^2} + \ldots \geq 1. \tag{2.43}$$

But the sequence $(\varepsilon_{m_1+1}, \varepsilon_{m_1+2}, \varepsilon_{m_1+3}, \ldots) <_{lex} \overline{\omega}$, and we can apply the same argument inductively, defining

$$\forall_{i\in\mathbb{N}} \quad m_{i+1} := m_i + \min\{k \in \mathbb{N} \mid \varepsilon_{m_i+k} < \omega_k\}. \tag{2.44}$$

The sequence $\{m_i\}_{i\in\mathbb{N}}$ is an increasing sequence of integers, therefore it is unbounded. Since $\overline{\varepsilon} \in \Sigma^+_{[\beta]}$ is a finite sequence, there exists some $i \in \mathbb{N}$ for which:

$$(\varepsilon_{m_i+1}, \varepsilon_{m_i+2}, \varepsilon_{m_i+3}, \ldots) = (0, 0, 0, \ldots),$$

and therefore (2.43) is impossible. □

Although in Lemma 2.6.1 we required the sequence $\overline{\varepsilon}$ to be finite, this condition isn't necessary:

Corollary 2.6.2. *Let* $\overline{\varepsilon} \in \Sigma^+_{[\beta]}$. *Then*

$$\forall_{k\in\mathbb{N}} \ \sigma^{k-1}(\overline{\varepsilon}) <_{lex} \overline{\omega} \quad \Rightarrow \quad 0 \leq \pi_\beta(\overline{\varepsilon}) < 1. \tag{2.45}$$

Proof. Let us first prove that $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} \leq 1$. In fact, if it was, there would exist some $m \in \mathbb{N}$ such that:

$$\sum_{k=1}^{m} \frac{\varepsilon_k}{\beta^k} > 1. \tag{2.46}$$

But $(\varepsilon_1, \ldots, \varepsilon_m, 0, 0, \ldots)$ satisfies the conditions of Lemma 2.6.1, therefore (2.46) is impossible. This proves that $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} \leq 1$.

Finally, let us prove that $\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} < 1$. By hypothesis $\overline{\varepsilon} <_{lex} \overline{\omega}$, so let us define $m := \min\{k \in \mathbb{N} \mid \varepsilon_k < \omega_k\}$. Then:

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} = \sum_{0<k<m} \frac{\varepsilon_k}{\beta^k} + \frac{\varepsilon_m}{\beta^m} + \frac{1}{\beta^m}\Big(\frac{\varepsilon_{m+1}}{\beta} + \frac{\varepsilon_{m+2}}{\beta^2} + \ldots\Big). \tag{2.47}$$

We've just proved that $\sum_{k=1}^{\infty} \frac{\varepsilon_{m+k}}{\beta^k} \leq 1$, and together with the fact that $\varepsilon_m + 1 \leq \omega_m$, we obtain from (2.47):

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} \leq \sum_{0<k<m} \frac{\varepsilon_k}{\beta^k} + \frac{\varepsilon_m}{\beta^m} + \frac{1}{\beta^m} \leq \sum_{k=1}^{m} \frac{\omega_k}{\beta^k} < 1. \tag{2.48}$$

□

Theorem 2.6.3. *A sequence $\overline{\varepsilon} \in \Sigma_{[\beta]}^{+}$ is the β-expansion of some $x \in [0,1)$ if and only if:*

$$\forall k \in \mathbb{N} \qquad \sigma^{k-1}(\overline{\varepsilon}) <_{lex} \overline{\omega}, \tag{2.49}$$

where $\overline{\omega}$ is the lexicographic supremum of X_β^+. Therefore:

$$X_\beta^+ = \{\overline{\varepsilon} \in \Sigma_{[\beta]}^{+} \mid \forall_{k \in \mathbb{N}} \quad \sigma^{k-1}(\overline{\varepsilon}) <_{lex} \overline{\omega}\}. \tag{2.50}$$

Proof. Let $\overline{\varepsilon} \in \Sigma_{[\beta]}^{+}$ be the β-expansion of some $x \in [0,1)$. This implies that:

$$\sigma^{k-1}(\overline{\varepsilon}) = (\varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}, \ldots) = d_\beta(T_\beta^{k-1}(x)). \tag{2.51}$$

By definition, $\overline{\omega}$ is the lexicographic supremum of X_β^+, therefore $d_\beta(T_\beta^{k-1}(x)) <_{lex} \overline{\omega}$, hence (2.49) is a necessary condition for $\overline{\varepsilon}$ to be the β-expansion of some $x \in [0,1)$.

Let us now prove that (2.49) is also sufficient for $\overline{\varepsilon} \in X_\beta^+$. Suppose that $\overline{\varepsilon} \in \Sigma_{[\beta]}^{+}$ satisfies (2.49), but $\overline{\varepsilon} \notin X_\beta^+$. By Corollary 2.6.2, we know that $0 \leq x := \pi_\beta(\overline{\varepsilon}) < 1$. If $\overline{\varepsilon} \notin X_\beta^+$, $d_\beta(x) := (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \ldots) \neq \overline{\varepsilon} := (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$. Let

$$m := \min\{k \in \mathbb{N} \mid \varepsilon_k \neq \varepsilon'_k\}.$$

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} = \sum_{k=1}^{\infty} \frac{\varepsilon'_k}{\beta^k} \quad \text{and} \quad \forall_{0<k<m}\, \varepsilon_k = \varepsilon'_k \quad \Rightarrow \quad \sum_{k=m}^{\infty} \frac{\varepsilon_k}{\beta^k} = \sum_{k=m}^{\infty} \frac{\varepsilon'_k}{\beta^k} \tag{2.52}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{\varepsilon_{m+k-1}}{\beta^k} = \sum_{k=1}^{\infty} \frac{\varepsilon'_{m+k-1}}{\beta^k} = T_\beta^{m-1}(x), \tag{2.53}$$

where $d_\beta(T_\beta^{m-1}(x)) = (\varepsilon'_m, \varepsilon'_{m+1}, \varepsilon'_{m+2}, \ldots)$.

But $\varepsilon_m \neq \varepsilon'_m$, and so $|\varepsilon_m - \varepsilon'_m| \geq 1$, which implies:

$$\Big| \sum_{k=1}^{\infty} \frac{\varepsilon_{m+k}}{\beta^k} - \sum_{k=1}^{\infty} \frac{\varepsilon'_{m+k}}{\beta^k} \Big| \geq 1. \tag{2.54}$$

But (2.54) is impossible, given that $0 \leq \sum_{k=1}^{\infty} \frac{\varepsilon_{m+k}}{\beta^k} < 1$ (because of Corollary 2.6.2) and $0 \leq \sum_{k=1}^{\infty} \frac{\varepsilon'_{m+k}}{\beta^k} < 1$ (because $(\varepsilon'_{m+1}, \varepsilon'_{m+2}, \ldots) = d_\beta(T_\beta^m(x))$). This proves that the β-expansion of x cannot be another sequence different than $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ and therefore (2.49) is a sufficient condition for $\overline{\varepsilon} \in X_\beta^+$. $\square$

We will consider the closure of X_β^+ and define a subshift of $(\Sigma_{[\beta]}^+, \sigma)$, which is known as the one sided β-shift:

Definition 2.6.4. *The one sided β-shift $(V_\beta^+, \sigma_\beta)$ is the subshift of $(\Sigma_{[\beta]}^+, \sigma)$ defined by:*

$$V_\beta^+ := \overline{X_\beta^+} = \{\overline{\varepsilon} \in \Sigma_{[\beta]}^+ \mid \forall_{k \in \mathbb{N}} \quad \sigma^{k-1}(\overline{\varepsilon}) \leq_{lex} \overline{\omega}\}. \tag{2.55}$$

The closure of X_β^+ is obtained by replacing in (2.50) the strict inequality by a less or equal inequality. Consider $\overline{\varepsilon} \in \Sigma_{[\beta]}^+$ which is the limit of a sequence of finite β-expansions in X_β^+:

$$\overline{\varepsilon} = \lim_{k \to \infty} (\varepsilon_1, \ldots, \varepsilon_k, 0, 0, \ldots), \tag{2.56}$$

but it isn't the β-expansion of any $x \in [0, 1)$:

$$\exists_{k \in \mathbb{N}} \, (\sigma^{k-1}(\overline{\varepsilon})) \geq_{lex} \overline{\omega}. \tag{2.57}$$

Since $(\varepsilon_1, \ldots, \varepsilon_k, 0, 0, \ldots) <_{lex} \overline{\omega}$, given that $\overline{\omega}$ is the *lexicographic supremum* of all β-expansions, then $\lim_{k \to \infty}(\varepsilon_1, \ldots, \varepsilon_k, 0, 0, \ldots) \not>_{lex} \overline{\omega}$. This means that if $\overline{\varepsilon} \notin X_\beta^+$, then $\exists_{k \in \mathbb{N}}$ such that $\sigma^{k+1}(\overline{\varepsilon}) = \overline{\omega}$. We are dealing here with two alternative expansions of x in base β:

$$\frac{\varepsilon_1}{\beta} + \ldots + \frac{\varepsilon_{k-1}}{\beta^{k-1}} + \frac{\omega_1}{\beta^k} + \frac{\omega_2}{\beta^{k+1}} + \ldots = \frac{\varepsilon_1}{\beta} + \ldots + \frac{\varepsilon_{k-1}}{\beta^{k-1}} + \frac{1}{\beta^{k-1}} = x,$$

but only one of them is the β-expansion of x: $d_\beta(x) = (\varepsilon_1, \ldots, \varepsilon_{k-1} + 1, 0, 0, \ldots)$.

Hence V_β^+ contains the β-expansion of every $x \in [0, 1)$, and additionally one other sequence which is not in X_β^+, for each $x \in [0, 1)$ which as a finite β-expansion: if $x \in (0, 1)$ and $d_\beta(x) = (\varepsilon_1, \ldots, \varepsilon_k, 0, 0, \ldots)$, with $\varepsilon_k \neq 0$, then the alternative $\overline{\varepsilon}' := (\varepsilon_1, \ldots, \varepsilon_k - 1, \omega_1, \omega_2, \omega_3, \ldots) \in V_\beta^+$ is not a β-expansion, but $\pi_\beta(\overline{\varepsilon}') = x$. The

β-expansion of 0, $d_\beta(0) = (0,0,\ldots)$ should be considered separately: we could consider $\overline{\omega} = (\omega_1, \omega_2, \ldots) \in V_\beta^+$ to be its alternative representation in base β, if we agree to identify the end points of the unit interval.

Finally, we recall that:

$$d_\beta\big([0,1)\big) \;\subset\; \overline{d_\beta\big([0,1)\big)} \;\subseteq\; \big\{0,1,\ldots,[\beta]\big\}^{\mathbb{N}}, \tag{2.58}$$

or equivalently, using the notation that we have defined:

$$X_\beta^+ \;\subset\; V_\beta^+ \;\subseteq\; \Sigma_{[\beta]}^+. \tag{2.59}$$

Chapter 3

Pisot numbers, Salem numbers and periodic points

3.1 Introduction

In the previous chapter we have given a description of the sequences which are β-expansions of numbers in the unit interval, when $\beta > 1$ and $\beta \notin \mathbb{N}$. We saw how non-trivial this problem was, as opposed to the case when $\beta > 1$ was an integer. Despite these results which had been obtained by Parry in [Par60], the periodic properties of the β-transformation (or equivalently, of the β-shift) still remained an open problem. It was well known that if $\beta > 1$ was an integer, then the set of eventually periodic points for T_β was the set of all rationals in the unit interval. Would it be possible then to describe the set of eventually periodic points when $\beta > 1$ was not an integer? Or equivalently, which numbers in the unit interval are represented by the eventually periodic β-expansions?

It was not until 1977 that Bertrand [Ber77] discovered a partial answer to this question, giving a complete characterization of the set of periodic points when $\beta > 1$ was an algebraic number of a particular type: a *Pisot* number. Almost at the same time, Schmidt [Sch80] not only obtained the same result, but also completed the picture, bringing into the answer another class of algebraic numbers: *Salem* numbers. Although

Schmidt could not give a complete characterization of the set of eventually periodic points for the β-transformation in the *Salem* case, he did however conjecture that a similar statement could be made for the *Salem* case as in the *Pisot* case. This is a long standing conjecture, which is often cited in the area, for instance in [Boy89], [Boy96], [Boy97], [Har06], [BBLT06], [Sch06], [Aki98].

We will introduce the basic concepts from algebraic number theory that will be needed for the exposition of the main results. After that, we will summarize the basic ideas behind the description of the set of eventually periodic points, and the two main theorems concerning eventually periodic points for the β-transformation, *Pisot* numbers and *Salem* numbers, as well as the unsolved conjecture.

3.2 Algebraic numbers

A root of a non-zero polynomial with rational coefficients is called an *algebraic number*. We say that a polynomial with rational coefficients is *irreducible* if it cannot be factored as the product of non-trivial polynomials with rational coefficients. Each *algebraic number* ξ determines a family of irreducible polynomials $\{p_s(x)\}_{s\in\mathbb{Q}}$ which are rational multiples of each other, and $p_s(\xi) = 0$. A polynomial whose leading coefficient is 1 is called a *monic polynomial*, and there exists only one such polynomial in the family of irreducible polynomials associated to ξ:

Definition 3.2.1. *Given an algebraic number ξ, its minimal polynomial $p(x)$ is the unique irreducible monic polynomial:*

$$p(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0, \quad a_{d-1}, \ldots, a_0 \in \mathbb{Q}, \tag{3.1}$$

such that $p(\xi) = 0$. The degree of ξ is the degree d of the minimal polynomial $p(x)$.

If all coefficients of the *minimal polynomial* of ξ are integers, then we say that ξ is an *algebraic integer*. In this case, in order to emphasize that all coefficients are integers, we write:

$$p(x) = x^d + c_{d-1}x^{d-1} + \ldots + c_1 x + c_0, \tag{3.2}$$

and it is understood that $c_0, \ldots, c_{d-1} \in \mathbb{Z}$.

3.3 Pisot numbers and Salem numbers

There are two types of algebraic integers in which we will be interested:

Definition 3.3.1. *Let $\beta > 1$ be an* algebraic integer *with minimal polynomial (3.2) of degree $d \geq 1$. $p(x)$ has d complex roots:*

$$\theta_1, \ldots, \theta_d \in \mathbb{C}, \quad and \quad p(x) = \prod_{1 \leq i \leq d} (x - \theta_i). \tag{3.3}$$

Let $\theta_1 := \beta > 1$. We say that:

1. *β is a* Pisot *number if:* $\forall_{1<i\leq d}, \quad |\theta_i| < 1$;

2. *β is a* Salem *number if:* $\forall_{1<i\leq d}, \quad |\theta_i| \leq 1$ *and* $\exists_{1<i\leq d} \ : \ |\theta_i| = 1$.

We point out that a *Pisot* number is sometimes equivalently called a *Pisot-Vijayaraghavan* number, or a *PV* number.

Any integer number $n > 1$ is an algebraic integer with minimal polynomial $p(x) = x - n$, which has no additional roots, and therefore all integers $n > 1$ are *Pisot* numbers. In order to find non-trivial *Pisot* numbers, we need to consider minimal polynomials of degree $d \geq 2$. We present some examples of minimal polynomials $p(x)$ and the corresponding *Pisot* numbers β in Table 3.1.

$p(x)$	β
$x^2 - x - 1$	$(1+\sqrt{5})/2$
$x^3 - x - 1$	$1.3247\ldots$
$x^3 - 2x^2 + x - 1$	$1.7548\ldots$
$x^2 - 4x + 2$	$2+\sqrt{2}$

Table 3.1: Examples of Pisot numbers and their minimal polynomials

Notice that $2+\sqrt{2}$ is an example of a *Pisot* number which has a minimal polynomial with a coefficient $|c_0| \neq 1$. In general, if $\beta > 1$ satisfies:

$$p(\beta) = \beta^d + c_{d-1}\beta^{d-1} + \ldots + c_1\beta + c_0 = 0, \tag{3.4}$$

then:

$$\beta^{d-1} + c_{d-1}\beta^{d-2} + \ldots + c_1 + c_0\beta^{-1} = 0, \tag{3.5}$$

and therefore we can express β^{-1} as:

$$\beta^{-1} = -c_0^{-1}\big(\beta^{d-1} + c_{d-1}\beta^{d-2} + \ldots + c_1\big). \tag{3.6}$$

If $c_0 = \pm 1$, then $\beta^{-1} \in \mathbb{Z}[\beta]$, and therefore β has an inverse in the ring $\mathbb{Z}[\beta]$, that is, β is a unit. If $|c_0| > 1$, then β is not a unit in $\mathbb{Z}[\beta]$.

Proposition 3.3.2. *Let $\beta > 1$ be a* Salem *number with minimal polynomial $p(x)$. The roots of $p(x)$ are:*

$$\theta_1, \theta_2, \theta_3, \ldots, \theta_d \in \mathbb{C}, \tag{3.7}$$

for some even $d \geq 4$. We define $\theta_1 := \beta$, $\theta_2 := \beta^{-1}$. The remaining roots $\theta_3, \ldots, \theta_d$ are complex numbers with modulus 1 and there exists an even number of them (they appear as complex conjugate pairs). Furthermore, $p(x)$ is a reciprocal polynomial, that is: $p(x) = x^d p(x^{-1})$.

Proof. Suppose that $\beta > 1$ is a *Salem* number with minimal polynomial (3.2). There exists at least some $\theta_i \in \mathbb{C}$ such that $|\theta_i| = 1$ and $p(\theta_i) = 0$. Furthermore, $\theta_i \neq \pm 1$, because $p(x)$ is an irreducible polynomial. This proves that θ_i has a non-zero imaginary part, and therefore the complex conjugate of θ_i, which we denote by $\overline{\theta_i}$, is another root of $p(x)$. This proves that the roots of $p(x)$ with modulus 1 appear as complex conjugate pairs, and there must exist at least one such pair.

If $|\theta_i| = 1$ then $\overline{\theta_i} = \theta_i^{-1}$ and so:

$$p(\theta_i) = 0, \quad \text{and} \quad p(\theta_i^{-1}) = 0. \tag{3.8}$$

Let us note that:

$$p(x^{-1}) = 0 \Leftrightarrow \underbrace{x^d p(x^{-1})}_{q(x)} = 0. \tag{3.9}$$

Since $q(x)$ is a polynomial of degree d and $q(\theta_i) = 0$, then all other conjugate roots of θ_i (that is, θ_j, for $j \neq i$ such that $p(\theta_j) = 0$) must also be roots of $q(x)$. But $q(x) = 0 \Leftrightarrow p(x^{-1}) = 0$, which means that if θ_j is a root of $p(x)$, so is θ_j^{-1}. Since β is the only root of $p(x)$ with modulus greater than 1, then β^{-1} is the only root of $p(x)$ with modulus less than 1. Furthermore,

$$c_0 = (-1)^d \prod_{i=1}^{d} \theta_i = \beta \cdot \beta^{-1} = 1, \tag{3.10}$$

since d is even and if $|\theta_i| = 1$, then $\theta_i \cdot \overline{\theta_i} = 1$.

$$q(x) = x^d p(x^{-1}) = c_0 x^d + c_1 x^{d-1} + \ldots + c_{d-1} x + 1, \tag{3.11}$$

but $c_0 = 1$, therefore $q(x)$ is a monic polynomial of degree d and has the same roots as $p(x)$. This implies that $p(x) = q(x)$, therefore $p(x) = x^d p(x^{-1})$, which is equivalent to the following condition on the coefficients of $p(x)$: $c_i = c_{d-i}$, for all $1 \leq i < d$. We say that $p(x)$ is a *reciprocal* polynomial. □

Since the minimal polynomial of any *Salem* number has $c_0 = 1$, then any *Salem* number is a unit in the ring $\mathbb{Z}[\beta]$ (compare with our discussion of the *Pisot* case, and in particular (3.6)). We present some examples of minimal polynomials $p(x)$ and the corresponding *Salem* numbers β in Table 3.2.

$p(x)$	β
$x^4 - x^3 - x^2 - x + 1$	$1.7220\ldots$
$x^4 - 2x^3 + x^2 - 2x + 1$	$1.8832\ldots$
$x^4 - 10x^3 - 10x + 1$	$10.0971\ldots$
$x^6 - x^4 - x^3 - x^2 + 1$	$1.4012\ldots$

Table 3.2: Examples of Salem numbers and their minimal polynomials

3.4 Periodic points under the β-transformation

We are interested in the periodic properties of points of the unit interval under iteration by the β-transformation.

Definition 3.4.1. *The orbit of $x \in [0,1)$ under the map T_β is defined as:*

$$\mathcal{O}(x) := \{T_\beta^k(x) : k \geq 0\} \subset [0,1). \tag{3.12}$$

If $\mathcal{O}(x)$ is an infinite set of points, then x is not a periodic point. Conversely, if $\mathcal{O}(x)$ is a finite set, then x is an eventually periodic point:

Definition 3.4.2. *The set of* eventually periodic *points for T_β is defined as:*

$$Per(T_\beta) := \big\{x \in [0,1) \mid \exists_{m\geq 0,\, p>0} : T_\beta^{p+m}(x) = T_\beta^m(x)\big\}. \tag{3.13}$$

If $m \geq 0$ and $p > 0$ are the minimum integers such that $T_\beta^{p+m}(x) = T_\beta^m(x)$, then m is the pre-period and p is the period of x.

Let $\mathbb{Q}(\beta)$ be the smallest field extension of $\mathbb{Q}$ containing β. If $\beta \in \mathbb{Q}$, then $\mathbb{Q}(\beta) = \mathbb{Q}$, but if β is an algebraic number of degree $d \geq 2$, then $\mathbb{Q}(\beta)$ is a non-trivial algebraic field extension.

Proposition 3.4.3.

$$Per(T_\beta) \subseteq [0,1) \cap \mathbb{Q}(\beta). \tag{3.14}$$

Proof. Let $x \in [0,1)$. If $x \in Per(T_\beta)$ then there exist $n > m > 0$ such that:

$$T_\beta^n(x) = T_\beta^m(x). \tag{3.15}$$

For all $i > 0$ let us write $T_\beta^i(x) = \beta T_\beta^{i-1}(x) - \varepsilon_i$, and $\varepsilon_i := [\beta T_\beta^{i-1}(x)]$.

$$T_\beta^i(x) = \beta(\ldots(\beta(\beta x - \varepsilon_1) - \varepsilon_2)\ldots) - \varepsilon_i = \beta^i x - \sum_{k=1}^{i} \varepsilon_k \beta^{i-k}. \tag{3.16}$$

From (3.15) and (3.16) we obtain:

$$\beta^n x - \sum_{k=1}^{n} \varepsilon_k \beta^{n-k} = \beta^m x - \sum_{k=1}^{m} \varepsilon_k \beta^{m-k}, \tag{3.17}$$

and so

$$x = \frac{\sum_{k=1}^{n} \varepsilon_k \beta^{n-k} - \sum_{k=1}^{m} \varepsilon_k \beta^{m-k}}{\beta^n - \beta^m}, \tag{3.18}$$

which proves that $x \in \mathbb{Q}(\beta)$, and therefore $Per(T_\beta) \subseteq [0,1) \cap \mathbb{Q}(\beta)$. □

Taking into account Proposition 3.4.3, if we want to study the set of eventually periodic points for the β-transformation, if suffices to consider the restriction of T_β to $[0,1) \cap \mathbb{Q}(\beta)$.

