MA3H1 TOPICS IN NUMBER THEORY
EXAMPLE SHEET 6

(1) Solve the following congruences
   (i) $x^2 \equiv 3 \pmod{7^3}$,
   (ii) $x^2 \equiv -2 \pmod{3^4}$,
   (iii) $x^3 + x \equiv 2 \pmod{225}$.

(2) Let $\alpha = -7425/16$. Calculate $|\alpha|_p$ for all primes p.

(3) Let x be a non-zero rational. Prove the **Product Formula**:
$$|x| \prod_{p \in \mathbb{P}} |x|_p = 1.$$

(4) Let $|c|_p < 1$. Show that
$$1 + c + c^2 + \cdots = \frac{1}{1-c}$$
   in $\mathbb{Q}_p$. Hence find an integer a such that $|4a + 1|_5 \leq 5^{-3}$.

(5) (i) Let p be an odd prime and a a non-zero integer. Show that a is a square in $\mathbb{Z}_p$ if
      and only if $a = p^{2r}b$ for some integer r where $\left(\frac{b}{p}\right) = 1$.
   (ii) Let a be a non-zero integer. Show that a is a square in $\mathbb{Z}_2$ if and only if $a = 2^{2r}b$
      where $b \equiv 1 \pmod 8$.
   (iii) Show that the equation
$$(X^2 - 2)(X^2 - 17)(X^2 - 34) = 0$$
      has solutions in $\mathbb{Z}_p$ for all primes p. Does it have solutions in $\mathbb{Z}$?

(6) For which values of p does the following series converge in $\mathbb{Q}_p$?
   (i) $1 + (15/7) + (15/7)^2 + (15/7)^3 + \cdots$.
   (ii) $1! + 2! + 3! + 4! + \cdots$.