MA3H1 Topics in Number Theory
Example Sheet 5

You should attempt all the questions on this sheet, but questions 1,2,3 will marked for credit, and must be handed in by **3pm Friday, week 9**.

(1) Which of the following are lattices in $\mathbb{Z}^2$? What is the index?
   (i) $\{(x, y) \in \mathbb{Z}^2 : x + y = 1\}$.
   (ii) $\{(x, y) \in \mathbb{Z}^2 : x + y = 0\}$.
   (iii) $\{(x, y) \in \mathbb{Z}^2 : 2 \mid x\}$.
   (iv) $\{(x, y) \in \mathbb{Z}^2 : x \equiv y \pmod 3\}$.
   (v) $\{(x, y) \in \mathbb{Z}^2 : x \equiv y \pmod 3, \qquad x \equiv 2y \pmod 5\}$.

(2) Which of the following are convex? Which of the following are symmetric?
   (i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 0\}$.
   (ii) $\{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 < 1\}$.
   (iii) $\{(x, y, z) \in \mathbb{R}^3 : 3x^2 + 5y^2 + 7z^2 < 1\}$.

(3) Let p be an odd prime satisfying $\left(\frac{-2}{p}\right) = 1$. Show that there are integers x, y such that $x^2 + 2y^2 = p$.

(4) Find an odd prime p for which $\left(\frac{-5}{p}\right) = 1$ but which is not of the shape $x^2 + 5y^2$ with x, $y \in \mathbb{Z}$.

(5) Let $p \equiv 1 \pmod 3$ be prime. Show that there is some $f \in \mathbb{Z}$ such that $f^2 + f + 1 \equiv 0 \pmod p$. Show that $p = x^2 + xy + y^2$ for some x, $y \in \mathbb{Z}$.