Definition 3.4.4. *We define the set of β-expansions of every $x \in [0,1) \cap \mathbb{Q}(\beta)$ as*

$$\widetilde{X}_\beta^+ := d_\beta\big([0,1) \cap \mathbb{Q}(\beta)\big). \tag{3.19}$$

$(\sigma, \widetilde{X}_\beta^+)$ is a subshift of (σ, X_β^+) and we have the following:

Proposition 3.4.5. $(\sigma, \widetilde{X}_\beta^+)$ *is semi-conjugate to* $\big(T_\beta, [0,1) \cap \mathbb{Q}(\beta)\big)$*:*

$$\begin{array}{ccc} \widetilde{X}_\beta^+ & \xrightarrow{\quad\sigma\quad} & \widetilde{X}_\beta^+ \\ \Big\downarrow{\scriptstyle \pi_\beta} & & \Big\downarrow{\scriptstyle \pi_\beta} \\ [0,1) \cap \mathbb{Q}(\beta) & \xrightarrow[T_\beta]{} & [0,1) \cap \mathbb{Q}(\beta) \end{array} \tag{3.20}$$

and the semi-conjugacy π_β is a bijection.

Proof. This follows from Proposition 2.3.3 together with (3.19). □

If $\beta > 1$ was an integer, Proposition 3.4.3 merely asserts that $Per(T_\beta) \subseteq [0,1) \cap \mathbb{Q}$. In fact, we actually know that in this case $Per(T_\beta) = [0,1) \cap \mathbb{Q}$. If $\beta > 1$ was an integer, then choose any rational $\frac{p}{q} \in [0,1) \cap \mathbb{Q}$. As we work out the orbit $T_\beta^k(\frac{p}{p})$, we realize that we are multiplying a rational number by an integer, and subtracting some integer in order to keep the outcome in the unit interval. Therefore, the iterations under T_β of the rational $\frac{p}{q}$ are rationals with a minimum denominator which divides q. In other words, $T_\beta^k(\frac{p}{q}) \in \{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}\}$, which is a finite set and this proves that the orbit of any rational point must be eventually periodic.

The natural question that follows is an attempt to generalize this result: do there exist non-integer $\beta > 1$ for which $Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$? Two of Schmidt's theorems in [Sch80] give a partial answer to this question:

Theorem 3.4.6. *Let $\beta > 1$ be a real number.*

If $[0,1) \cap \mathbb{Q} \subseteq Per(T_\beta)$, then β is either a Pisot *or a* Salem *number.*

Proof. See [Sch80], [pg. 272]. □

This theorem is consistent with the previously known description of $Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$ when $\beta > 1$ is an integer. If $\beta > 1$ is an integer, then the hypothesis of Theorem 3.4.6 is satisfied, and it is true that any integer $\beta > 1$ is a (trivial) *Pisot* number.

It turns out that when β is any *Pisot* number, a complete description of $Per(T_\beta)$ can be given, as it was shown in [Ber77] and [Sch80]:

Theorem 3.4.7. *Let β be a* Pisot *number. Then $Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$.*

Proof. See [Sch80] [pg. 274] or [Ber77]. □

We shall make a few remarks about the main idea behind the proof of Theorem 3.4.7. If $\beta > 1$ is an algebraic number and $x \in [0,1) \cap \mathbb{Q}(\beta)$, then the map T_β multiplies x by β and subtracts a suitable integer in $\{0, 1, \ldots, [\beta]\}$. We can represent x in a unique way as a sum of d powers of β weighted by rational coefficients. If we fix a minimum denominator $q > 0$ and write x as:

$$\frac{p_1 + p_2\beta + \ldots + p_d\beta^{d-1}}{q}, \quad \text{with} \quad p_1, p_2, \ldots, p_d \in \mathbb{Z}, \tag{3.21}$$

then as we iterate x under T_β, we keep on obtaining points in the unit interval which can be rewritten in the form (3.21). In fact, the minimum common denominator cannot increase (and always divides q), and we can use the identity $\beta^d = -(c_{d-1}\beta^{d-1} + \ldots + c_1\beta + c_0)$ to reduce the sum of powers of β to another of type (3.21). The difficulty in this case consists in the fact that there exist infinitely many d-tuples $(p_1, p_2, \ldots, p_d) \in$

$\mathbb{Z}^d$ defining numbers in $[0,1) \cap \mathbb{Q}(\beta)$ for the same fixed denominator q. We need some additional condition that will impose a constraint on the set of possible d-tuples $(p_1, p_2, \ldots, p_d) \in \mathbb{Z}^d$ corresponding to $T_\beta^k(x)$, namely, that such a set should be bounded and hence finite. The *Pisot* condition, which implies that all other conjugate roots of β have modulus strictly less than 1, provides a sufficient argument which guarantees the finiteness of the orbit of any $x \in [0,1) \cap \mathbb{Q}(\beta)$.

However, if β is a *Salem* number, the existence of complex conjugate roots with modulus 1 (whose powers always have modulus 1, and therefore don't converge to zero) doesn't allow us to use the same algebraic argument that works in the *Pisot* case. It is possible that if β is a *Salem* number then there might exist $x \in [0,1) \cap \mathbb{Q}(\beta)$ such that $\mathcal{O}(x)$ is an infinite set, hence $Per(T_\beta) \neq [0,1) \cap \mathbb{Q}(\beta)$. However, in [Sch80] Schmidt claimed the existence of computational evidence against that possibility, and made the following conjecture:

Conjecture 3.4.8. *Let β be a* Salem *number. Then $Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$.*

This conjecture is quoted in [Boy89], [Boy96], [Boy97], [Har06], [BBLT06], [Sch06], [Aki98], and it still remains an open problem. If it is indeed true, then together with Theorem 3.4.6 and Theorem 3.4.7 we could state the following (which was formulated in [Sch80] as well):

Conjecture 3.4.9. *$Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$ if and only if β is either a* Pisot *or a* Salem *number.*

Chapter 4

An equivalent system for studying periodic points

4.1 Introduction

In this chapter, we assume that $\beta > 1$ is a non-trivial *Pisot* number (not an integer) or a *Salem* number. Instead of considering the β-transformation defined on the unit interval, we will only be concerned with its restriction to $[0,1) \cap \mathbb{Q}(\beta)$. This choice is justified by the fact that $Per(T_\beta) \subseteq [0,1) \cap \mathbb{Q}(\beta)$, therefore $[0,1) \cap \mathbb{Q}(\beta)$ is an appropriate domain for studying the set of eventually periodic points for the β-transformation.

If β is an algebraic number of degree d, then any $x \in \mathbb{Q}(\beta)$ can be expressed as the sum of d terms consisting of consecutive powers of β weighted by rational coefficients. For instance:

$$\forall x \in \mathbb{Q}(\beta), \quad x = x_1 + x_2\beta + \ldots + x_d\beta^{d-1}, \tag{4.1}$$

with coefficients $x_k \in \mathbb{Q}$. For each $x \in \mathbb{Q}(\beta)$, the coefficients $x_1, \ldots, x_d$ are uniquely defined, because if there existed a different set of coefficients $y_1, \ldots, y_d \in \mathbb{Q}$ such that:

$$x_1 + x_2\beta + \ldots + x_d\beta^{d-1} = y_1 + y_2\beta + \ldots + y_d\beta^{d-1}, \tag{4.2}$$

then β would be the root of a polynomial with rational coefficients $(y_i - x_i)$ and of degree less than d, which is impossible since β is an algebraic number of degree d.

We will show that $\mathbb{Q}(\beta)$ is a d-dimensional vector space (under scalar multiplication over the rationals) which is isomorphic to $\mathbb{Q}^d$ (d-dimensional vector space under scalar multiplication by rationals as well). Furthermore, we can choose a vector space isomorphism between $\mathbb{Q}(\beta)$ and $\mathbb{Q}^d$ and this will induce a field structure on $\mathbb{Q}^d$. This isomorphism establishes a bijection between $[0,1) \cap \mathbb{Q}(\beta) \subset \mathbb{Q}(\beta)$ and a subset $L \subset \mathbb{Q}^d$, and therefore it induces a map $\widetilde{C} : L \to L$ which is semi-conjugate to the map $T_\beta : [0,1) \cap \mathbb{Q}(\beta) \to [0,1) \cap \mathbb{Q}(\beta)$.

Although the dynamical systems $(\widetilde{C}, L)$ and $(T_\beta, [0,1) \cap \mathbb{Q}(\beta))$ are merely semi-conjugate, this semi-conjugacy happens to be a bijection. Therefore we can say that $(\widetilde{C}, L)$ is equivalent to $(T_\beta, [0,1) \cap \mathbb{Q}(\beta))$, as far as the study of periodic points is concerned. Furthermore, $\widetilde{C} : L \to L$ is connected to the linear map defined by the *companion matrix* of the minimal polynomial of β, which is defined as:

$$C = \begin{pmatrix} 0 & \dots & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -c_{d-2} \\ 0 & \dots & 0 & 1 & -c_{d-1} \end{pmatrix}. \tag{4.3}$$

The roots of a monic polynomial $p(x)$ are the same as the eigenvalues of its *companion matrix* C. If $\beta > 1$ is a *Pisot* or a *Salem* number, then the linear map C expands by the eigenvalue β along the corresponding eigendirection, and contracts and/or rotates on the remaining eigenspaces. This provides a geometric visualization in $L \subset \mathbb{Q}^d$ of the dynamics of T_β restricted to $\mathbb{Q}(\beta) \cap [0,1)$.

If the *companion matrix* has determinant ± 1, then $C \in GL(d, \mathbb{Z})$ and it defines a toral automorphism in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. This is always the case when β is a *Salem* number, but in the *Pisot* case we should require $c_0 = \pm 1$ (this excludes the *Pisot* numbers which aren't units in $\mathbb{Z}[\beta]$). When $|\det C| = 1$ the dynamical system $(\widetilde{C}, L)$ can always be factored to the toral automorphism defined by C, and more specifically, to the restriction of the toral automorphism to the points of the d-torus with rational coordinates.

4.2 Isomorphic fields

Let $\mathbb{Q}(\beta)$ be the algebraic field extension of $\mathbb{Q}$ by the *Pisot* or *Salem* number β. $\mathbb{Q}(\beta)$ is a d-dimensional vector space with scalars $\mathbb{Q}$, and we will choose an isomorphism to represent elements of $\mathbb{Q}(\beta)$ by points in $\mathbb{Q}^d$.

Proposition 4.2.1. *Let β be a* Pisot *or a* Salem *number with minimal polynomial of degree d. Then $\mathbb{Q}^d$ and $\mathbb{Q}(\beta)$ are d-dimensional vector spaces with scalar multiplication over $\mathbb{Q}$. Therefore they are isomorphic, and we define the following isomorphism:*

$$\begin{aligned} f: \quad \mathbb{Q}^d &\longrightarrow \mathbb{Q}(\beta) \\ \mathbf{x} &\longmapsto x := \bar{\beta} \cdot \mathbf{x} \end{aligned} \tag{4.4}$$

with $\bar{\beta} = (1, \beta, \ldots, \beta^{d-1})$ and $\mathbf{x} := (x_1, x_2, \ldots, x_d)^t \in \mathbb{Q}^d$.

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ be the canonical basis for $\mathbb{Q}^d$:

$$\mathbb{Q}^d = \{x_1\mathbf{v}_1 + \ldots + x_d\mathbf{v}_d \mid x_1, \ldots, x_d \in \mathbb{Q}\}. \tag{4.5}$$

Since β is an algebraic number of degree d, each $x \in \mathbb{Q}(\beta)$ admits a unique decomposition of the type

$$x = x_1 + x_2\beta + \ldots + x_d\beta^{d-1}, \qquad \text{with} \quad x_1, \ldots, x_d \in \mathbb{Q}. \tag{4.6}$$

Let $\mathbf{u}_1 = 1, \mathbf{u}_2 = \beta, \ldots, \mathbf{u}_d = \beta^{d-1}$, and so $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ is a basis for $\mathbb{Q}(\beta)$:

$$\mathbb{Q}(\beta) = \{x_1\mathbf{u}_1 + \ldots + x_d\mathbf{u}_d \mid x_1, \ldots, x_d \in \mathbb{Q}\}. \tag{4.7}$$

Define an isomorphism $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ by $f(\mathbf{v}_k) = \mathbf{u}_k, \ \forall_{1 \leq k \leq d}$. Therefore,

$\forall\, \mathbf{x} := (x_1, \ldots, x_d)^t \in \mathbb{Q}^d$ we have

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{v}_1 + \ldots x_d\mathbf{v}_d) \\ &= x_1 f(\mathbf{v}_1) + \ldots + x_d f(\mathbf{v}_d) \\ &= x_1\, 1 + x_2\, \beta + \ldots + x_d\, \beta^{d-1} \\ &= \bar{\beta} \cdot \mathbf{x}. \end{aligned} \tag{4.8}$$

□

Let us point out that although the isomorphism $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ has the explicit definition (4.4), its inverse $f^{-1} : \mathbb{Q}(\beta) \to \mathbb{Q}^d$ is defined implicitly: for each $x \in \mathbb{Q}(\beta)$, $f^{-1}(x)$ is the unique $\mathbf{x} \in \mathbb{Q}^d$ such that $x = \bar{\beta} \cdot \mathbf{x}$ (of course, if x had been defined in the canonical form $x = x_1 + x_2\beta + \ldots x_d\beta^{d-1}$, then $f^{-1}(x) = (x_1, \ldots, x_d)^t$).

Since $\mathbb{Q}(\beta)$ is not only a vector space but also a field, we can use the isomorphism $f^{-1} : \mathbb{Q}(\beta) \to \mathbb{Q}^d$ to define a multiplication $\otimes : \mathbb{Q}^d \times \mathbb{Q}^d \to \mathbb{Q}^d$, such that $\left(\mathbb{Q}^d, +, \otimes\right)$ is a field isomorphic to $\left(\mathbb{Q}(\beta), +, \times\right)$:

Proposition 4.2.2. *Define a multiplication* $\otimes : \mathbb{Q}^d \times \mathbb{Q}^d \to \mathbb{Q}^d$ *by*

$$\mathbf{x} \otimes \mathbf{y} := f^{-1}(f(\mathbf{x}) \times f(\mathbf{y})), \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{Q}^d. \tag{4.9}$$

Then $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ *is a field isomorphism between* $\left(\mathbb{Q}^d, +, \otimes\right)$ *and* $\left(\mathbb{Q}(\beta), +, \times\right)$*.*
The unit element of $\left(\mathbb{Q}^d, +, \otimes\right)$ *is* $\overline{1} := (1, 0, \ldots, 0)^t \in \mathbb{Z}^d$*.*

Proof. Since $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ is a vector space isomorphism, in order to endow $\mathbb{Q}^d$ with a multiplication $\otimes$ such that $(\mathbb{Q}^d, +, \otimes)$ is a field isomorphic to $(\mathbb{Q}(\beta), +, \times)$, it is necessary and sufficient to have:

$$f(\mathbf{x} \otimes \mathbf{y}) = f(\mathbf{x}) \times f(\mathbf{y})), \tag{4.10}$$

which is equivalent to (4.9), since f is invertible.

Finally, the unit element of $\left(\mathbb{Q}^d, +, \otimes\right)$ is $f^{-1}(1) = (1, 0, \ldots, 0)^t \in \mathbb{Z}^d$. □

The main reason why we are interested in the field $\left(\mathbb{Q}^d, +, \otimes\right)$ is in order to reproduce in it the field operations of $(\mathbb{Q}(\beta), +, \times)$. In particular, this should provide a way of defining a map in a subset of $\mathbb{Q}^d$, which is equivalent to the β-transformation applied to any $x \in [0,1) \cap \mathbb{Q}(\beta)$. Indeed, the β-transformation uses the multiplication by β in the field $\mathbb{Q}(\beta)$, followed by a subtraction by an integer multiple of the unit element of the field, and therefore we should expect a similar definition for an equivalent map in $(\mathbb{Q}^d, +, \otimes)$. However, there are two technical difficulties: not only $f^{-1}(x)$ cannot be defined explicitly, but also the definition of the multiplication in $\mathbb{Q}^d$ seems rather strange. We will shortly see how these apparent difficulties can be overcome.

Let $\mathbb{Q}(C) \subset GL(d, \mathbb{Q})$ be the field extension of $\mathbb{Q}$ by the matrix C (under matrix addition, and matrix multiplication). The unit element of $\mathbb{Q}(C)$ is the identity matrix $Id \in GL(d, \mathbb{Q})$.

Proposition 4.2.3. *$\mathbb{Q}(C)$ is a d-dimensional vector space with scalars $\mathbb{Q}$, and admits a basis $\{Id, C, \dots, C^{d-1}\}$:*

$$\mathbb{Q}(C) = \{x_1 Id + x_2 C + \ldots + x_d C^{d-1} \ | \ x_1, \dots, x_d \in \mathbb{Q}\}. \qquad (4.11)$$

Proof. According to the *Cayley-Hamilton* theorem (see [SM88]), C is a root of its characteristic polynomial : $C^d + c_{d-1}C^{d-1} + \ldots + c_1 C + c_0 Id = 0$. Therefore any power of C greater than or equal to d can be rearranged as a sum of terms containing non-negative powers of C such as (4.11). Furthermore, the minimal polynomial is irreducible, hence $\{Id,\ C,\ \dots\ , C^{d-1}\}$ are linearly independent. $\square$

Theorem 4.2.4. *The fields $\left(\mathbb{Q}(\beta), +, \times\right)$ and $\left(\mathbb{Q}(C), +, \times\right)$ are isomorphic, and the field isomorphism:*

$$\begin{array}{rccc} F: & \mathbb{Q}(\beta) & \longrightarrow & \mathbb{Q}(C) \\ & x & \longmapsto & A := F(x), \end{array} \qquad (4.12)$$

is completely defined by the equation $F(\beta) = C$.

Proof. $\mathbb{Q}(\beta)$ is isomorphic to $\mathbb{Q}[x]/p(x)$, because $p(\beta) = 0$. But $\mathbb{Q}(C)$ is also isomorphic to $\mathbb{Q}[x]/p(x)$, because $p(C) = 0$ (according to the *Cayley-Hamilton* theorem). Therefore we define a field isomorphism between $\mathbb{Q}(\beta)$ and $\mathbb{Q}(C)$ by sending β to C. $\square$

Although the product of matrices in not commutative in general, it is in the field $\mathbb{Q}(C)$, because the powers of C commute.

If $x \in \mathbb{Q}(\beta)$ is defined as

$$x := \sum_{i=-m}^{n} a_i \beta^{-i}, \quad \text{with } \forall_{-m \leq i \leq n}\, a_i \in \mathbb{Q}, \tag{4.13}$$

then Theorem 4.2.4 gives a practical way to calculate $F(x) \in \mathbb{Q}(C)$:

$$\begin{aligned} F(x) &= \sum_{i=-m}^{n} F(a_i \beta^{-i}) = \sum_{i=-m}^{n} a_i\, F(\beta^{-i}) \\ &= \sum_{i=-m}^{n} a_i F(\beta)^{-i} = \sum_{i=-m}^{n} a_i C^{-i}. \end{aligned} \tag{4.14}$$

An alternative way of defining $F(x)$ is:

Proposition 4.2.5. *Let $x \in \mathbb{Q}(\beta)$ be defined as $x = x_1 + x_2\beta + \ldots + x_d\beta^{d-1}$, and therefore $\mathbf{x} := f^{-1}(x) = (x_1, \ldots, x_d)^t \in \mathbb{Q}^d$.*

Then $F(x) := x_1 Id + x_2 C + \ldots + x_d C^{d-1} = A(x)$, with

$$A(x) = \begin{pmatrix} | & | & & | \\ \mathbf{x} & C\mathbf{x} & \cdots & C^{d-1}\mathbf{x} \\ | & | & & | \end{pmatrix}. \tag{4.15}$$

Proof. Let us fix the following basis for $\mathbb{Q}(\beta)$:

$$\{\mathbf{u}_1 = 1, \mathbf{u}_2 = \beta, \ldots, \mathbf{u}_d = \beta^{d-1}\}. \tag{4.16}$$

According to [Wik07], each $x \in \mathbb{Q}(\beta)$ admits a unique representation by a d-dimensional matrix $A(x) = \{a_{ij}\}_{1 \leq i,j \leq d}$ with rational coefficients determined by:

$$\forall_{1 \leq j \leq d} \quad \mathbf{u}_j x = \sum_{i=1}^{d} a_{ij} \mathbf{u}_i \tag{4.17}$$

$$\beta^{j-1} x = \sum_{i=1}^{d} a_{ij} \beta^{i-1}, \tag{4.18}$$

and this is called the *regular representation* of $\mathbb{Q}(\beta)$. The column j of the matrix $A(x)$ contains the coordinates of $\beta^{j-1}x$ in the basis (4.16). Therefore if $f(\mathbf{x}) = x$ then the first column of $A(x)$ is equal to $\mathbf{x}$. Furthermore

$$\begin{aligned} \beta f(\mathbf{x}) &= \beta(\bar{\beta} \cdot \mathbf{x}) \\ &= \beta(x_1 + x_2\beta + \ldots + x_d\beta^{d-1}) \\ &= -c_0x_d + (x_1 - c_1x_d)\beta + \ldots + (x_{d-1} - c_{d-1}x_d)\beta^{d-1} \\ &= \bar{\beta} \cdot C\mathbf{x} \\ &= f(C\mathbf{x}), \end{aligned} \tag{4.19}$$

which proves that the second column of $A(x)$ is $C\mathbf{x}$, and the same argument applies for obtaining the following columns. This proves (4.15).

Finally, $F(x) = A(x)$, because $A(1) = Id,\ A(\beta) = C,\ \ldots,\ A(\beta^{d-1}) = C^{d-1}$ hence the *regular representation* of $\mathbb{Q}(\beta)$ is the isomorphism $F : \mathbb{Q}(\beta) \to \mathbb{Q}(C)$. □

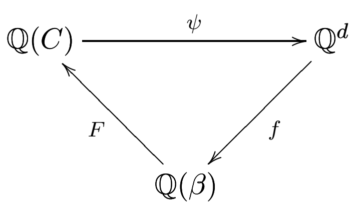
Definition 4.2.6. *Let*

$$\begin{array}{rccc} \psi : & \mathbb{Q}(C) & \longrightarrow & \mathbb{Q}^d \\ & A & \longmapsto & \mathbf{x} := A\bar{1}, \end{array} \tag{4.20}$$

with $\bar{1} := (1, 0, \ldots, 0)^t \in \mathbb{Z}^d$.

The map $\psi : \mathbb{Q}(C) \to \mathbb{Q}^d$ sends each matrix A to the column vector contained in its first column.

Proposition 4.2.7. $\big(\mathbb{Q}(C), +, \times\big)$, $\big(\mathbb{Q}^d, +, \otimes\big)$ *and* $\big(\mathbb{Q}(\beta), +, \times\big)$ *are d-dimensional isomorphic fields:*



(4.21)

and the isomorphisms are defined in (4.4), (4.12) and (4.20).

Proof. We defined the field isomorphisms $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ and $F : \mathbb{Q}(\beta) \to \mathbb{Q}(C)$, therefore the three of them are isomorphic. In order to prove that $\psi : \mathbb{Q}(C) \to \mathbb{Q}^d$ is the other field isomorphism it suffices to prove that:

$$\begin{aligned} \psi^{-1}(\mathbf{x}) &= F \circ f(\mathbf{x}) \\ &= F(\bar{\beta} \cdot \mathbf{x}) \\ &= A(x), \end{aligned} \tag{4.22}$$

where $A(x)$ contains the vector $\mathbf{x}$ in its first column. □

4.3 The equivalent system

We will need two main results in order to define an equivalent system for studying periodic points for the β-transformation of the unit interval. The first consists of choosing a suitable phase space, with points which are in one-to-one correspondence with points in $[0,1) \cap \mathbb{Q}(\beta)$. The second, requires carefully defining a map in that phase space, in such a way that it reproduces (through a bijective semi-conjugacy) the dynamics of the restriction of the β-transformation to $[0,1) \cap \mathbb{Q}(\beta)$.

Proposition 4.3.1. *Let β be a* Pisot *number or a* Salem *number of degree $d \geq 2$, and $\bar{\beta} := (1, \beta, \ldots, \beta^{d-1}) \in \mathbb{R}^d$. The isomorphism $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ is a bijection between*

$$L := \{\mathbf{x} \in \mathbb{Q}^d \mid 0 \leq \ \bar{\beta} \cdot \mathbf{x} \ < 1\} \qquad \textit{and} \qquad [0,1) \cap \mathbb{Q}(\beta). \tag{4.23}$$

Proof. Let us consider $f : \mathbb{Q}^d \to \mathbb{Q}(\beta)$ as defined in (4.4), which is an isomorphism between $\mathbb{Q}^d$ and $\mathbb{Q}(\beta)$, according to Proposition 4.2.1. Since $[0,1) \cap \mathbb{Q}(\beta)$ is a subset of $\mathbb{Q}(\beta)$, there exists a unique $L \subset \mathbb{Q}^d$ such that $f : L \to [0,1) \cap \mathbb{Q}(\beta)$ is a bijection. Equivalently,

$$\begin{aligned} L &:= f^{-1}\big([0,1) \cap \mathbb{Q}(\beta)\big) \\ &= \{\mathbf{x} \in \mathbb{Q}^d \mid 0 \leq f(\mathbf{x}) < 1\} \\ &= \{\mathbf{x} \in \mathbb{Q}^d \mid 0 \leq \bar{\beta} \cdot \mathbf{x} < 1\}, \end{aligned} \tag{4.24}$$

which proves what we wanted. □

This was the first step that we needed. Before defining a suitable map in L, we should make some comments about the geometry of the set $L \subset \mathbb{Q}^d \subset \mathbb{R}^d$. The definition of L in (4.23) is equivalent to $L := Y \cap \mathbb{Q}^d$, where

$$Y := \{(y_1, \ldots, y_d) \in \mathbb{R}^d \ | \ 0 \leq y_1 + y_2\beta + \ldots + y_d\beta^{d-1} < 1\}. \quad (4.25)$$

It is an elementary result from vector analysis that the inequalities in (4.25) defining $Y \subset \mathbb{R}^d$ have the following geometrical meaning: Y is the set of points in $\mathbb{R}^d$ comprised between two parallel $(d-1)$-dimensional planes (whose direction is defined by the orthogonal vector $(1, \beta, \ldots, \beta^{d-1}) \in \mathbb{R}^d$), including the plane containing the origin, and excluding the other plane (which contains, for instance, the point $\overline{1} = (1, 0, \ldots, 0)^t \in \mathbb{Z}^d$). If we intersect Y with $\mathbb{Q}^d$ we obtain L (and therefore, L is not closed).

The geometry of L is related to the geometry of the eigenspaces of the linear map $C : \mathbb{R}^d \to \mathbb{R}^d$ (where C is the companion matrix of the minimal polynomial of β). Let us decompose $\mathbb{R}^d$ into three C-invariant subspaces:

$$\mathbb{R}^d = E^s \oplus E^c \oplus E^u, \quad (4.26)$$

where E^u is the 1-dimensional eigenspace associated to the unique expanding eigenvalue β; E^s is the stable subspace which is the direct sum of the eigenspaces associated to the eigenvalues of C with modulus less than 1 (in the *Salem* case, E^s is 1-dimensional); and E^c is the direct sum of the eigenspaces associated to the eigenvalues of C with modulus 1 ($E^c = \varnothing$ in the *Pisot* case).

Proposition 4.3.2. *Let β be a* Pisot *or a* Salem *number of degree $d \geq 2$. Consider the decomposition of $\mathbb{R}^d$ as in (4.26). Then*

$$E^s \oplus E^c \ \perp \ (1, \beta, \ldots, \beta^{d-1}). \quad (4.27)$$

Proof. The vector $\bar{\beta} = (1, \beta, \ldots, \beta^{d-1})$ is a left eigenvector of C, associated to the eigenvalue β:

$$
\begin{aligned}
\bar{\beta} \cdot C &= (1, \beta, \ldots, \beta^{d-1}) \begin{pmatrix} 0 & \ldots & \ldots & 0 & -c_0 \\ 1 & 0 & \ldots & 0 & -c_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -c_{d-2} \\ 0 & \ldots & 0 & 1 & -c_{d-1} \end{pmatrix} \\
&= (\beta, \ldots, \beta^{d-1}, -c_0 - c_1\beta - \ldots - c_{d-1}\beta^{d-1}) \\
&= (\beta, \ldots, \beta^{d-1}, \beta^d) \\
&= \beta \, \bar{\beta}. \qquad (4.28)
\end{aligned}
$$

Choose any $\mathbf{x} \in \mathbb{R}^d$ and write $\mathbf{x} = \mathbf{x}_{s,c} + \mathbf{x}_u$, with $\mathbf{x}_{s,c} \in E^s \oplus E^c$ and $\mathbf{x}_u \in E^u$. According to (4.28), we have

$$
\begin{aligned}
\bar{\beta} \cdot C(\mathbf{x}) &= \beta \, \bar{\beta} \cdot \mathbf{x} \\
\bar{\beta} \cdot C(\mathbf{x}_{s,c} + \mathbf{x}_u) &= \beta \, \bar{\beta} \cdot (\mathbf{x}_{s,c} + \mathbf{x}_u). \qquad (4.29)
\end{aligned}
$$

Since $\mathbf{x}_u \in E^u$, which is the eigenspace associated to the eigenvalue β, we have $C(\mathbf{x}_{s,c} + \mathbf{x}_u) = C(\mathbf{x}_{s,c}) + \beta \mathbf{x}_u$, and from (4.29) we obtain

$$\bar{\beta} \cdot C(\mathbf{x}_{s,c}) + \bar{\beta} \cdot (\beta \mathbf{x}_u) = \beta\bar{\beta} \cdot \mathbf{x}_{s,c} + \beta\bar{\beta} \cdot \mathbf{x}_u, \qquad (4.30)$$

which is equivalent to $\bar{\beta} \cdot C(\mathbf{x}_{s,c}) = \bar{\beta} \cdot (\beta \mathbf{x}_{s,c})$ and to:

$$\bar{\beta} \cdot \big((C - \beta Id)\mathbf{x}_{s,c}\big) = 0, \qquad (4.31)$$

for any $\mathbf{x}_{s,c} \in E^s \oplus E^c$. But the restriction of the linear map $(C - \beta Id)$ to the invariant subspace $E^s \oplus E^c$ is invertible, therefore from (4.31) it follows that

$$\forall \, \mathbf{v} \in E^s \oplus E^c, \quad \bar{\beta} \cdot \mathbf{v} = 0, \quad \text{hence} \quad E^s \oplus E^c \perp \bar{\beta}. \qquad (4.32)$$

□

We can finally state the relation between the geometry of L and the C-invariant subspaces of $\mathbb{R}^d$:

Corollary 4.3.3. *The set $L \subset \mathbb{Q}^d$ is contained in the region between two parallel planes in $\mathbb{R}^d$, which are:*

$$E^s \oplus E^c \qquad and \qquad \overline{1} + E^s \oplus E^c. \tag{4.33}$$

Proof. The $(d-1)$-dimensional planes which are the boundary of Y are orthogonal to $\bar{\beta}$, as we have seen in (4.25). $E^s \oplus E^c$ is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ (because E^u is 1-dimensional), and according to Proposition 4.3.2, this subspace is orthogonal to $\bar{\beta}$. Therefore $E^s \oplus E^c$ is the $(d-1)$-dimensional plane which is orthogonal to $\bar{\beta}$ and contains the origin, whereas its translation by the vector $\overline{1} := (1, 0, \ldots, 0)^t \in \mathbb{Z}^d$ is the other parallel plane (containing $\overline{1}$) which together form the boundary of Y. ☐

We now state the second main result which we need for the definition of our equivalent system for studying periodic points for the β-transformation:

Theorem 4.3.4. *The map defined by*

$$\begin{array}{rccc} \widetilde{C}: & L & \longrightarrow & L \\ & \mathbf{x} & \longmapsto & C\mathbf{x} - \varepsilon_1(\mathbf{x})\overline{1}, \end{array} \tag{4.34}$$

with $\varepsilon_1(\mathbf{x}) := [\beta f(\mathbf{x})] = [\beta\,\bar{\beta} \cdot \mathbf{x}] \in \{0, 1, \ldots, [\beta]\}$ is semi-conjugate to the restriction of the β-transformation to $[0,1) \cap \mathbb{Q}(\beta)$:

$$\begin{array}{rccc} T_\beta: & [0,1) \cap \mathbb{Q}(\beta) & \longrightarrow & [0,1) \cap \mathbb{Q}(\beta) \\ & x & \longmapsto & \beta x - [\beta x]. \end{array} \tag{4.35}$$

The semi-conjugacy of the two systems is shown in the following commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\ \widetilde{C}\ } & L \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ [0,1) \cap \mathbb{Q}(\beta) & \xrightarrow[T_\beta]{} & [0,1) \cap \mathbb{Q}(\beta) \end{array} \tag{4.36}$$

where the semi-conjugacy $f: L \to [0,1) \cap \mathbb{Q}(\beta)$ is a bijection (as defined in (4.4)).

Proof. According to (4.4), $f : L \to [0, 1) \cap \mathbb{Q}(\beta)$ is a continuous bijection. Let us now prove that the diagram (4.36) is commutative. Using the definition of $\widetilde{C}$ (4.34), we have:

$$f \circ \widetilde{C}(\mathbf{x}) = f\big(C\mathbf{x} - \varepsilon_1(\mathbf{x})\overline{1}\big). \tag{4.37}$$

f is a linear map (vector space isomorphism) between $\mathbb{Q}^d$ and $\mathbb{Q}(\beta)$ (with scalars $\mathbb{Q}$). Since $\forall \mathbf{x} \in L,\ \ \varepsilon_1(\mathbf{x}) \in \{0, 1, \ldots, [\beta]\} \subset \mathbb{Q}$, we use the linearity of f to obtain:

$$f \circ \widetilde{C}(\mathbf{x}) = f\big(C\mathbf{x}\big) - \varepsilon_1(\mathbf{x}) f\big(\overline{1}\big). \tag{4.38}$$

Let us use (4.19) to simplify $f\big(C\mathbf{x}\big) = \beta f(\mathbf{x})$. Furthermore, $f(\overline{1}) = \bar{\beta} \cdot \overline{1} = 1$ and $\varepsilon_1(\mathbf{x}) := [\beta f(\mathbf{x})]$, therefore:

$$\begin{aligned} f \circ \widetilde{C}(\mathbf{x}) &= \beta f(\mathbf{x}) - \varepsilon_1(\mathbf{x}) \\ &= \beta f(\mathbf{x}) - [\beta f(\mathbf{x})]. \end{aligned} \tag{4.39}$$

On the other hand,

$$\begin{aligned} T_\beta \circ f(\mathbf{x}) &= T_\beta\big(f(\mathbf{x})\big) \\ &= \beta f(\mathbf{x}) - [\beta f(\mathbf{x})], \end{aligned} \tag{4.40}$$

which proves that $f \circ \widetilde{C}(\mathbf{x}) = T_\beta \circ f(\mathbf{x})$ and the diagram is commutative. □

Proposition 4.3.1 together with Theorem 4.3.4 provide us with a non-invertible dynamical system $(\widetilde{C}, L)$ which is equivalent to the β-transformation of the unit interval, as far as the study of periodic points is concerned.

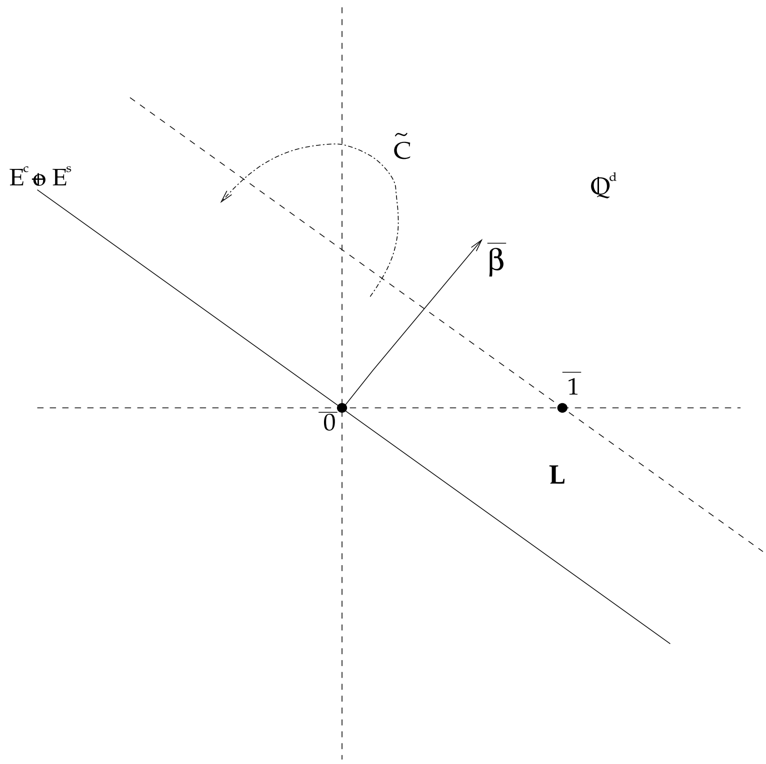


Figure 4.1: The set L and the map $\widetilde{C}$

Chapter 5

Periodic points

5.1 Introduction

In this chapter, we consider the dynamical system $\big(\widetilde{C}, L\big)$ which is semi-conjugate to $(T_\beta, [0,1)\cap\mathbb{Q}(\beta))$. Given that the semi-conjugacy f is a bijection, $\big(\widetilde{C}, L\big)$ is an equivalent system for studying periodic points for the β-transformation.

We assume that $\beta > 1$ is a *Pisot* number (having minimal polynomial $p(x)$ with $c_0 = \pm 1$) or a *Salem* number, therefore $|c_0| = |\det C| = 1$ and $C \in GL(d, \mathbb{Z})$. It is a standard result in the theory of *Dynamical Systems* that the linear map C defines a toral automorphism on the d-dimensional torus (see, for example, [KH95]):

Definition 5.1.1. *Let $\beta > 1$ be a* Pisot *number (with a minimal polynomial $p(x)$ having $c_0 = \pm 1$) or a* Salem *number. The companion matrix of the minimal polynomial of β defines a toral automorphism in $\mathbb{R}^d/\mathbb{Z}^d$:*

$$\begin{aligned} \overline{C}: \quad \mathbb{R}^d/\mathbb{Z}^d &\longrightarrow \mathbb{R}^d/\mathbb{Z}^d \\ \mathbf{x} + \mathbb{Z}^d &\longmapsto C\mathbf{x} + \mathbb{Z}^d. \end{aligned} \tag{5.1}$$

The similarity between (5.1) and (4.34) suggests that the map $\widetilde{C}$ can be factored to the toral automorphism $\overline{C}$. Given that the domain of the map $\widetilde{C}$ is $L \subset \mathbb{Q}^d$, when we factor $\widetilde{C}$ to $\overline{C}$, it suffices to consider the restriction of the toral automorphism to $\mathbb{Q}^d/\mathbb{Z}^d \subset \mathbb{T}^d$ (which is the set of periodic points for the toral automorphism $\overline{C}$):

Proposition 5.1.2. *The dynamical systems* $\widetilde{C} : L \to L$ *and* $\overline{C} : \mathbb{Q}^d/\mathbb{Z}^d \to \mathbb{Q}^d/\mathbb{Z}^d$ *are topologically semi-conjugate:*

$$\begin{array}{ccc} L & \xrightarrow{\widetilde{C}} & L \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{Q}^d/\mathbb{Z}^d & \xrightarrow[\overline{C}]{} & \mathbb{Q}^d/\mathbb{Z}^d \end{array} \tag{5.2}$$

and the semi-conjugacy is the projection into the torus:

$$\begin{array}{rccc} \pi : & L \subset \mathbb{Q}^d & \longrightarrow & \mathbb{Q}^d/\mathbb{Z}^d \\ & \mathbf{x} & \longmapsto & \mathbf{x} + \mathbb{Z}^d. \end{array} \tag{5.3}$$

Proof. Let us prove that the diagram commutes. $\forall \mathbf{x} \in L,\ \ \varepsilon_1(\mathbf{x})\overline{1} \in \mathbb{Z}^d$, therefore

$$\begin{aligned} \pi \circ \widetilde{C}(\mathbf{x}) &= \pi\big(C\mathbf{x} - \varepsilon_1(\mathbf{x})\overline{1}\big) \\ &= C\mathbf{x} - \varepsilon_1(\mathbf{x})\overline{1} + \mathbb{Z}^d \\ &= C\mathbf{x} + \mathbb{Z}^d. \end{aligned} \tag{5.4}$$

On the other hand,

$$\begin{aligned} \overline{C} \circ \pi(\mathbf{x}) &= \overline{C}(\mathbf{x} + \mathbb{Z}^d) \\ &= C\mathbf{x} + \mathbb{Z}^d, \end{aligned} \tag{5.5}$$

hence $\pi \circ \widetilde{C}(\mathbf{x}) = \overline{C} \circ \pi(\mathbf{x}),\ \ \forall \mathbf{x} \in L$.

Finally, $\pi : L \to \mathbb{Q}^d/\mathbb{Z}^d$ is a continuous surjection, therefore π is a topological semi-conjugacy. □

We will be interested in the factorization of $\widetilde{C}$-orbits to $\overline{C}$-orbits by the semi-conjugacy π. We recall that any $\overline{C}$-orbit in $\mathbb{Q}^d/\mathbb{Z}^d$ is periodic (in the strict sense), and therefore any $\widetilde{C}$-orbit (whether it is eventually periodic or possibly non-periodic) is mapped by π into a periodic $\overline{C}$-orbit.

5.2 Partitioning L according to periodic properties

The map $\widetilde{C}$ is not invertible, and we define the set of *eventually periodic* points as:

Definition 5.2.1. *The set of* eventually periodic *points for* $\widetilde{C}$ *is:*

$$Per(\widetilde{C}) := \{\mathbf{x} \in L \subset \mathbb{Q}^d \mid \exists_{m\geq 0,\, p>0} \ : \ \widetilde{C}^{p+m}(\mathbf{x}) = \widetilde{C}^m(\mathbf{x})\}. \tag{5.6}$$

If $\mathbf{x} \in Per(\widetilde{C})$ we are interested in the minimum number of iterates m that it takes for $\widetilde{C}^m(\mathbf{x})$ to be periodic in the strict sense:

Definition 5.2.2. *Let* $\mathbf{x} \in Per(\widetilde{C}) \subseteq L$. *The pre-period of* $\mathbf{x}$ *is*

$$m := \min\{k \in \mathbb{N}_0 \ \mid \ \exists_{p\in\mathbb{N}} : \widetilde{C}^{p+k}(\mathbf{x}) = \widetilde{C}^k(\mathbf{x})\}. \tag{5.7}$$

If $\mathbf{x} \in Per(\widetilde{C})$ *has pre-period* $m = 0$, *we say that* $\mathbf{x}$ *is strictly periodic.*

For each $\mathbf{x} \in Per(\widetilde{C})$ we define its (minimum) period as the (minimum) period of the periodic component of the $\widetilde{C}$-orbit of $\mathbf{x}$:

Definition 5.2.3. *Let* $\mathbf{x} \in Per(\widetilde{C}) \subseteq L$. *The (minimum) period of* $\mathbf{x}$ *is*

$$p := \min\{k \in \mathbb{N} \ \mid \ \exists_{m\in\mathbb{N}} : \widetilde{C}^{k+m}(\mathbf{x}) = \widetilde{C}^m(\mathbf{x})\}. \tag{5.8}$$

We can partition L into the disjoint union of three sets distinguishing points according to their periodic properties:

Definition 5.2.4. *The set* L *is the disjoint union of three sets:*

$$L = \underbrace{P \ \cup \ E}_{Per(\widetilde{C})} \cup N, \tag{5.9}$$

where:

1. P *is the set of strictly periodic points (pre-period* $m = 0$*).*

2. E *is the set of eventually periodic points with pre-period* $0 < m < \infty$.

3. $N := L \setminus Per(\widetilde{C})$ *is the set of non-periodic points.*

According to Theorem 3.4.7 and Theorem 4.3.4, if β is a *Pisot* number then $Per(\widetilde{C}) = L$ and $N = \varnothing$. The same would be true in the *Salem* case, should Schmidt's Conjecture 3.4.8 be true. Since we do not know whether or not that is the case, we assume the possibility that $N \neq \varnothing$ in the *Salem* case.

The set $Per(\widetilde{C}) = P \cup E$ is $\widetilde{C}$-invariant:

$$\forall \mathbf{x} \in Per(\widetilde{C}),\ \forall n \in \mathbb{N}, \quad \widetilde{C}^n(\mathbf{x}) \in Per(\widetilde{C}), \tag{5.10}$$

and so is N (though this is trivial in the *Pisot* case):

$$\forall \mathbf{x} \in N,\ \forall n \in \mathbb{N}, \quad \widetilde{C}^n(\mathbf{x}) \in N. \tag{5.11}$$

For each $\mathbf{x} \in E$, there exists $m \in \mathbb{N}$ such that $\widetilde{C}^m(\mathbf{x}) \in P$, therefore P can be considered as an attractor (in the sense that we are suggesting) for the orbit of any $\mathbf{x} \in Per(\widetilde{C})$.

If β is *Salem* and $N \neq \varnothing$, then the orbit of any $\mathbf{x} \in N$ is non-periodic and it contains an infinite number of points. Moreover, as we choose $\mathbf{x} \in N \subset L \subset \mathbb{Q}^d$ we fix a minimum common denominator $q > 0$ for the rational coordinates $(x_1, \ldots, x_d)^t \in \mathbb{Q}^d$. Given that $\widetilde{C}$ does not increase q, we can say that for any $n \in \mathbb{N}$, $\widetilde{C}^n(\mathbf{x}) \in \left(\mathbb{Z}^d/q\right) \cap L$. But $\left(\mathbb{Z}^d/q\right)$ is a lattice, hence any non-periodic (infinite) orbit must be unbounded. This proves that if $N \neq \varnothing$, then

$$\forall \mathbf{x} \in N,\ \forall K > 0,\ \exists k \in \mathbb{N} \quad : \quad n > k \Rightarrow |\widetilde{C}^n(\mathbf{x})| > K. \tag{5.12}$$

5.3 Symbolic dynamics

We can use the subshift $(\sigma, \widetilde{X}_\beta^+)$ to describe the dynamics of $(\widetilde{C}, L)$.

If we consider Proposition 3.4.5 and Theorem 4.3.4 together, we have:

$$\begin{array}{ccc} \widetilde{X}_\beta^+ & \xrightarrow{\sigma} & \widetilde{X}_\beta^+ \\ {\scriptstyle \pi_\beta}\downarrow & & \downarrow{\scriptstyle \pi_\beta} \\ [0,1)\cap\mathbb{Q}(\beta) & \xrightarrow[T_\beta]{} & [0,1)\cap\mathbb{Q}(\beta) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ L & \xrightarrow[\widetilde{C}]{} & L \end{array} \tag{5.13}$$

where both $\pi_\beta : \widetilde{X}_\beta^+ \to [0,1)\cap\mathbb{Q}(\beta)$ and $f : L \to [0,1)\cap\mathbb{Q}(\beta)$ are bijective topological semi-conjugacies (their inverses are not continuous). Let us define a bijection $\Pi_\beta : \widetilde{X}_\beta^+ \to L$ by $\Pi_\beta := f^{-1} \circ \pi_\beta$. This allows us to write the simplified commutative diagram:

$$\begin{array}{ccc} \widetilde{X}_\beta^+ & \xrightarrow{\sigma} & \widetilde{X}_\beta^+ \\ {\scriptstyle \Pi_\beta}\downarrow & & \downarrow{\scriptstyle \Pi_\beta} \\ L & \xrightarrow[\widetilde{C}]{} & L \end{array} \tag{5.14}$$

Although $\Pi_\beta : \widetilde{X}_\beta^+ \to L$ is not a semi-conjugacy (because it is not continuous), it is a bijection and carries the dynamics of the shift $(\sigma, \widetilde{X}_\beta^+)$ to $(\widetilde{C}, L)$.

Let $\mathbf{s} \in \mathbb{R}^d$ be the lift of the fundamental homoclinic point of $\overline{C}$ (this is a concept introduced in [Sid01] or [Sid03]; see also Appendix C), whose $\mathbb{Z}^d$-coordinate is $\overline{1}$:

$$\mathbf{s} := (\overline{1} + E^s \oplus E^c) \cap E^u. \tag{5.15}$$

Note that $\mathbf{s}$ belongs to the intersection of E^u with the boundary of Y , and that $\bar{\beta}\cdot\mathbf{s} = 1$. We can map any $x \in [0,1)\cap\mathbb{Q}(\beta)$ to $(x\,\mathbf{s}) \in \mathbb{R}^d$.

Proposition 5.3.1. *Let* $\mathbf{s}$ *be defined by (5.15). Then for each* $x \in [0,1)\cap\mathbb{Q}(\beta)$

$$\mathbf{x} := (x\,\mathbf{s} + E^s \oplus E^c) \cap \mathbb{Q}^d = f^{-1}(x). \tag{5.16}$$

Proof. For each $x \in [0,1)\cap\mathbb{Q}(\beta)$ there exists a unique $\mathbf{x} \in L \subset \mathbb{Q}^d$ such that $f(\mathbf{x}) = x$. Consider (4.26) and write $\mathbf{x} = \mathbf{x}_{s,c} + \mathbf{x}_u$, with $\mathbf{x}_{s,c} \in E^s \oplus E^c$ and $\mathbf{x}_u \in E^u$. Since

$x = f(\mathbf{x})$, we have:

$$\begin{aligned} x &= \bar{\beta} \cdot (\mathbf{x}_{s,c} + \mathbf{x}_u) \\ &= \underbrace{\bar{\beta} \cdot \mathbf{x}_{s,c}}_{0} + \bar{\beta} \cdot \mathbf{x}_u. \end{aligned} \tag{5.17}$$

On the other hand $\bar{\beta} \cdot (x\,\mathbf{s}) = x\,\underbrace{\bar{\beta} \cdot \mathbf{s}}_{1} = x$. This means that $\bar{\beta} \cdot (x\,\mathbf{s}) = \bar{\beta} \cdot \mathbf{x}_u$ and $(x\,\mathbf{s}) \in E^u$, hence $\mathbf{x}_u = x\,\mathbf{s}$ and

$$\underbrace{(\mathbf{x}_{s,c} + \mathbf{x}_u)}_{\mathbf{x}} \in (x\,\mathbf{s} + E^s \oplus E^c) \cap \mathbb{Q}^d. \tag{5.18}$$

In fact, there cannot exist other $\mathbf{v} \neq \mathbf{x}$ such that $\mathbf{v} \in (x\,\mathbf{s} + E^s \oplus E^c) \cap \mathbb{Q}^d$, because any plane parallel to $E^s \oplus E^c$ is orthogonal to $\bar{\beta} = (1, \beta, \ldots, \beta^{d-1})$, and therefore it cannot contain a non-zero vector with rational coordinates, or equivalently, it cannot contain two different points with rational coordinates. Therefore $\mathbf{x} := (x\mathbf{s} + E^s \oplus E^c) \cap \mathbb{Q}^d$ is well defined and $\mathbf{x} = f^{-1}(x)$. □

Proposition 5.3.2. *The bijection in (5.14)*

$$\begin{aligned} \Pi_\beta : \quad \widetilde{X}_\beta^+ &\longrightarrow L \\ \overline{\varepsilon} &\longmapsto \mathbf{x} := \Pi_\beta(\overline{\varepsilon}), \end{aligned} \tag{5.19}$$

can be defined as:

$$\Pi_\beta(\overline{\varepsilon}) := \left(\sum_{k=1}^{\infty} \varepsilon_k\, C^{-k}(\mathbf{s}) \;+\; E^s \oplus E^c \right) \cap \mathbb{Q}^d. \tag{5.20}$$

Proof. Let $\overline{\varepsilon} \in \widetilde{X}_\beta^+$ and $x := d_\beta(\overline{\varepsilon}) = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}$. By definition

$$\begin{aligned} \Pi_\beta(\overline{\varepsilon}) &= f^{-1} \circ d_\beta(\overline{\varepsilon}) \\ &= f^{-1}(x), \end{aligned} \tag{5.21}$$

and together with Proposition 5.3.1 we obtain

$$\Pi_\beta(\overline{\varepsilon}) = (x\,\mathbf{s} + E^s \oplus E^c) \cap \mathbb{Q}^d. \tag{5.22}$$

But $x\,\mathbf{s} = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}\,\mathbf{s} = \sum_{k=1}^{\infty} \varepsilon_k C^{-k}(\mathbf{s})$, because $\mathbf{s} \in E^u$. Replacing this in (5.22) gives (5.20). □

We can give an alternative definition for Π_β, provided that $\overline{\varepsilon} \in \widetilde{X}_\beta^+$ is an eventually periodic sequence:

Proposition 5.3.3. *Let $\overline{\varepsilon} \in \widetilde{X}_\beta^+$ be eventually periodic:*

$$\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m, \overline{\varepsilon_{m+1}, \ldots, \varepsilon_{m+p}}, \ldots). \tag{5.23}$$

Then

$$\Pi_\beta(\overline{\varepsilon}) = \sum_{1 \le k \le m} \varepsilon_k C^{-k}\,\overline{1} \;+\; (C^p - Id)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} C^{p-m-k}\,\overline{1}. \tag{5.24}$$

Proof. Let us start by writing the β-expansion of $x := \pi_\beta(\overline{\varepsilon})$:

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} = \sum_{1 \le k \le m} \frac{\varepsilon_k}{\beta^k} \;+\; \frac{1}{\beta^m}\left(\sum_{k=1}^{\infty} \frac{\varepsilon_{m+k}}{\beta^k} \right). \tag{5.25}$$

The infinite sum in (5.25) can be simplified:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\varepsilon_{m+k}}{\beta^k} &= \left(\frac{\varepsilon_{m+1}}{\beta} + \cdots + \frac{\varepsilon_{m+p}}{\beta^p} \right) \;+\; \left(\frac{\varepsilon_{m+1}}{\beta^{p+1}} + \cdots + \frac{\varepsilon_{m+p}}{\beta^{2p}} \right) \;+\; \cdots \\
&= \left(\frac{\varepsilon_{m+1}}{\beta} + \cdots + \frac{\varepsilon_{m+p}}{\beta^p} \right) \sum_{n=0}^{\infty} \left(\frac{1}{\beta^p} \right)^n \\
&= \left(\sum_{k=1}^{p} \frac{\varepsilon_{m+k}}{\beta^k} \right) \frac{\beta^p}{\beta^p - 1} = (\beta^p - 1)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} \beta^{p-k}.
\end{aligned} \tag{5.26}$$

If we replace (5.26) in (5.25), we have:

$$\begin{aligned}
x &= \sum_{1 \le k \le m} \frac{\varepsilon_k}{\beta^k} \;+\; \frac{1}{\beta^m}\Big((\beta^p - 1)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} \beta^{p-k} \Big) \\
&= \sum_{k=1}^{m} \frac{\varepsilon_k}{\beta^k} \;+\; (\beta^p - 1)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} \beta^{p-m-k}.
\end{aligned} \tag{5.27}$$

We can use the field isomorphism $F : \mathbb{Q}(\beta) \to \mathbb{Q}(C)$ to obtain:

$$\begin{aligned}
F(x) &= F\left(\sum_{1 \le k \le m} \frac{\varepsilon_k}{\beta^k} \right) \;+\; F\big((\beta^p - 1)^{-1} \big)\, F\left(\sum_{k=1}^{p} \varepsilon_{m+k} \beta^{p-m-k} \right) \\
&= \sum_{1 \le k \le m} \varepsilon_k C^{-k} \;+\; (C^p - Id)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} C^{p-m-k}.
\end{aligned} \tag{5.28}$$

Finally, we recall that $\Pi_\beta(\overline{\varepsilon}) = \mathbf{x}$ and $\mathbf{x} = f^{-1}(x) = \psi \circ F(x)$, therefore:

$$\Pi_\beta(\overline{\varepsilon}) = \sum_{1\le k\le m} \varepsilon_k C^{-k}\,\overline{1} \;+\; (C^p - Id)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} C^{p-m-k}\,\overline{1}. \tag{5.29}$$

□

Corollary 5.3.4. *If $\mathbf{x} \in L$ is strictly p-periodic, then there exists a unique strictly p-periodic sequence $\overline{\varepsilon} := (\overline{\varepsilon_1, \ldots, \varepsilon_p}, \ldots) \in \widetilde{X}^+_\beta$, such that $\mathbf{x} := \Pi_\beta(\overline{\varepsilon})$ and:*

$$\begin{aligned}\Pi_\beta(\overline{\varepsilon}) &= (C^p - Id)^{-1} \sum_{k=1}^{p} \varepsilon_k C^{p-k}\overline{1} &(5.30)\\ &= (C^p - Id)^{-1} \begin{pmatrix} | & & | & | \\ C^{p-1}\overline{1} & \cdots & C\overline{1} & \overline{1} \\ | & & | & | \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_p \end{pmatrix}. &(5.31)\end{aligned}$$

Given that in the *Pisot* case, every $\mathbf{x} \in L$ is eventually periodic, we have an explicit definition of Π_β for every $\overline{\varepsilon} \in \widetilde{X}^+_\beta$. However, in the *Salem* case Proposition 5.3.3 may not always be useful (if there exist non-periodic sequences in $\widetilde{X}^+_\beta$).

5.4 Strictly periodic orbits

If $\mathbf{x} \in L$ is eventually p-periodic, there exists a unique eventually p-periodic sequence $\overline{\varepsilon} \in \widetilde{X}^+_\beta$, such that $\Pi_\beta(\overline{\varepsilon}) = \mathbf{x}$. Together with Proposition 5.3.3 we can say that if $\mathbf{x} \in L$ is eventually p-periodic, then $\mathbf{x} \in (C^p - Id)^{-1}(\mathbb{Z}^d)$:

Proposition 5.4.1. *Let $Per_p(\widetilde{C})$ be the set of eventually p-periodic points for $\widetilde{C}$. Then*

$$Per_p(\widetilde{C}) \;\subseteq\; L \cap (C^p - Id)^{-1}(\mathbb{Z}^d)\,. \tag{5.32}$$

Proof. Let $\mathbf{x} \in Per_p(\widetilde{C})$, $\overline{\varepsilon} \in \widetilde{X}^+_\beta$ and $\Pi_\beta(\overline{\varepsilon}) = \mathbf{x}$. Proposition 5.3.3 allows us to write $\mathbf{x}$ as in (5.24), which contains two terms:

$$\begin{aligned}\sum_{1\le k\le m} \varepsilon_k C^{-k}\,\overline{1} \quad &\in \quad \mathbb{Z}^d \subset (C^p - Id)^{-1}(\mathbb{Z}^d)\\ (C^p - Id)^{-1} \sum_{k=1}^{p} \varepsilon_{m+k} C^{p-m-k}\,\overline{1} \quad &\in \quad (C^p - Id)^{-1}(\mathbb{Z}^d), &(5.33)\end{aligned}$$

therefore $\mathbf{x} \in (C^p - Id)^{-1}(\mathbb{Z}^d)$. $\square$

If $\mathbf{x} \in Per(\widetilde{C}) \cap (C^p - Id)^{-1}(\mathbb{Z}^d)$, it does not necessarily follow that $\mathbf{x}$ is eventually p-periodic. However, there must exist $k \in \mathbb{N}$ such that $\mathbf{x}$ is kp-periodic, because $\pi(\mathbf{x})$ is p-periodic for $\overline{C}$.

Proposition 5.4.2. *For each $p \in \mathbb{N}$, let $P_p := P \cap (C^p - Id)^{-1}(\mathbb{Z}^d)$. Then*

$$\forall \mathbf{x} \in Per_p(\widetilde{C}) \quad \exists\, m \in \mathbb{N}_0 \quad : \quad \widetilde{C}^m(\mathbf{x}) \in P_p. \tag{5.34}$$

Furthermore the set of strictly periodic points is

$$P = \bigcup_{p=1}^{\infty} P_p. \tag{5.35}$$

In the Pisot *case, P_p contains a finite number of strictly p-periodic $\widetilde{C}$-orbits, and therefore P_p is a finite set. Moreover, P is a bounded set.*

Conjecture 5.4.3. *Consider the previous Proposition. If β is* Salem *, then P_p is an infinite and unbounded set. Furthermore, P is unbounded.*

We propose this conjecture, taking into account the computer simulations that we have carried out for a particular *Salem* case in Chapter 6.3 and Appendix A.

5.5 Correspondence between orbits

In this section, we will mention some of the differences between the *Pisot* and the *Salem* cases, concerning the set P and the projection $\pi : P \to \mathbb{Q}^d/\mathbb{Z}^d$.

The restriction of $\widetilde{C}$ to P is invertible. Furthermore, we have the following:

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{C}} & P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{Q}^d/\mathbb{Z}^d & \xrightarrow[\overline{C}]{} & \mathbb{Q}^d/\mathbb{Z}^d \end{array} \tag{5.36}$$

Proposition 5.5.1. *If β is* Pisot *(with $c_0 = \pm 1$) then $\pi : P \to \mathbb{Q}^d/\mathbb{Z}^d$ is surjective.*

Proof. If β if *Pisot* , then $Per(\widetilde{C}) = L$ and $\forall \mathbf{x} \in L \; \exists_{k \in \mathbb{N}} \; : \widetilde{C}^k(\mathbf{x}) \in P$. In order to prove that $\pi : P \to \mathbb{Q}^d/\mathbb{Z}^d$ is surjective, let us choose any $(\mathbf{x} + \mathbb{Z}^d) \in \mathbb{Q}^d/\mathbb{Z}^d$. Define $\mathbf{x}' := \mathbf{x} - [\bar{\beta} \cdot \mathbf{x}]\bar{1}$, and note that $\mathbf{x}' \in L$ and $\pi(\mathbf{x}') = \mathbf{x} + \mathbb{Z}^d$. If $\mathbf{x}' \in P$, then we have found a pre-image for $(\mathbf{x} + \mathbb{Z}^d)$. If $\mathbf{x}' \in E$ and has pre-period m, then $\mathbf{x}'' := \widetilde{C}^m(\mathbf{x}') \in P$. Since the $\widetilde{C}$-orbit of $\mathbf{x}''$ in P is strictly periodic and is factored by π to the $\overline{C}$-orbit of $\mathbf{x} + \mathbb{Z}^d$, there exists $i \in \mathbb{N}_0$ such that $\pi \circ \widetilde{C}^i(\mathbf{x}'') = \mathbf{x} + \mathbb{Z}^d$, and $\widetilde{C}^i(\mathbf{x}'') \in P$. ☐

If β is *Salem* , we cannot say whether or not $\pi : P \to \mathbb{Q}^d/\mathbb{Z}^d$ is surjective: if Schmidt's Conjecture 3.4.8 is true, then Proposition 5.5.1 would also be true for any *Salem* number. However, if Conjecture 3.4.8 is false, there could exist some $(\mathbf{x} + \mathbb{Z}^d) \in \mathbb{Q}^d/\mathbb{Z}^d$ such that any pre-image $\pi^{-1}(\mathbf{x} + \mathbb{Z}^d)$ is non-periodic, and therefore $\pi^{-1}(\mathbf{x} + \mathbb{Z}^d) \cap P = \varnothing$.

Definition 5.5.2. *Define the set of strictly periodic points with integer coordinates for $\widetilde{C}$ as:*

$$Z := P \cap \mathbb{Z}^d. \tag{5.37}$$

If the β-expansion of 1 is eventually periodic (this is always the case if β is *Pisot* , and it is not known for the *Salem* case) with pre-period m and period p, then the strictly periodic component of the orbit of $\bar{1}$ belongs to Z:

$$\left\{\widetilde{C}^n(\bar{1})\right\}_{m \leq n < m+p} \subseteq Z. \tag{5.38}$$

Furthermore, in the *Pisot* case Z must be a finite set, because there exist only a finite number of points with integer coordinates in the bounded set P. We conjecture that in the *Salem* case Z contains an infinite number of points, based on the computations of the example in Chapter 6.3 and Appendix A, in which $Z = \pi^{-1}(\mathbf{v}_0 + \mathbb{Z}^4)$.

5.6 Counting periodic points

The description of the periodic points for a toral automorphism is a simple exercise:

Proposition 5.6.1. *The set of p-periodic points for $(\overline{C}, \mathbb{Q}^d/\mathbb{Z}^d)$ is:*

$$\begin{aligned} Per_p(\overline{C}) &:= \Big\{(\mathbf{x}+\mathbb{Z}^d) \in \mathbb{Q}^d/\mathbb{Z}^d \ \mid\ \overline{C}^p(\mathbf{x}+\mathbb{Z}^d) = (\mathbf{x}+\mathbb{Z}^d)\Big\} \\ &= (C^p - Id)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d. \end{aligned} \tag{5.39}$$

Proof. According to the definitions, $(\mathbf{x}+\mathbb{Z}^d) \in \mathbb{Q}^d/\mathbb{Z}^d$ is a p-periodic point for the toral automorphism $\overline{C}$ if and only if $\overline{C}^p(\mathbf{x}+\mathbb{Z}^d) = (\mathbf{x}+\mathbb{Z}^d)$. This is equivalent to $(C^p\mathbf{x}+\mathbb{Z}^d) = (\mathbf{x}+\mathbb{Z}^d)$ and also $(C^p - Id)\mathbf{x} \in \mathbb{Z}^d$. Since $\det(C^p - Id) \neq 0$, $(C^p - Id)$ is invertible and $(C^p - Id)\mathbf{x} \in \mathbb{Z}^d$ is equivalent to $\mathbf{x} \in (C^p - Id)^{-1}(\mathbb{Z}^d)$, and therefore $Per_p(\overline{C}) = (C^p - Id)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d$. □

Corollary 5.6.2. *The number of p-periodic points for the toral automorphism $\overline{C}$ is:*

$$\big|Per_p(\overline{C})\big| = |\det(C^p - Id)| \tag{5.40}$$

$$= \prod_{i=1}^{d} |\theta_i^p - 1|, \tag{5.41}$$

where θ_i are the roots of the minimal polynomial (3.2) of β.

Proof. The set of p-periodic points for the toral automorphism is the finite subgroup of the torus:

$$(C^p - Id)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d, \tag{5.42}$$

which is isomorphic to

$$\mathbb{Z}^d/(C^p - Id)(\mathbb{Z}^d), \tag{5.43}$$

and the number of points of this subgroup is $|\det(C^p - Id)|$. Now C^p has eigenvalues $\{\theta_i^p\}_{1\leq i\leq d}$, so $(C^p - I)$ has eigenvalues $\{\theta_i^p - 1\}_{1\leq i\leq d}$ and therefore

$$|\det(C^p - I)| = \prod_{i=1}^{d} |\theta_i^p - 1|. \tag{5.44}$$

□

Since $\pi : L \to \mathbb{Q}^d/\mathbb{Z}^d$ factors the non-invertible dynamical system $(\widetilde{C}, L)$ to $(\overline{C}, \mathbb{Q}^d/\mathbb{Z}^d)$, it can either preserve the period of the periodic component of the eventually periodic $\widetilde{C}$-orbit, or factor that period. π maps an eventually p-periodic $\widetilde{C}$-orbit to a $\overline{C}$-orbit which is also p-periodic (though p isn't necessarily the minimum period).

Since π maps p-periodic points under the map $\widetilde{C}$ into p-periodic points under $\overline{C}$, it is immediate that $Per_p(\widetilde{C}) \subset \big(L \cap (C^p - Id)^{-1}(\mathbb{Z}^d)\big)$.

Corollary 5.6.3. *For each $p \in \mathbb{N}$, we have an upper bound for the number of strictly p-periodic points for $\widetilde{C}$:*

$$|Per_p(\widetilde{C})| < ([\beta] + 1)^p \tag{5.45}$$

Proof. There is a bijection between the strictly p-periodic sequences in $\widetilde{X}^+_\beta$ and the strictly p-periodic points for $\widetilde{C}$. But $\widetilde{X}^+_\beta \subset \{0, 1, \ldots, [\beta]\}^{\mathbb{N}}$, where there exist $([\beta]+1)^p$ p-periodic sequences with $[\beta]+1$ symbols, and this is an upper bound for $|Per_p(\widetilde{C})|$. ☐

The growth of $|Per_p(\overline{C})| = |\det(C^p - Id)| = \prod_{i=1}^d |\theta_i^p - 1|$. And in the *Pisot* case, this is $\prod_{i=1}^d |\theta_i^p - 1| \approx \beta^p$.

In the *Salem* case, this product has an irregular pattern, because some times the powers of the roots of modulus 1 are close to 1, therefore $|\theta_i^p - 1|$ is small.

5.7 Schmidt's proof translated into the notation $(\widetilde{C}, L)$

In this section, we rewrite some of the important results in [Sch80] into our notation.

Lemma 5.7.1. *Let $\mathbf{x} \in L \subset \mathbb{Q}^d$. The following conditions are equivalent:*

1. $\mathbf{x} \in Per(\widetilde{C})$
2. $\exists_{K_1>0},\ \forall_{n\in\mathbb{N}},\ |\widetilde{C}^n(\mathbf{x})| < K_1$
3. $\exists_{K_2>0},\ \forall_{1\le i\le d},\ \forall_{n\in\mathbb{N}},\ \ |(1, \theta_i, \ldots, \theta_i^{d-1}) \cdot \widetilde{C}^n(\mathbf{x})| < K_2$

Proof. If $\mathbf{x} \in Per(\widetilde{C})$ then $\{\widetilde{C}^n(\mathbf{x})\}_{n\in\mathbb{N}}$ is a finite, and therefore bounded set. This proves that $(1) \Rightarrow (2)$.

Suppose $\mathbf{x} \in L \subset \mathbb{Q}^d$ has rational coordinates with a minimum positive common denominator $q > 0$. The minimum positive denominator of $\widetilde{C}^n(\mathbf{x})$ cannot be greater than q, therefore the $\widetilde{C}$-orbit of $\mathbf{x}$ belongs to a lattice in $\mathbb{Q}^d$ defined by q. If (2) holds, then the $\widetilde{C}$-orbit of $\mathbf{x}$ is bounded, and since it belongs to a lattice, it must be eventually periodic: $\mathbf{x} \in Per(\widetilde{C})$. This proves that $(2) \Rightarrow (1)$.

There exists some $B > 0$ such that $\forall_{1 \le i \le d}$, we have $|(1, \theta_i, \ldots, \theta_i^{d-1})| < B$, therefore if (2) holds, then (3) also holds. To see that (3) implies (2), we write:

$$\begin{pmatrix} x_{\theta_1}^{(n)} \\ x_{\theta_2}^{(n)} \\ \vdots \\ x_{\theta_d}^{(n)} \end{pmatrix} = \begin{pmatrix} 1 & \theta_1 & \ldots & \theta_1^{d-1} \\ 1 & \theta_2 & \ldots & \theta_2^{d-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \theta_d & \ldots & \theta_d^{d-1} \end{pmatrix} \widetilde{C}^n(\mathbf{x}) \tag{5.46}$$

But the matrix $\{\theta_i^{j-1}\}_{1 \le i \le d}$ is invertible, therefore we can multiply (5.46) by the inverse of the matrix and express $\widetilde{C}^n(\mathbf{x})$ as the product of a matrix by a bounded quantity. Therefore (3) implies (2).

□

The Theorem 3.4.7 becomes:

Theorem 5.7.2. *Let β be a* Pisot *number. Then $Per(\widetilde{C}) = L$.*

Finally, Conjecture 3.4.8 becomes:

Conjecture 5.7.3. *Let β be a* Salem *number. Then $Per(\widetilde{C}) = L$.*

Chapter 6

Examples and computational results

6.1 Introduction

The main problem which served as motivation for our results was the description of $Per(T_\beta)$, which is a subset of $[0,1) \cap \mathbb{Q}(\beta)$, as we have seen in Proposition 3.4.3. If β is a *Pisot* number then Theorem 3.4.7 gives a complete description of $Per(T_\beta)$. The Conjecture 3.4.8 which was first suggested in [Sch80], claims that if β is a *Salem* number then the description of $Per(T_\beta)$ is the same as for the *Pisot* case.

So far, Conjecture 3.4.8 remains to be proved or disproved. [Boy89], [Boy96] and [Boy97] considers whether or not *Salem* numbers satisfy a weaker property than Conjecture 3.4.8. These works are based on explicit computation of the orbits of certain points, and also on statistical heuristic arguments, which according to Boyd suggest that Conjecture 3.4.8 should be false. We will summarize the main results and implications of [Boy89], [Boy96] and [Boy97].

For any $x \in [0,1) \cap \mathbb{Q}(\beta)$, the explicit computation of $\{T_\beta^n(x) \mid 0 < n < k\}$ can be implemented with the help of a mathematical package (for instance, Maple™). These computations can be done with absolute precision, because we can represent

$x \in [0,1) \cap \mathbb{Q}(\beta)$ by $\mathbf{x} \in L \subset \mathbb{Q}^d$, and compute the corresponding $\widetilde{C}$-orbit. But in such case, we are dealing with points in $\mathbb{Q}^d$, and therefore our computations will be symbolic, rather than numerical approximations. We will show some computational results that we have obtained for the *Salem* case.

The explicit computation of T_β-orbits of points $x \in [0,1) \cap \mathbb{Q}(\beta)$ cannot be used to prove Conjecture 3.4.8, because that would require explicitly computing infinitely many orbits. On the other hand, if Conjecture 3.4.8 was false, and for a given *Salem* number β we wanted to prove that the orbit of some $x \in [0,1) \cap \mathbb{Q}(\beta)$ was infinite, then explicit computation of that orbit would be of no use for proving that the orbit was infinite.

6.2 Boyd's results

The approach of [Boy89], [Boy96] and [Boy97] studies a particular implication of Conjecture 3.4.8:

Proposition 6.2.1. *If $Per(T_\beta) = [0,1) \cap \mathbb{Q}(\beta)$ then the β-expansion of 1 is eventually periodic.*

Proof. Given that

$$1 = \frac{[\beta] + \{\beta\}}{\beta}, \tag{6.1}$$

the β-expansion of 1 is:

$$d_\beta(1) = ([\beta], \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots), \tag{6.2}$$

where $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) := d_\beta(\{\beta\})$. But $\{\beta\} \in [0,1) \cap \mathbb{Q}(\beta)$, therefore it is eventually periodic. □

If the β-expansion of 1 is eventually periodic (or equivalently, if $\{T_\beta^n(1) \mid n \in \mathbb{N}\}$ is finite) then we say that β is a beta-number (this definition was introduced in [Par60]). According to Theorem 3.4.7, all *Pisot* numbers are beta-numbers, and if Conjecture 3.4.8 is true, then all *Salem* numbers would also be beta-numbers.

In [Boy89], it was proved that every *Salem* number of degree 4 is a beta-number. If β is a *Salem* number of degree 4, then its minimal polynomial is a reciprocal polynomial of degree 4, which is determined by two coefficients $a, b \in \mathbb{Z}$ (not every pair $(a, b) \in \mathbb{Z}^2$ defines a *Salem* number, though):

$$p(x) = x^4 + ax^3 + bx^2 + ax + 1. \tag{6.3}$$

[Boy89] proved that for fixed a, the period of the orbit of 1 is bounded by $2|a|$. However, this does not prove 3.4.8 for *Salem* numbers of degree 4. It merely proves that such *Salem* numbers are beta-numbers, or in other words, that the T_β-orbit of 1 is eventually periodic.

[Boy96] deals with the same problem for *Salem* numbers of degree 6. Extensive computations were carried out, trying to compute the T_β-orbits of 1 for several hundreds of different *Salem* numbers. For some *Salem* numbers, the complete orbit was found to be finite, whereas in other cases, it wasn't possible to determine whether or not the orbit was finite (given that after several thousands of iterations, a periodic pattern was not obtained). It was claimed that for such cases, this did not mean that the T_β-orbit of 1 was infinite, but possibly simply its period was too big.

[Boy96] also developed a probabilistic model for the distribution of the T_β-orbit of 1 for *Salem* numbers of any degree d (which is necessarily even and $d \geq 4$). It was claimed that this heuristic probabilistic argument provided numerical data supporting the conjecture that all *Salem* numbers of degree 6 should be beta-numbers, but the same should not hold for *Salem* numbers of degree higher than 6. This contradicts Schmidt's Conjecture 3.4.8.

6.3 Salem example of degree 4

We would like to observe different aspects in the dynamical behaviour of $\widetilde{C}$-orbits in the *Salem* case (in comparison with the *Pisot* case, which is better understood). One of the immediate problems we face, is the dimension of the space $\mathbb{Q}^d$ which embeds L, because *Salem* numbers have a minimal polynomial of degree $d \geq 4$. If we want to plot the orbits, we have to project them into a subspace of dimension less than 4. We will choose to project them into the 2-dimensional subspace E^c (which is the eigenspace of C corresponding to the eigenvalues of modulus 1).

Consider the *Salem* number β of degree 4 defined by the minimal polynomial

$$p(x) = x^4 - 10x^3 - 10x + 1. \tag{6.4}$$

The companion matrix for $p(x)$ is:

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \end{pmatrix}. \tag{6.5}$$

We can work out a numerical approximation for $\beta \simeq 10.097114224488$, and therefore $[\beta] = 10$ and $\{\beta\} \simeq 0.097114224488$. The β-expansion of 1 is

$$d_\beta(1) = (10, 0, 9, 9, 0, 9, 9, \ldots) = (10, \overline{0, 9, 9}, \ldots). \tag{6.6}$$

We shall be considering the following diagram:

$$\begin{array}{ccc} \widetilde{X}_\beta^+ & \xrightarrow{\sigma} & \widetilde{X}_\beta^+ \\ \Big\downarrow{\scriptstyle \Pi_\beta} & & \Big\downarrow{\scriptstyle \Pi_\beta} \\ L & \xrightarrow[\tilde{C}]{} & L \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{Q}^d/\mathbb{Z}^d & \xrightarrow[\overline{C}]{} & \mathbb{Q}^d/\mathbb{Z}^d \end{array} \tag{6.7}$$

We recall that $\Pi_\beta : \widetilde{X}_\beta^+ \to L$ maps the β-expansion of each $x \in [0, 1) \cap \mathbb{Q}(\beta)$ to $\mathbf{x} \in L \subset \mathbb{Q}^d$, such that $\mathbf{x} := f^{-1}(x)$.

We consider the fixed points for the (non-hyperbolic) toral automorphism $\overline{C}$:

$$Fix(\overline{C}) = Per_1(\overline{C}) := (C - Id)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d. \tag{6.8}$$

We can compute:

$$(C-Id)^{-1} = \frac{1}{18}\begin{pmatrix} -19 & -1 & -1 & -1 \\ -9 & -9 & 9 & 9 \\ -9 & -9 & -9 & 9 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6.9}$$

and $|Fix(\overline{C})| = |C - Id| = 18$. We can think of $Fix(\overline{C})$ as a finite subgroup of $\mathbb{T}^d$, in this case of order 18.

Let us define $\mathbf{v}_1 := \frac{1}{18}(-19,-9,-9,1)^t$, and $\forall_{0\leq k<18}\ \ \mathbf{v}_k := k\mathbf{v}_1$.

Proposition 6.3.1. *$(C-Id)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d$ is cyclic subgroup of $\mathbb{T}^d$. We can choose $\mathbf{v}_1+\mathbb{Z}^d$ as a generator, and we have:*

$$Fix(\overline{C}) = (C-Id)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d = \big\{(k\mathbf{v}_1)+\mathbb{Z}^d \mid 0 \leq k < 18\big\}. \tag{6.10}$$

Proof. If suffices to check that for any i, j such that $0 < i < j < 18$, $\mathbf{v}_j - \mathbf{v}_i = \mathbf{v}_{j-i} \notin \mathbb{Z}^d$, therefore $\mathbf{v}_j + \mathbb{Z}^d$ and $\mathbf{v}_i + \mathbb{Z}^d$ are different elements in $\mathbb{T}^d$. Furthermore, $18\mathbf{v}_1 \in \mathbb{Z}^d$ (as we would expect, because the order of the subgroup is 18), therefore $\mathbf{v}_1 + \mathbb{Z}^d$ is a generator for $Fix(\overline{C})$. □

On the other hand, we can also enumerate every fixed point for $\widetilde{C}$. Each $\mathbf{x} \in Fix(\widetilde{C}) = Per_1(\widetilde{C})$ can be enumerated using the bijection $\Pi_\beta(\overline{\varepsilon})$, given that $\overline{\varepsilon} \in \widetilde{X}_\beta^+$ and $\sigma(\overline{\varepsilon}) = \overline{\varepsilon}$. Since $\widetilde{X}_\beta^+ \subset \Sigma_{[\beta]}^+$, there are at the most $[\beta]$ different sequences in $\widetilde{X}_\beta^+$ with period 1. If we define $\overline{\varepsilon} := (\overline{\varepsilon_1},\ldots)$ with $\varepsilon_1 \in \{0,1,\ldots,([\beta]-1)\}$ we can use (5.30):

$$\begin{aligned} \Pi_\beta(\overline{\varepsilon}) &= (C-Id)^{-1}\varepsilon_1\overline{1} \\ &= \varepsilon_1\mathbf{v}_1. \end{aligned} \tag{6.11}$$

In our case $0 \leq \varepsilon_1 = k < 10$, and therefore there exist 10 fixed points for $\widetilde{C}$. If we compare (6.11) and (6.10), we immediately see that:

$$\pi(\varepsilon_1\mathbf{v}_1) = k\mathbf{v}_1 + \mathbb{Z}^d = \mathbf{v}_k + \mathbb{Z}^d, \tag{6.12}$$

and therefore each of the 10 fixed points for $\widetilde{C}$ is mapped into a distinct fixed point for $\overline{C}$. But there exist 8 additional fixed points for $\overline{C}$, namely $\{\mathbf{v}_k + \mathbb{Z}^d \mid 10 \leq k < 18\}$ which don't have a pre-image in L which is 1-periodic.

If Conjecture 3.4.8 is true, then any point $\mathbf{x}+\mathbb{Z}^d \in \mathbb{Q}^d/\mathbb{Z}^d$ has at least a strictly periodic lift $\pi^{-1}(\mathbf{x}+\mathbb{Z}^d)$. In our case, we would like to investigate which strictly periodic images do the fixed points admit.

Each $\mathbf{v}_j + \mathbb{Z}^d$ (for $0 \leq j < 18$) has infinitely many pre-images in L:

$$\pi^{-1}\big(\mathbf{v}_j + \mathbb{Z}^d\big) = L \cap (\mathbf{v}_j + \mathbb{Z}^d), \tag{6.13}$$

though we are interested in those pre-images which are strictly-periodic:

$$\pi^{-1}\big(\mathbf{v}_j + \mathbb{Z}^d\big) \cap P. \tag{6.14}$$

6.3.1 Computing pre-images within a bounded region with Maple

We consider the factor map $\pi : L \to \mathbb{Q}^d/\mathbb{Z}^d$, such that $\pi(\mathbf{x}) := \mathbf{x} + \mathbb{Z}^d$. In what follows, when we mention a *pre-image* of a point $\mathbf{x}+\mathbb{Z}^d$, we mean any $\mathbf{v} \in L$ such that $\pi(\mathbf{v}) = \mathbf{x} + \mathbb{Z}^d$, or in other words, such that $(\mathbf{v} - \mathbf{x}) \in \mathbb{Z}^d$.

For any $M \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{Q}^d$, define:

$$B_M(\mathbf{x}) := \{\mathbf{x} + \mathbf{n} \mid \mathbf{n} := (n_1, \ldots, n_d) \in \mathbb{N}^d \text{ and } \forall_{1\leq i\leq d} \ \ |n_i| \leq M\}. \tag{6.15}$$

For any $\mathbf{x} + \mathbb{Z}^d \in \mathbb{Q}^d/\mathbb{Z}^d$, if we choose some $M \in \mathbb{N}$ we define a finite set of pre-images

$$B_M(\mathbf{x}) \cap L \subset \pi^{-1}(\mathbf{x} + \mathbb{Z}^d). \tag{6.16}$$

If $\mathbf{x} \in \mathbb{Q}^d$ is small compared to $M \in \mathbb{N}$, we can interpret $B_M(\mathbf{x}) \cap L$ as the set of pre-images of $\mathbf{x} + \mathbb{Z}^d$ which have maximum (infinity) norm approximately less than M.

6.3.2 Pre-images of $\mathbf{v}_0 + \mathbb{Z}^4$

Let us choose a fixed point for $\overline{C}$, for instance:

$$\mathbf{v}_0 + \mathbb{Z}^4 \ \in \mathbb{Q}^4/\mathbb{Z}^4. \tag{6.17}$$

We define how big the bounded region is by setting $M = 100$, and compute the finite set of pre-images $B_{100}(\mathbf{v}_0) \cap L$ with Maple Routine 1.

Given this set of pre-images of $\mathbf{v}_0 + \mathbb{Z}^4$, we want to consider only those which are strictly $\widetilde{C}$-periodic, and we can do that using Routine 2 to compute the list

6.3.3 Generating projections of the orbits

We have computed a bounded $\widetilde{C}$-invariant set of pre-images of the fixed points for $\overline{C}$. The computation of the orbits is a straightforward procedure with the help of Maple. But if we want to display those orbits, we need to project them into a subspace of dimension less than 4. We will define a change of basis for $\mathbb{R}^4$ as follows.

$$E^u = \langle \ \underbrace{(-1, \beta^3 - 10\beta^2, \beta^2 - 10\beta, \beta)^t}_{\mathbf{v}_u} \ \rangle \tag{6.18}$$

because $C\mathbf{v}_u = \beta\mathbf{v}_u$. We can obtain a stable eigenvalue for C replacing β by β^{-1} in (6.18) and simplifying the expression using the equation $p(\beta) = 0$. We obtain, for instance:

$$E^s = \langle \ \underbrace{(\beta^3, 10\beta - 1, 10\beta^2 - \beta, -\beta^2)^t}_{\mathbf{v}_s} \ \rangle, \tag{6.19}$$

and we can confirm that $C\mathbf{v}_s = \beta^{-1}\mathbf{v}_s$.

In order to find two generators for E^c, we proceeded in the following way:

$$E^s \oplus E^c = \langle \ (\beta, -1, 0, 0)^t \ , \ (0, \beta, -1, 0)^t \ , \ (0, 0, \beta, -1)^t \ \rangle, \tag{6.20}$$

because these three vectors are linearly independent and orthogonal to $\bar{\beta}$.

Consider the change of basis defined by the matrix:

$$S' = \begin{pmatrix} \beta & 0 & 0 & -1 \\ -1 & \beta & 0 & \beta^3 - 10\beta^2 \\ 0 & -1 & \beta & \beta^2 - 10\beta \\ 0 & 0 & -1 & \beta \end{pmatrix}, \tag{6.21}$$

where the first three column vectors generate $E^s \oplus E^c$, whereas the last one generates E^u. We have numerically iterated by the linear map C a point $\mathbf{x} \in E^s \oplus E^c$. It approximates E^c and we considered the first three coordinates in the basis defined by S'. The external product of the vectors consisting of these three coordinates for consecutive C-iterates of $\mathbf{x}$ converges to a vector in $\mathbb{R}^3$ which is orthogonal to E^c. The numerical results that we obtained suggested that the orthogonal vector was $(\beta^2, \beta, 1)$, and therefore E^c should be generated, for instance, by $(1, -\beta, 0)$ and $(0, 1, -\beta)$, which in terms of the basis defined by S' corresponds to:

$$\begin{aligned} \mathbf{v}_{c1} &:= 1 \cdot (\beta, -1, 0, 0)^t - \beta(0, \beta, -1, 0)^t = (\beta, -1 - \beta^2, \beta, 0)^t, \\ \mathbf{v}_{c2} &:= 1 \cdot (0, \beta, -1, 0)^t - \beta(0, 0, \beta, -1)^t = (0, \beta, -1 - \beta^2, \beta)^t. \end{aligned} \tag{6.22}$$

This means that

$$E^c = \langle \underbrace{(\beta, -1, -\beta^3, \beta^2)^t}_{\mathbf{v}_{c1}}, \underbrace{(0, \beta, -1 - \beta^2, \beta)^t}_{\mathbf{v}_{c2}} \rangle, \tag{6.23}$$

and we can confirm that $C\mathbf{v}_{c1} = \mathbf{v}_{c2}$ and $C\mathbf{v}_{c2} = \mathbf{v}_{c1} + (\beta^3 - 10\beta^2 - \beta)\mathbf{v}_{c2}$, therefore $\langle \mathbf{v}_{c1}, \mathbf{v}_{c1} \rangle = E^c$ because this is a 2-dimensional invariant subspace orthogonal to $\bar{\beta}$.

The change of basis matrix

$$S = \begin{pmatrix} \beta & 0 & \beta^2 & -1 \\ -1 - \beta^2 & \beta & 10 - \beta^{-1} & \beta^3 - 10\beta^2 \\ \beta & -1 - \beta^2 & 10\beta - 1 & \beta^2 - 10\beta \\ 0 & \beta & -\beta & \beta \end{pmatrix} \tag{6.24}$$

contains $\mathbf{v}_{c1}, \mathbf{v}_{c2}$ in the first two columns, and therefore if we want to project a point $\mathbf{x} \in \mathbb{R}^4$ into E^c, we take the first two coordinates of $S^{-1}\mathbf{x}$. This is what we do in order to obtain the orbits shown in Appendix A, generated with the Maple code shown in Appendix B.5.

6.3.4 Interpretation of the plots

Let us consider the projections of the strictly periodic orbits that are shown in Appendix A.

The circular shapes that we observe, reflect the rotating behaviour of the orbits for the map $\widetilde{C}$. Any strictly periodic point $\mathbf{x} \in L$ has a component along the stable and unstable directions (E^s and E^u) which are bounded and rather small, as the absolute distance of the point to the origin increases. If a strictly periodic point is very far from the origin, then its component along E^c must be big. When we iterate such point by $\widetilde{C}$, its E^c-coordinates are rotated around an ellipse by an irrational angle (this corresponds to the linear map C), and subsequently translated by a variable bounded vector, whose contribution is less relevant the farther we get away from the origin. This explains why a slightly square shape can be observed for orbits which are nearer to the origin, whereas this shape becomes more evenly circular as the orbit is farther away from the origin.

The tones of gray correspond to different periods, where black corresponds to the lowest period (1) and the lightest shade of gray corresponds to the highest period.

Tables 6.1 and 6.2 show the number of strictly periodic orbits found within the bounded region (defined by the parameter $M = 100$), according to their periods and to which fixed point $\mathbf{v}_k + \mathbb{Z}^4$ for the toral automorphism it is mapped by the factor map π.

period	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$
1	1	1	1	1	1	1	1	1	1	1
3	1									1
4			2	2	6	6	2	2		
7	1	1							1	1
11	1	1	1					1	1	1
15	10	7	5	7	10	11	7	5	7	10
19	2	1		1	2	2	1		1	1
26				1	2	2	1			
30										
34	1	2	6	1	3	3	1	6	3	1
49	1	1	4	1	2	2	1	4	1	1
53	1									1
60		2							2	
64	1	1	1	1		1	1	1	1	1
79				2			2			
83										
94										
98			1					1		
109	1			1			1			1
128										
158				1					1	
total orbits $\rightarrow$	21	17	21	19	26	28	18	21	19	20

Table 6.1: Number of $\widetilde{C}$-orbits which are pre-images of $\mathbf{v}_k + \mathbb{Z}^4$

period	$\mathbf{v}_{10}$	$\mathbf{v}_{11}$	$\mathbf{v}_{12}$	$\mathbf{v}_{13}$	$\mathbf{v}_{14}$	$\mathbf{v}_{15}$	$\mathbf{v}_{16}$	$\mathbf{v}_{17}$
1								
3								
4	1	1	4	4	4	4	1	1
7	1							1
11	1	1		1	1		1	1
15	7	12	8	5	4	8	12	7
19	4	1	1	2	3	1	1	4
26				1	1			
30				1	1			
34			3	1	1	3		
49	1	3	2	1		2	3	1
53								
60	1			1	1			1
64	1	1	2	1	1	2	1	1
79	1							1
83	1		1	1	1	1		1
94				1	1			
98	1							1
109		1					1	
128		1	1			1	1	
158							1	
total orbits $\rightarrow$	20	21	22	20	19	22	22	20

Table 6.2: Number of $\widetilde{C}$-orbits which are pre-images of $\mathbf{v}_k + \mathbb{Z}^4$

Appendix A

Pictures for the Salem example

In this appendix, we present the plots generated with Maple while studying some periodic orbits of the map $\widetilde{C}$ for the *Salem* number defined by the polynomial $p(x) = x^4 - 10x^3 - 10x + 1$. Each page contains two plots: the top one shows the projection into E^c of the strictly periodic points for $\widetilde{C}$ which are pre-images of the fixed point $\mathbf{v}_k + \mathbb{Z}^4$ for the toral automorphism $\overline{C}$. We limit our plot to a bounded region in L (depending on a parameter $M = 100$), and that is the reason why the circular regions containing the points are bounded.

The plot on the bottom half of each page is related to the one above it: each strictly periodic orbit is represented by a unique point, whose horizontal coordinate is proportional to the average distance to the origin of its projection to E^c, whereas the the vertical coordinate represents the period of the orbit.

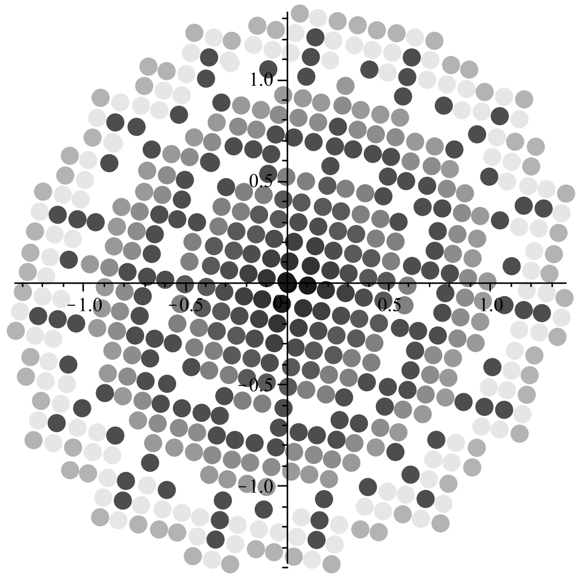


Figure A.1: Projection of pre-images of $\mathbf{v}_0 + \mathbb{Z}^4$ in E^c

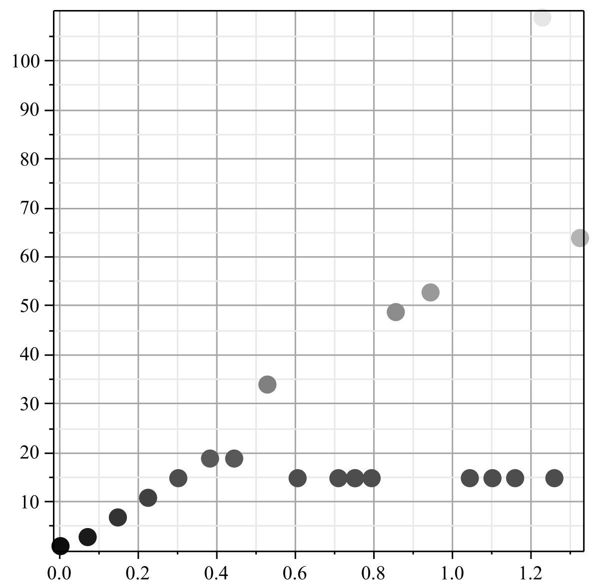


Figure A.2: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_0 + \mathbb{Z}^4)$

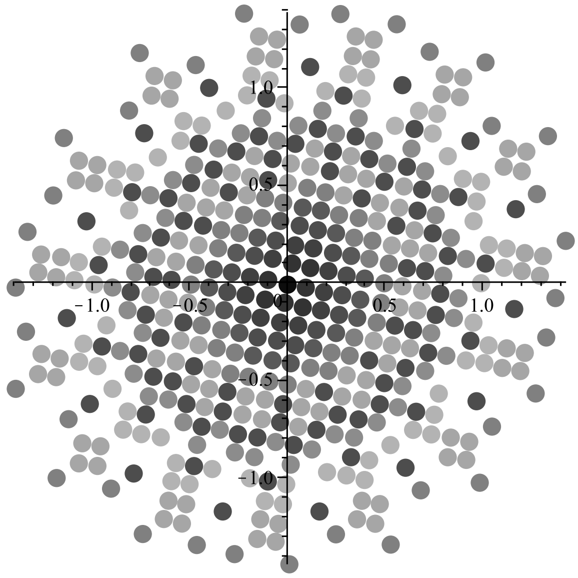


Figure A.3: Projection of pre-images of $\mathbf{v}_1 + \mathbb{Z}^4$ in E^c

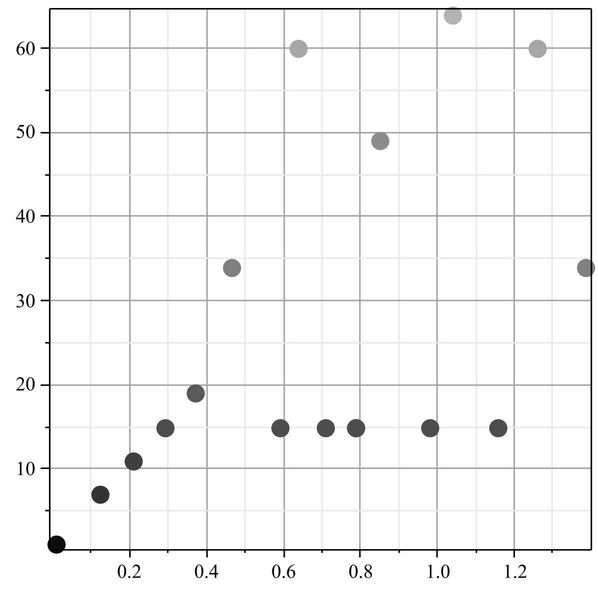


Figure A.4: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_1 + \mathbb{Z}^4)$

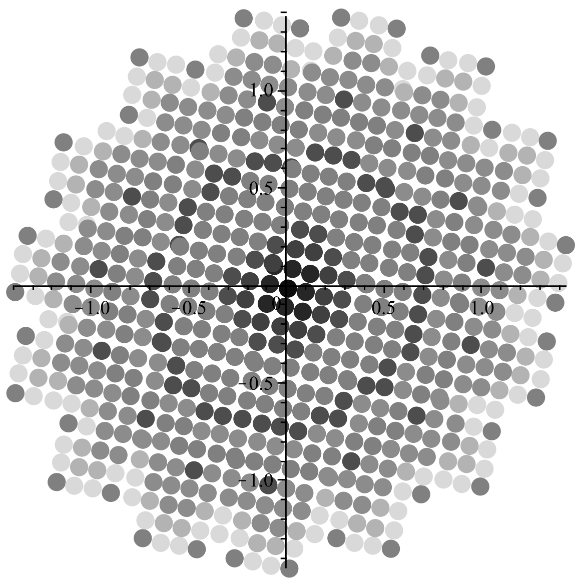


Figure A.5: Projection of pre-images of $\mathbf{v}_2 + \mathbb{Z}^4$ in E^c

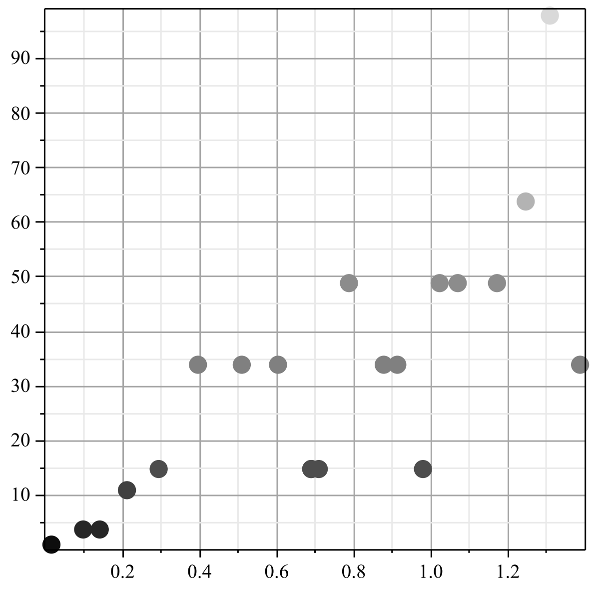


Figure A.6: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_2 + \mathbb{Z}^4)$

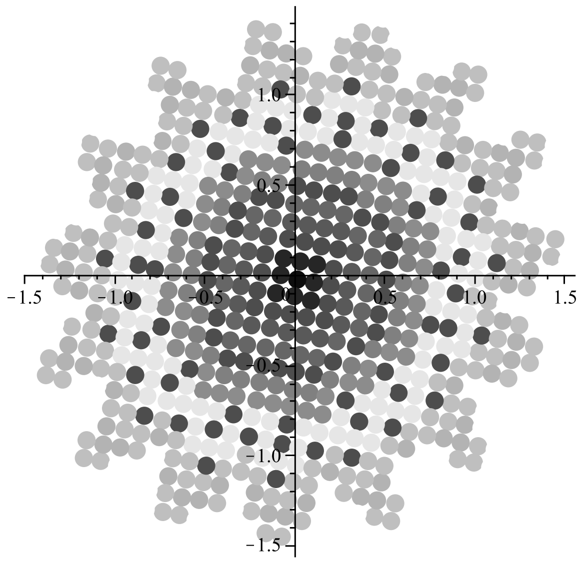


Figure A.7: Projection of pre-images of $\mathbf{v}_3 + \mathbb{Z}^4$ in E^c

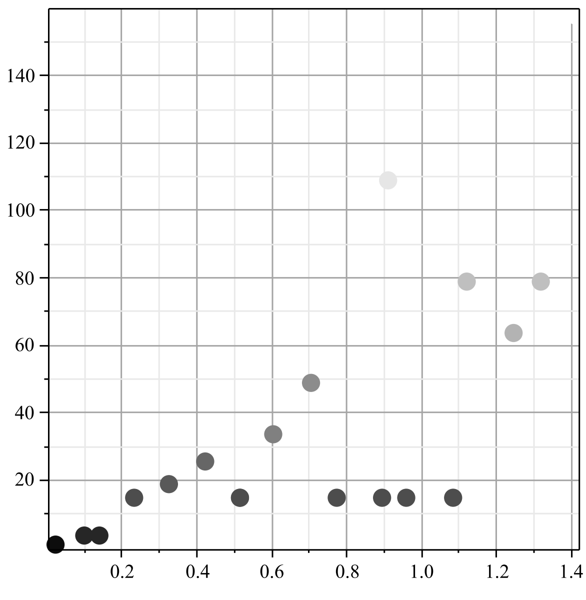


Figure A.8: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_3 + \mathbb{Z}^4)$

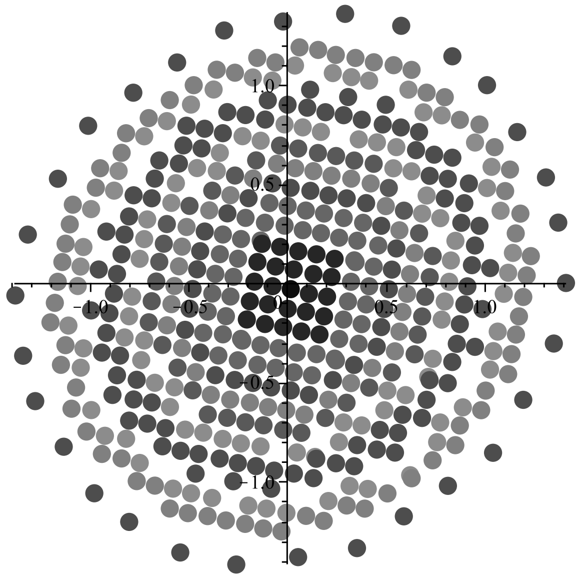


Figure A.9: Projection of pre-images of $\mathbf{v}_4 + \mathbb{Z}^4$ in E^c

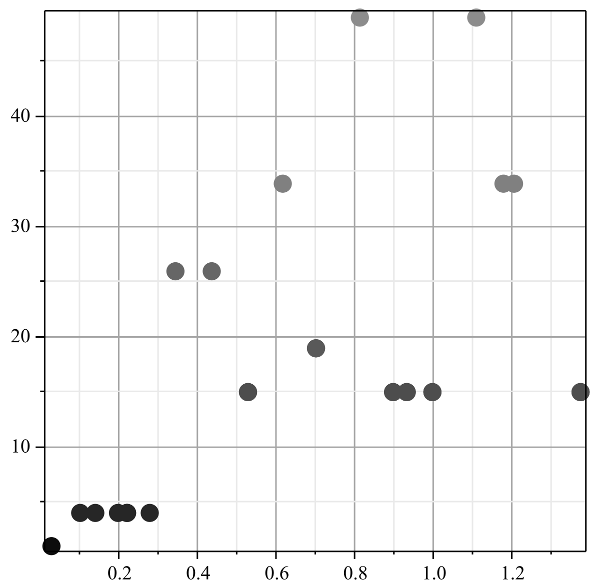


Figure A.10: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_4 + \mathbb{Z}^4)$

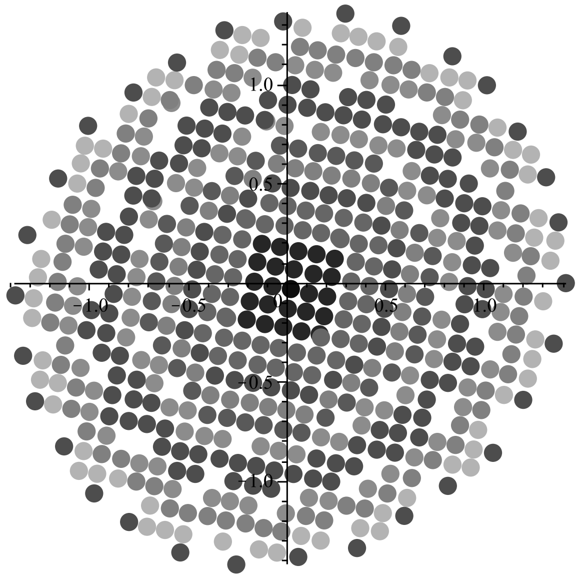


Figure A.11: Projection of pre-images of $\mathbf{v}_5 + \mathbb{Z}^4$ in E^c

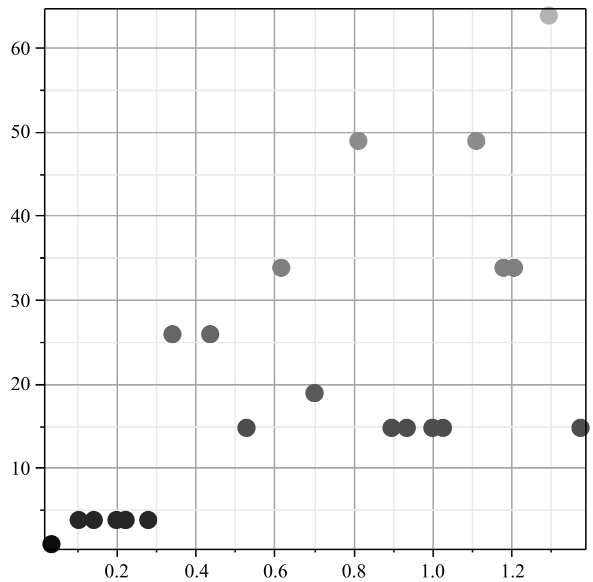


Figure A.12: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_5 + \mathbb{Z}^4)$

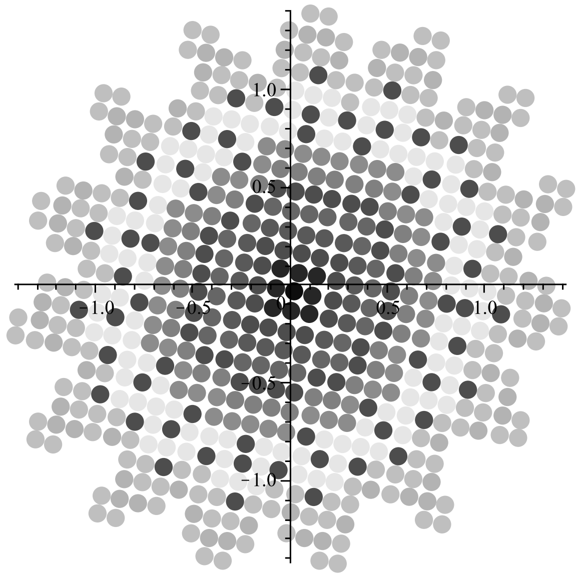


Figure A.13: Projection of pre-images of $\mathbf{v}_6 + \mathbb{Z}^4$ in E^c

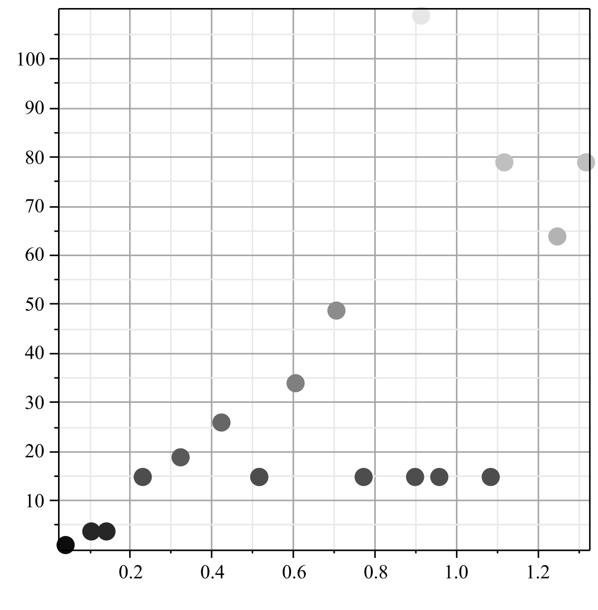


Figure A.14: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_6 + \mathbb{Z}^4)$

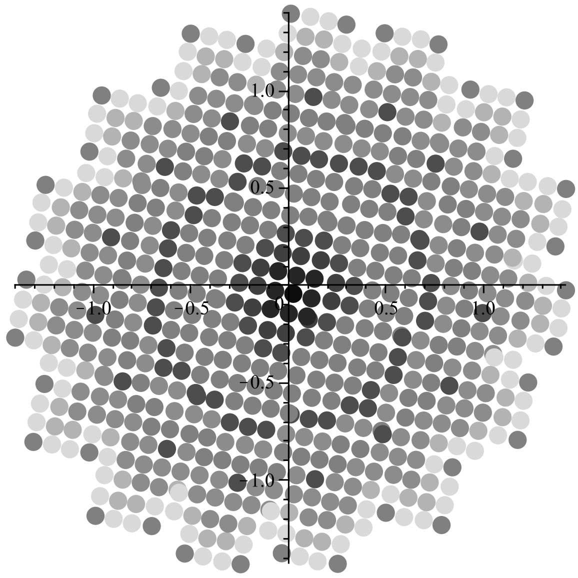


Figure A.15: Projection of pre-images of $\mathbf{v}_7 + \mathbb{Z}^4$ in E^c

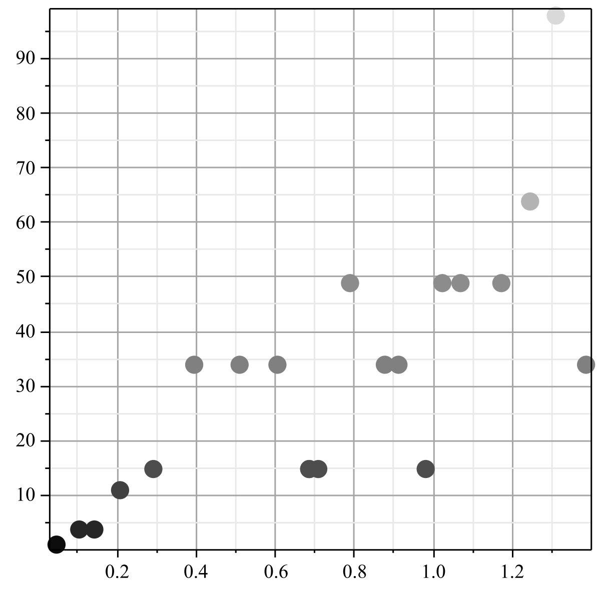


Figure A.16: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_7 + \mathbb{Z}^4)$

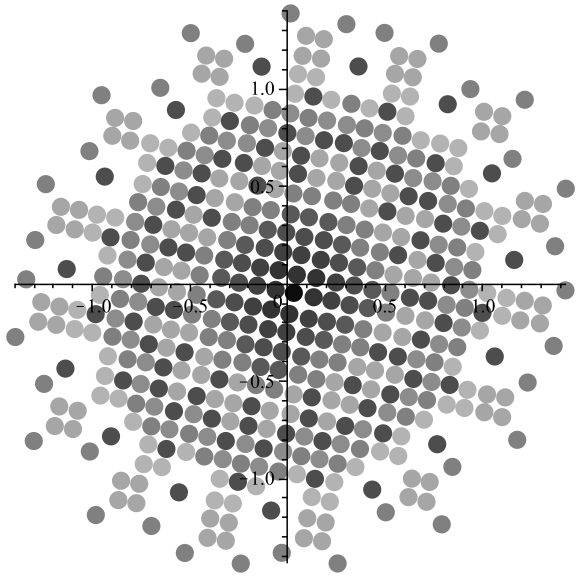


Figure A.17: Projection of pre-images of $\mathbf{v}_8 + \mathbb{Z}^4$ in E^c

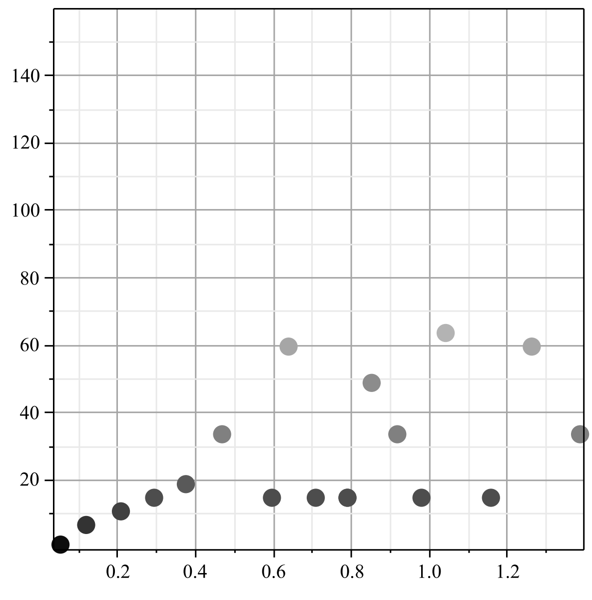


Figure A.18: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_8 + \mathbb{Z}^4)$

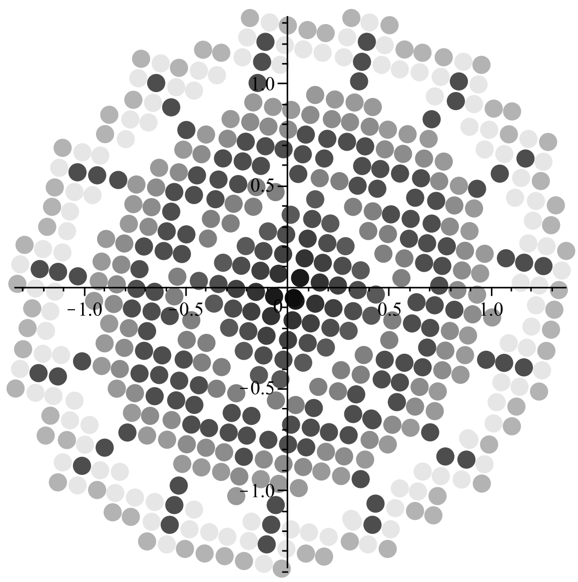


Figure A.19: Projection of pre-images of $\mathbf{v}_9 + \mathbb{Z}^4$ in E^c

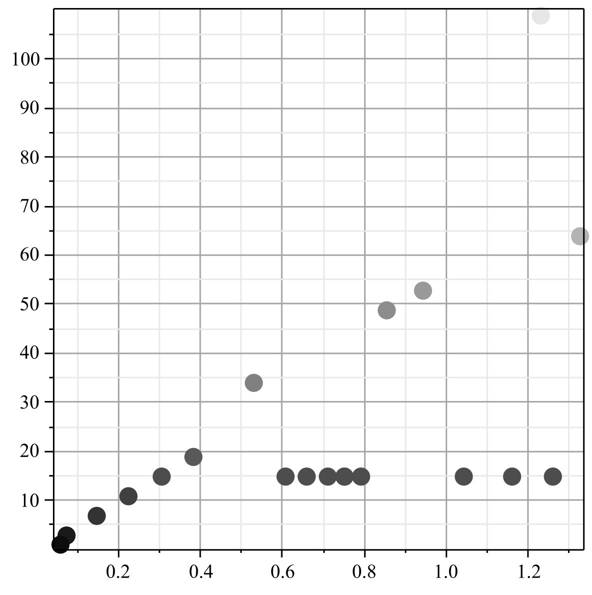


Figure A.20: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_9 + \mathbb{Z}^4)$

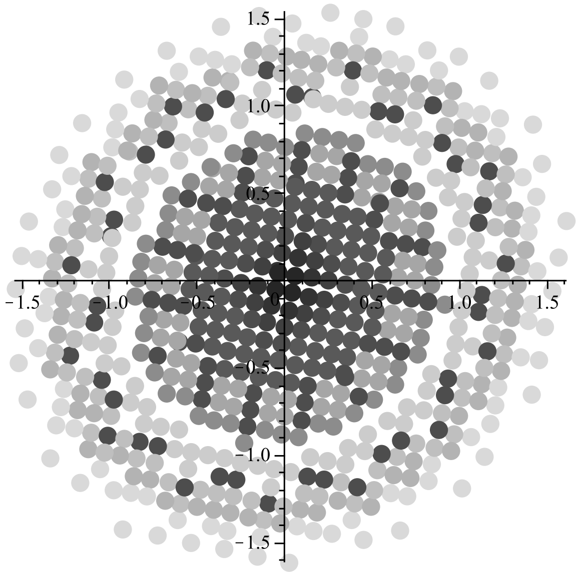


Figure A.21: Projection of pre-images of $\mathbf{v}_{10} + \mathbb{Z}^4$ in E^c

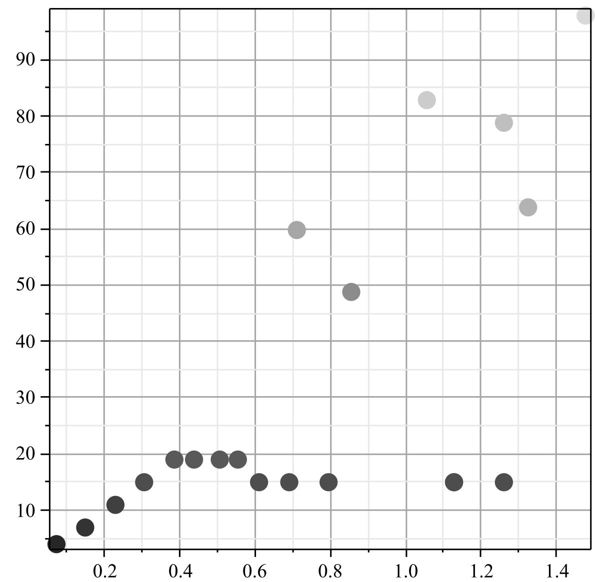


Figure A.22: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{10} + \mathbb{Z}^4)$

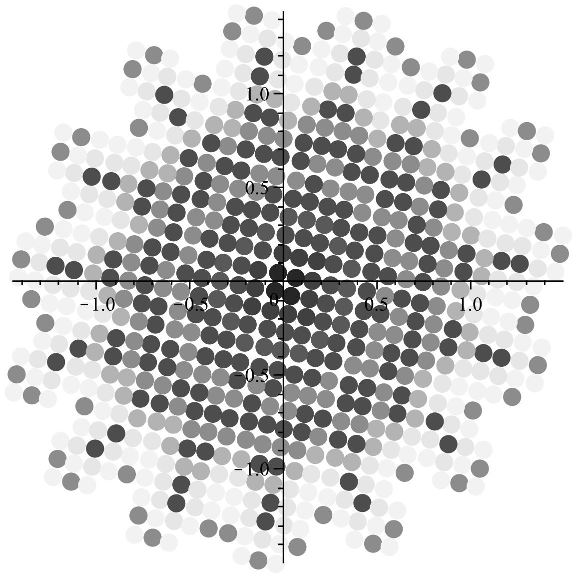


Figure A.23: Projection of pre-images of $\mathbf{v}_{11} + \mathbb{Z}^4$ in E^c

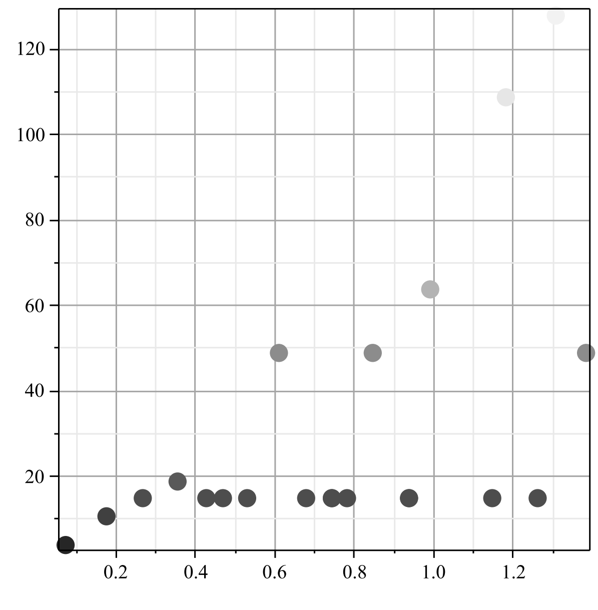


Figure A.24: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{11} + \mathbb{Z}^4)$

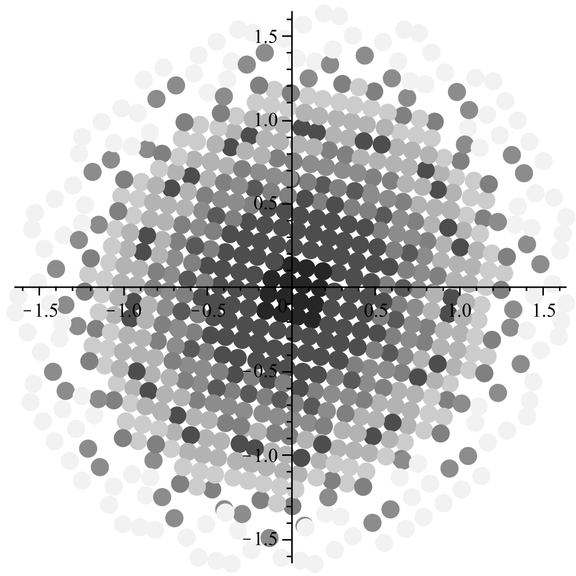


Figure A.25: Projection of pre-images of $\mathbf{v}_{12} + \mathbb{Z}^4$ in E^c

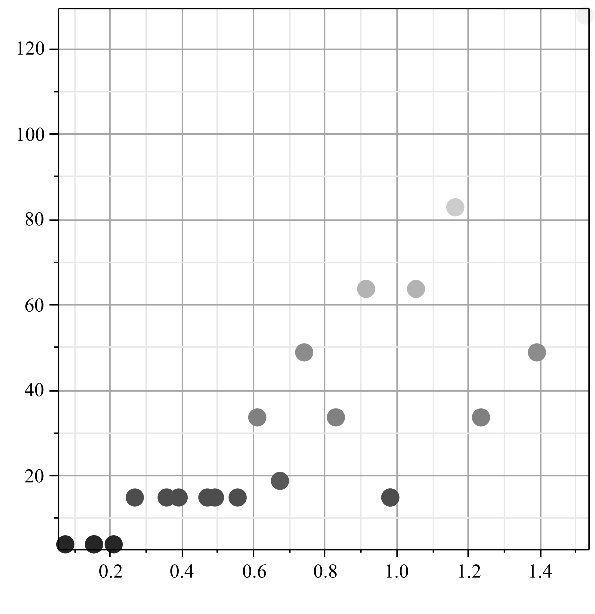


Figure A.26: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{12} + \mathbb{Z}^4)$

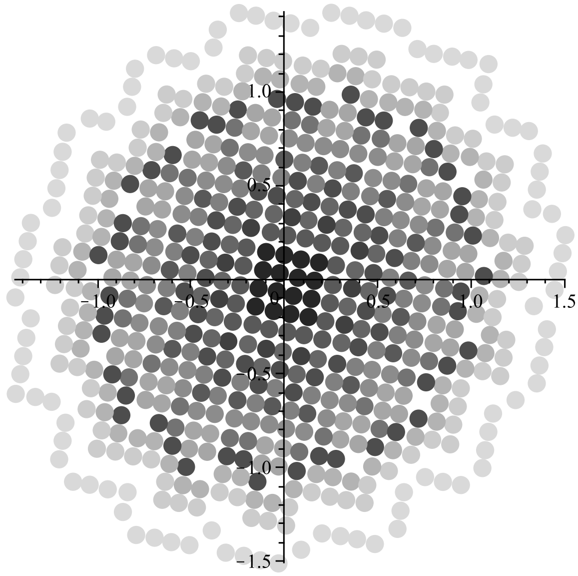


Figure A.27: Projection of pre-images of $\mathbf{v}_{13} + \mathbb{Z}^4$ in E^c

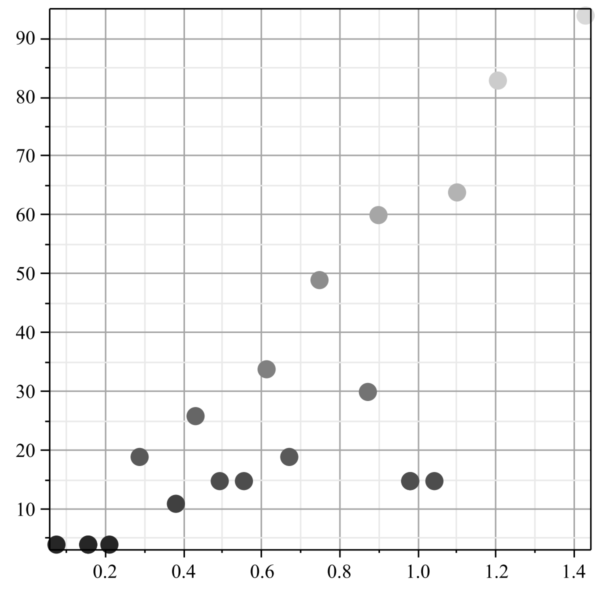


Figure A.28: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{13} + \mathbb{Z}^4)$

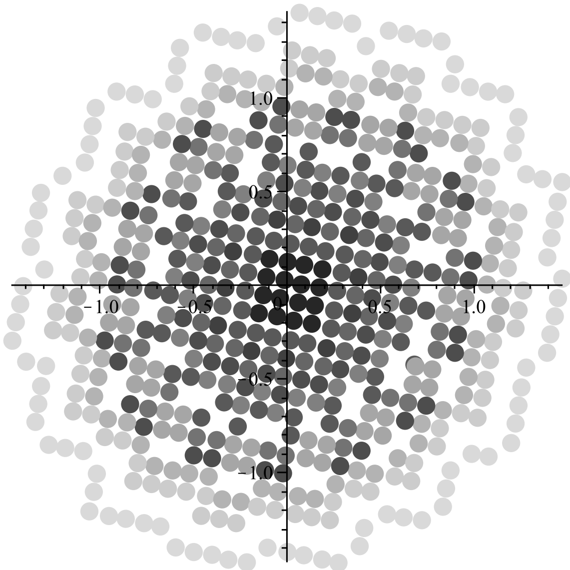


Figure A.29: Projection of pre-images of $\mathbf{v}_{14} + \mathbb{Z}^4$ in E^c

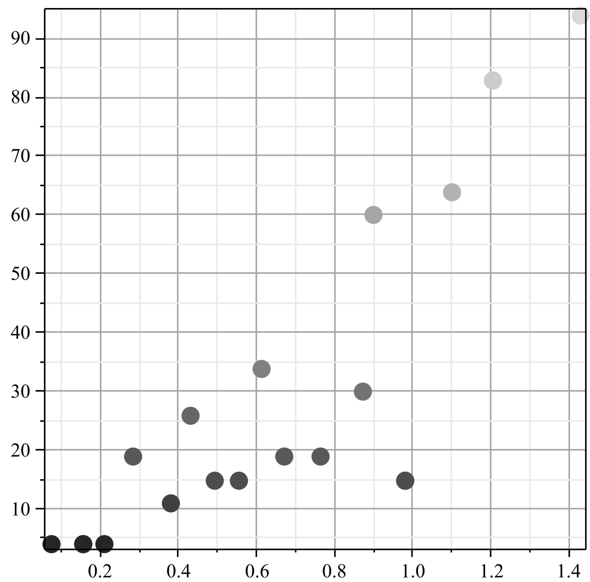


Figure A.30: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{14} + \mathbb{Z}^4)$

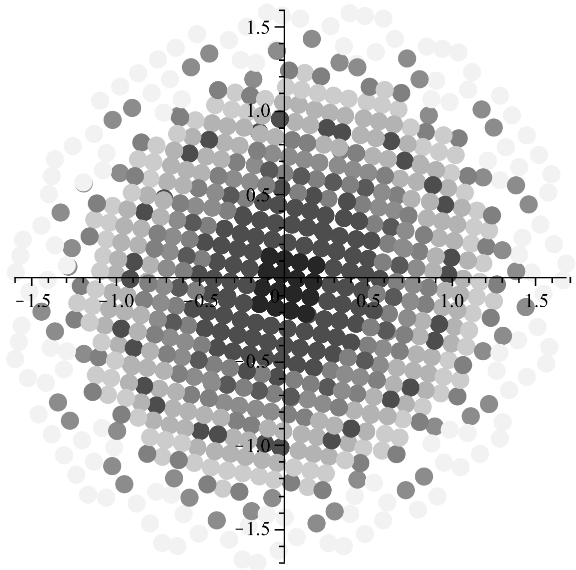


Figure A.31: Projection of pre-images of $\mathbf{v}_{15} + \mathbb{Z}^4$ in E^c

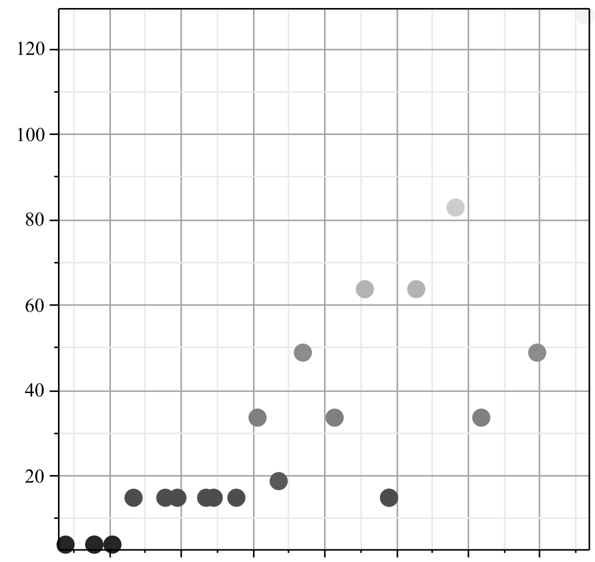


Figure A.32: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{15} + \mathbb{Z}^4)$

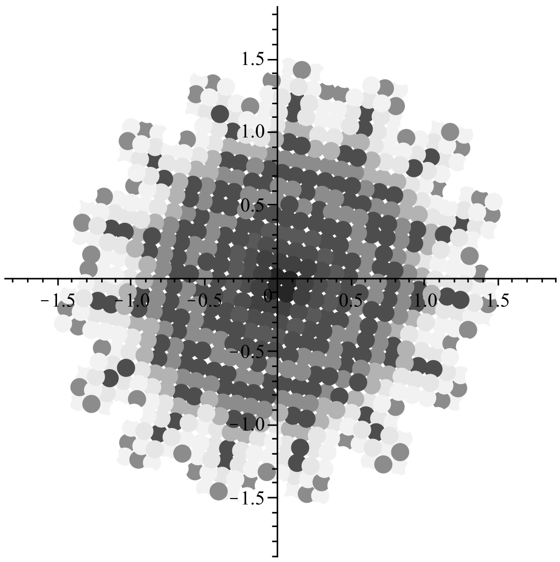


Figure A.33: Projection of pre-images of $\mathbf{v}_{16} + \mathbb{Z}^4$ in E^c

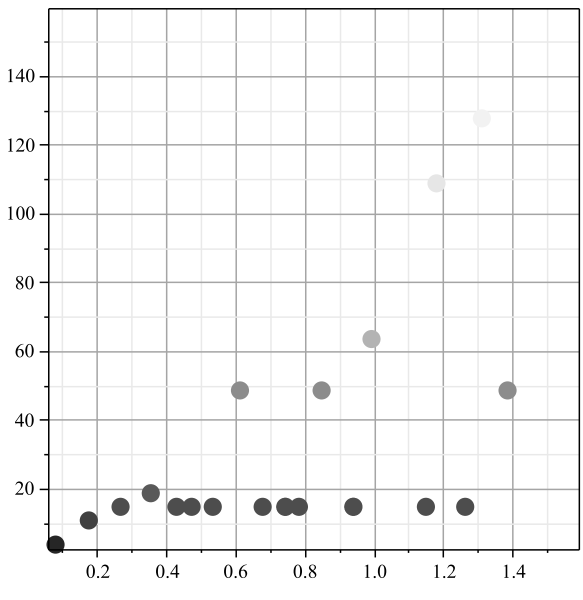


Figure A.34: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{16} + \mathbb{Z}^4)$

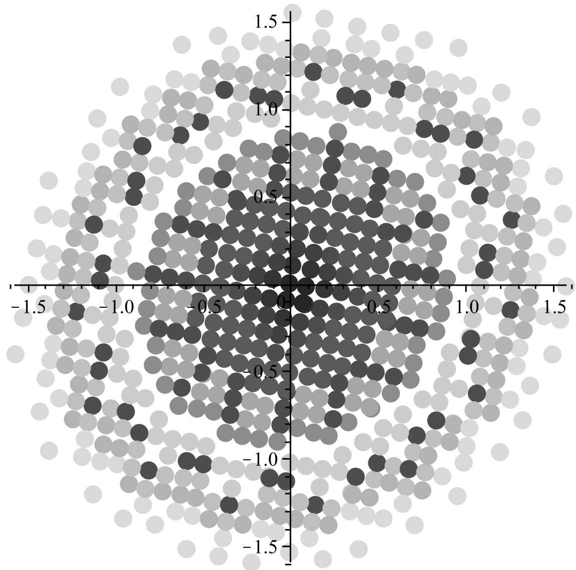


Figure A.35: Projection of pre-images of $\mathbf{v}_{17} + \mathbb{Z}^4$ in E^c

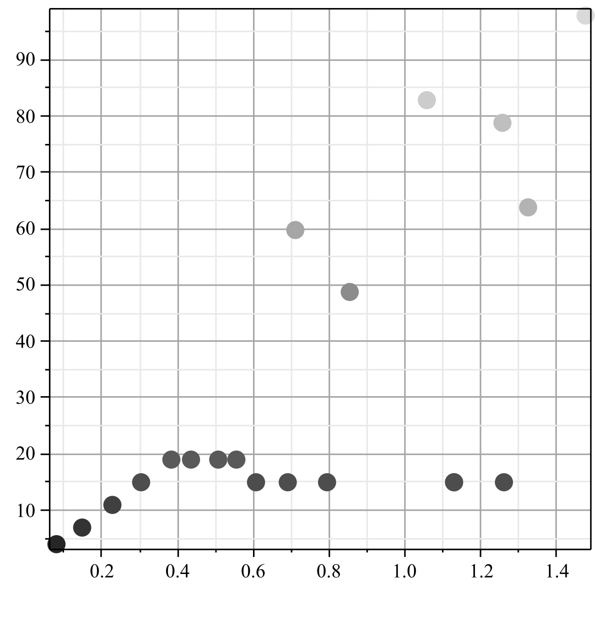


Figure A.36: Average orbit distance versus period for each orbit $\pi^{-1}(\mathbf{v}_{17} + \mathbb{Z}^4)$

Appendix B

Maple code

The following code is a common environment, which should precede any of the Maple programmes that we present afterwards. It loads the Maple packages 'LinearAlgebra' and 'plots', initializes the constants β, β^2 and β^3 and defines the companion matrix C:

```
with(LinearAlgebra):
with(plots):
beta := 10.097114224488101025834954968367621419O2:
beta2:=beta*beta:
beta3:=beta*beta*beta:
C:=Matrix([[0,0,0,-1],[1,0,0,10],[0,1,0,0],[0,0,1,10]]):
```

B.1 Implementation of the greedy algorithm

The following routine uses the *greedy* algorithm to compute the orbit of the point provided as an argument:

```
beta_expansion:=proc(u)
local symbols,orbit,i,j,k,v,repeat,temp;
global C,beta,beta2,beta3:
#Initializations:
```

```
orbit:=[u]; symbols:=[]; v:=u;

#Compute the orbit and symbols up to 10000 iterations,
#or stop once it repeats itself (eventually periodic)
for i from 1 to 10000 do
v:=convert(C.Vector(v),'list'):
k:=floor(v[1]+v[2]*beta+v[3]*beta2+v[4]*beta3):
v[1]:=v[1]-k:
repeat:=0; # check if v has already occurred in the orbit
for j from 1 to nops(orbit) do
if (orbit[j]=v) then repeat:=j; end if;
end do;
symbols:=[op(symbols),k];
if (repeat > 0) then break; end if;
orbit:=[op(orbit),v];
end do;
# end of iterations

if (repeat=0) then
print("Could not find a finite orbit up to 10000 iterations.");
else
  print("Preperiod m = ", repeat-1);
  if (repeat>1) then print(seq(symbols[j],j=1..repeat-1)); end if;
  print("Period p = ", nops(symbols)-repeat+1);
  print(seq(symbols[j],j=repeat..nops(symbols)));
end if;
end proc;
```

The usage is: *beta-expansion(*$[1, 0, 0, 0]$*)*, in order to generate the symbols which are the β-expansion of $[1, 0, 0, 0]$ (or any other vector that we choose instead), output the pre-period m and the period p.

B.2 Routine 1

This Maple routine computes the set of pre-images $B_M(\mathbf{v})$ (for a given $M \in \mathbb{N}$ and $\mathbf{v} \in \mathbb{Q}^4$, and considering the Salem number β). It stores this set of pre-images in a list *points* and saves that list into a file *preim-v0-M100.m*.

```
Routine 1


points:=[]:
M:=100; #range
v:=0 * [-19/18,-9/18,-9/18,1/18]; #set v0=0
x:=v[1]+v[2]*beta+v[3]*beta2+v[4]*beta3:  #\bbb.v
for i from -M to M do
  for j from -M to M do
    for k from -M to M do
      for l from -M to M do
        temp:=i+j*beta+k*beta2+l*beta3+x;
        if ((temp >= 0) and (temp < 1))
 then points:=[op(points), [i,j,k,l]+v]:
        end if;
      end do;
    end do;
  end do;
end do;
save points, "preim_v0_M100.m";
```

B.3 Routine 2

This Maple routine reads the list of pre-images of $\mathbf{v}_k + \mathbb{Z}^4$ within a bounded region defined by $M = 100$, which had been computed in Routine 1 and saved in the file *preim-vk-M100.m*.

Next, it computes the pre-period m and the period p of each point (it looks for eventually periodicity up to a maximum of 10000 iterations). Those points which are strictly periodic ($m = 0$) are stored in the list *strper-points*, and their respective periods are stored in the list *periods*. These two lists are saved in the file *str-per-vk-M100.m*, for subsequent use.

```
Routine 2
# compute period
Compute_period:=proc(u)
local orbit,i,j,m,p,v;

#Initializations:
orbit:=[]: m:=0: p:=0: v:=u:
#Check for periodicity up to 10000 iterations
for i from 1 to 10000 while (p=0) do
orbit:=[op(orbit),v]:
v:=convert(C.Vector(v),'list'):
v[1]:=v[1]-floor(v[1]+v[2]*beta+v[3]*beta2+v[4]*beta3):
  for j from nops(orbit) by (-1) to 1 while (p=0) do
    if (orbit[j]=v) then
                    m:=j-1:
                    p:=nops(orbit)-m:
```

```
                    end if:
  end do:
end do:
return(m,p): #return pre-period m and period p
end proc:
# compute the strictly periodic pre-images of vk
read "preim_v17_M100.m";   #reads pre-images of vk
strper_points:=[]: periods:=[]:
for i to nops(points) do
(m,p):=Compute_period(points[i]):
   if ((m=0) and (p>0)) then
      strper_points:=[op(strper_points),points[i]]:
      periods:=[op(periods),p]:
   end if:
end do:
nops(strper_points);
save strper_points, periods, "strper_v17_M100.m";
periodlist:={}:
for i to nops(periods) do
periodlist:=periodlist union {periods[i]}:
end do:
periodlist;
```

B.4 Complete Orbits

Since the set of strictly periodic points found within the bounded region defined by $M = 100$ is not necessarily $\widetilde{C}$-invariant (because the orbit of such point may lie slightly outside the bounded region), we add to the list the possibly missing points of the orbits which are strictly periodic. This will make sure that our plots will show a circular boundary, instead of a rectangular one.

```
# compute period
compute_period:=proc(u)
local i,p,v;

#Initializations:
p:=0: v:=u:

#Check for periodicity up to 10000 iterations
for i from 1 to 10000 while (p=0) do

v:=convert(C.Vector(v),'list'):
v[1]:=v[1]-floor(v[1]+v[2]*beta+v[3]*beta2+v[4]*beta3):

if (v=u) then p:=i:
         end if:
end do:

return(p): #return period p
end proc:

# load  the variables "strper_points" and "periods"
read "mw_files/strper_v17_M100.m";
# save everything in a list
pointlist:={}:
for i to nops(strper_points) do
  v:=strper_points[i]:     #complete the orbit of v
  for j to periods[i] do
```

```
      pointlist:=pointlist union {v}:
      v:=convert(C.Vector(v),'list'):
      v[1]:=v[1]-floor(v[1]+v[2]*beta+v[3]*beta2+v[4]*beta3):
   end do:
end do:

strper_points:=[]:
periods:=[]:
for i to nops(pointlist) do
     strper_points:=[op(strper_points),pointlist[i]]:
     p:=compute_period(pointlist[i]):
     periods:=[op(periods),p]:
end do:
# this is the version of the file with completed orbits!
save strper_points, periods, "mw_files/strper_v17_M100.m";
```

B.5 Generate graphics

This is the Maple code that we used to generate the plots.

```
S:=Matrix([[beta,0,beta^2,-1],[-1-beta^2,beta,10-beta^(-1)
      ,beta^3-10*beta^2],[beta,-1-beta^2,10*beta-1,beta^2-10*beta],
      [0,beta,-beta,beta]]):
S_1:=Matrix([[1,0,0,0],[0,1,0,0]]).S^(-1);

# Read the lists "strper_points", and "periods"
read "strper_v4_M100.m";
newpoints:=seq(S_1.Vector(strper_points[i]),
i=1..nops(strper_points)):
```

```
newpoints2:=[seq([newpoints[i][1],newpoints[i][2]],
i=1..nops(strper_points))]:

graylist:=[seq(COLOR(RGB,n/20,n/20,n/20),n=1..20)]:

periodlist:={}:
for i to nops(strper_points) do
periodlist:=periodlist union {periods[i]}:
end do:
periodlist;
                         {1, 4, 15, 19, 26, 34, 49}
colorlist:=[]:
for l from 1 to nops(strper_points) do
p:=periods[l]:
 if (p=1) then colorlist:=[op(colorlist), graylist[1]]:
  elif (p=3) then colorlist:=[op(colorlist), graylist[2]]:
  elif (p=4) then colorlist:=[op(colorlist), graylist[3]]:
  elif (p=7) then colorlist:=[op(colorlist), graylist[4]]:
  elif (p=11) then colorlist:=[op(colorlist), graylist[5]]:
  elif (p=15) then colorlist:=[op(colorlist), graylist[6]]:
  elif (p=19) then colorlist:=[op(colorlist), graylist[7]]:
  elif (p=26) then colorlist:=[op(colorlist), graylist[8]]:
  elif (p=30) then colorlist:=[op(colorlist), graylist[9]]:
  elif (p=34) then colorlist:=[op(colorlist), graylist[10]]:
  elif (p=49) then colorlist:=[op(colorlist), graylist[11]]:
  elif (p=53) then colorlist:=[op(colorlist), graylist[12]]:
  elif (p=60) then colorlist:=[op(colorlist), graylist[13]]:
  elif (p=64) then colorlist:=[op(colorlist), graylist[14]]:
```

```
  elif (p=79) then colorlist:=[op(colorlist), graylist[15]]:
  elif (p=83) then colorlist:=[op(colorlist), graylist[16]]:
  elif (p=98) then colorlist:=[op(colorlist), graylist[17]]:
  elif (p=109) then colorlist:=[op(colorlist), graylist[18]]:
  elif (p=128) then colorlist:=[op(colorlist), graylist[19]]:
  elif (p=158) then colorlist:=[op(colorlist), graylist[20]]:
end if:
end do:
nops(colorlist);

pointplot(newpoints2,axes=NORMAL,connect=false,
          symbol=solidcircle,symbolsize=25,color=colorlist);

read "strper_v4_M100.m";
orbit_represent:=[]:
ppp_list:=[]:
average_dst:=[]:

for i to nops(strper_points) do
    if (strper_points[i] <> [0,0,0,0]) then
    orbit_represent:=[op(orbit_represent),strper_points[i]]:
    ppp_list:=[op(ppp_list),periods[i]]:
    u:=strper_points[i]:
    ddd:=0:
           for j to periods[i] do
                for k to nops(strper_points) do
                 if (strper_points[k]=u) then
                  strper_points:=subsop(k=[0,0,0,0],strper_points):
```

```
                 end if:
                end do:
           aux:=S_1.Vector(u):
           ddd:=ddd+sqrt(aux[1]^2+aux[2]^2):
           u:=convert(C.Vector(u),'list'):
           u[1]:=u[1]-floor(u[1]+u[2]*beta+u[3]*beta2+u[4]*beta3):
           end do:
    average_dst:=[op(average_dst),ddd/periods[i]]:
    end if:
end do:

colorlist:=[]:
for l from 1 to nops(orbit_represent) do
p:=ppp_list[l]:
 if (p=1) then colorlist:=[op(colorlist), graylist[1]]:
  elif (p=3) then colorlist:=[op(colorlist), graylist[2]]:
  elif (p=4) then colorlist:=[op(colorlist), graylist[3]]:
  elif (p=7) then colorlist:=[op(colorlist), graylist[4]]:
  elif (p=11) then colorlist:=[op(colorlist), graylist[5]]:
  elif (p=15) then colorlist:=[op(colorlist), graylist[6]]:
  elif (p=19) then colorlist:=[op(colorlist), graylist[7]]:
  elif (p=26) then colorlist:=[op(colorlist), graylist[8]]:
  elif (p=30) then colorlist:=[op(colorlist), graylist[9]]:
  elif (p=34) then colorlist:=[op(colorlist), graylist[10]]:
  elif (p=49) then colorlist:=[op(colorlist), graylist[11]]:
  elif (p=53) then colorlist:=[op(colorlist), graylist[12]]:
  elif (p=60) then colorlist:=[op(colorlist), graylist[13]]:
  elif (p=64) then colorlist:=[op(colorlist), graylist[14]]:
```

```
  elif (p=79) then colorlist:=[op(colorlist), graylist[15]]:
  elif (p=83) then colorlist:=[op(colorlist), graylist[16]]:
  elif (p=98) then colorlist:=[op(colorlist), graylist[17]]:
  elif (p=109) then colorlist:=[op(colorlist), graylist[18]]:
  elif (p=128) then colorlist:=[op(colorlist), graylist[19]]:
  elif (p=158) then colorlist:=[op(colorlist), graylist[20]]:
end if:
end do:
newpoints2:=[seq([average_dst[i],ppp_list[i]],
i=1..nops(orbit_represent))]:
pointplot(newpoints2,axes=boxed,connect=false,symbol=solidcircle,
                symbolsize=25, color=colorlist, gridlines=true);
```

Appendix C

Arithmetic codings for Pisot automorphisms

C.1 Introduction

We summarize in this chapter some results of an area known as arithmetic dynamics. This branch of symbolic dynamics was initially developed by Vershik and Sidorov in [Ver92] and [VS98] for the two-dimensional *Pisot* automorphism, and subsequently generalized by Sidorov in [Sid01] and [Sid03] to *Pisot* automorphisms of dimension higher than two.

C.2 Arithmetic coding

For some hyperbolic toral automorphisms $\overline{C} : \mathbb{T}^d \mapsto \mathbb{T}^d$ we can be establish a semi-conjugacy ϕ from the two-sided shift space (σ, X_β) to $(\overline{C}, \mathbb{T}^d)$. This semi-conjugacy is obtained in a different way than the semi-conjugacies obtained with Markov partitions, and it is called an *arithmetic coding*.

$$\begin{array}{ccc} X_\beta & \xrightarrow{\sigma} & X_\beta \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ \mathbb{T}^d & \xrightarrow[\overline{C}]{} & \mathbb{T}^d \end{array} \qquad \text{(C.1)}$$

This method was presented in [Sid01] for the 2-dimensional case, developing the results which had been achieved in [Ver92]. Their approach is rather axiomatic, in the sense the authors specify a priori which properties should a semi-conjugacy have in order to be called an *arithmetic encoding*. Afterwards they proceed into studying the problem of which toral automorphisms admit an *arithmetic coding* and how many different *arithmetic codings* can be found for such automorphisms. We will borrow the ideas from [Sid01], but present them in a more intuitive approach.

C.3 The semi-conjugacy in the Pisot case

Let us start off by extending the one sided β-shift (σ, X_β^+) to a two-sided β-shift (σ, X_β), in order to obtain an invertible shift. This two-sided space of sequences is defined as:

$$X_\beta := \{(\ldots 00\varepsilon_{-N}\varepsilon_{-N+1}\ldots\varepsilon_0.\varepsilon_1\varepsilon_2\ldots) \mid (\varepsilon_{-N}\varepsilon_{-N+1}\varepsilon_{-N+2}\ldots) \in X_\beta^+)\}. \qquad \text{(C.2)}$$

X_β is the two-sided space of sequences that can be obtained by shifting any one-sided sequences in X_β^+ an arbitrary (but finite) number of times to the left. In particular, these sequences are finite towards the left (for each sequence, $\exists N \in \mathbb{N}$ such that $\forall k < -N, \varepsilon_k = 0$). The two-sided β-shift is well defined in X_β and there exists a semi-conjugacy between (σ, X_β) and $(M_\beta, \mathbb{R}_0^+)$, (with $M_\beta(x) := \beta x$):

$$\begin{array}{ccc} X_\beta & \xrightarrow{\sigma} & X_\beta \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{R}_0^+ & \xrightarrow[M_\beta]{} & \mathbb{R}_0^+ \end{array} \qquad \text{(C.3)}$$

The semi-conjugacy is defined by:

$$f(\ldots 00\varepsilon_{-N}\varepsilon_{-N+1}\ldots\varepsilon_0.\varepsilon_1\ldots) := \sum_{k=-N}^{\infty}\frac{\varepsilon_k}{\beta^k}. \tag{C.4}$$

If β is *Pisot* then the toral automorphism $\overline{C}: \mathbb{T}^d \longrightarrow \mathbb{T}^d$ has a one-dimensional unstable manifold at the origin, which can be lifted to the universal cover $\mathbb{R}^d$ as a straight line through the origin which is the expanding eigenspace of the matrix C associated to the eigenvalue β:

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{C} & \mathbb{R}^d \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{T}^d & \xrightarrow[\overline{C}]{} & \mathbb{T}^d \end{array} \tag{C.5}$$

We would like this semi-conjugacy to be a map between compact spaces. However, given that X_β is not compact, we will compactify it by including sequences which are not finite to the left: $\overline{X_\beta}$. The two sided-shift $(\sigma, \overline{X_\beta})$ is a *sofic shift*.

We shall attempt to map sequences in X_β (which can be identified with $\mathbb{R}_0^+$) into the semi-straight line $E^u \subset \mathbb{R}^d$ which is the lift of the unstable manifold of the origin $W^u \subset \mathbb{T}^d$. The semi-conjugacy will be $\pi \circ f$:

$$\begin{array}{ccc} X_\beta & \xrightarrow{\sigma} & X_\beta \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ E^u \subset \mathbb{R}^d & \xrightarrow{C} & E^u \subset \mathbb{R}^d \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ W^u \subset \mathbb{T}^d & \xrightarrow[\overline{C}]{} & W^u \subset \mathbb{T}^d \end{array} \tag{C.6}$$

Let us start by relating homoclinic points (to the origin) in (σ, X_β) and homoclinic points (to the origin) in $(\overline{C}, \mathbb{T}^d)$. Fix one arbitrary homoclinic point for (σ, X_β) of a very elementary type, for instance: $\delta_1 = (\ldots 00.100\ldots)$. This corresponds to the number $x = \frac{1}{\beta} \in \mathbb{R}_0^+$, and we would like to map it to a homoclinic point $\mathbf{t} \in E_u \subset \mathbb{T}^d$.

The lift of $\mathbf{t}$, that is, $\mathbf{s} := \pi^{-1}(\mathbf{t}) \subset E^u$, represents distance β^{-1} from the origin along $E^u \subset \mathbb{R}^d$.

So, let us choose a homoclinic point $\mathbf{t} \in W^u \cap W^s$ and lift it to the universal cover $\mathbb{R}^d$, and define:

$$\phi_{\mathbf{t}}(\delta_1) := \pi \circ f(\delta_1) = \pi(\mathbf{s}) = \mathbf{t}, \tag{C.7}$$

and $\mathbf{s}$ is the lift of $\mathbf{t}$ into E^u.

Let us denote by δ_k the sequence of symbols having only one non-zero symbol 1 in the term of order k. Using the semi-conjugacy relation: $\phi_{\mathbf{t}} \circ \sigma = \overline{C} \circ \phi_{\mathbf{t}}$, we obtain:

$$\phi_{\mathbf{t}}(\delta_k) = \phi_{\mathbf{t}}(\sigma^{-k+1}(\delta_1)) = \overline{C}^{-k+1}(\phi_{\mathbf{t}}(\delta_1)) = \overline{C}^{-k+1}(\mathbf{t}). \tag{C.8}$$

This can also be expressed in terms of the universal cover:

$$f(\delta_k) = f(\sigma^{-k+1}(\delta_1)) = C^{-k+1}(f(\delta_1)) = C^{-k+1}(\mathbf{s}). \tag{C.9}$$

We stress that as we define $\phi_{\mathbf{t}}(\delta_k)$, we define how we map distances from R_0^+ into E^u and W^u. In particularly, we are defining where are numbers $\frac{1}{\beta^k}$ mapped to along E^u and W^u.

Any sequence $\overline{\varepsilon} \in \overline{X_\beta}$ which is finite to the left represents a number $x \in \mathbb{R}_0^+$:

$$x = \sum_{k=-N}^{\infty} \frac{\varepsilon_k}{\beta^k}. \tag{C.10}$$

We want to carry the arithmetic relations in $\mathbb{R}_0^+$ to E^u and W^u, therefore, for sequences which are finite to the left, we shall define:

$$f(\ldots 00\varepsilon_{-N}\varepsilon_{-N+1}\ldots\varepsilon_{-1}\varepsilon_0.\varepsilon_1\varepsilon_2\ldots) := \sum_{k=-N}^{\infty} \varepsilon_k C^{-k+1}(\mathbf{s}) = \left(\sum_{k=-N}^{\infty} \frac{\varepsilon_k}{\beta^{k-1}}\right)\mathbf{s}, \tag{C.11}$$

which corresponds to:

$$\phi_{\mathbf{t}}(\ldots 00\varepsilon_{-N}\varepsilon_{-N+1}\ldots\varepsilon_{-1}\varepsilon_0.\varepsilon_1\varepsilon_2\ldots) := \sum_{k=-N}^{\infty} \varepsilon_k \overline{C}^{-k+1}(\mathbf{t}). \tag{C.12}$$

If we want to extend the definition of $\phi_{\mathbf{t}}$ to all sequences in $\overline{X_\beta}$, we need to extend the definition to sequences which aren't finite to the left. f will not be well defined for such sequences, because

$$\sum_{k=-\infty}^{\infty} \varepsilon_k C^{-k+1}(\mathbf{s}) = \left(\sum_{k=-\infty}^{\infty} \frac{\varepsilon_k}{\beta^{k-1}} \right) \mathbf{s} \tag{C.13}$$

may be a sum which diverges to infinity. However, is $\mathbf{t}$ is a homoclinic point, the following is well defined:

$$\phi_{\mathbf{t}}(\varepsilon) := \lim_{N \longrightarrow \infty} \pi \left(\sum_{k=-N}^{\infty} \varepsilon_k C^{-k+1}(\mathbf{s}) \right) = \sum_{k=-\infty}^{\infty} \varepsilon_k \overline{C}^{-k+1}(\mathbf{t}). \tag{C.14}$$

We need β to be *Pisot* , because if it was *Salem* , then we wouldn't be able to find non-trivial homoclinic points $\mathbf{t}$. In the *Salem* case points $x \in \mathbb{T}^d$ can be decomposed into the sum of three components: in the stable, unstable and centre manifolds of the origin. Since the component in the centre manifold doesn't vanish, we do not have non-trivial homoclinic points as in the *Pisot* case.

Finally, in the *Pisot* case, we should remember that W^u is dense in $\mathbb{T}^d$, therefore the semi-conjugacy $\phi_{\mathbf{t}}$ is surjective. If $\mathbf{t}$ is chosen wisely (fundamental homoclinic point), then the semi-conjugacy is bijective almost everywhere (if fails to be bijective in a set of measure zero, where we have ambiguity in representing the same point by two different sequences).

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