MA136 Introduction to Abstract Algebra

Homework Assignment 4

Attempt all the questions on this sheet, and hand in the solutions to (A1)–(A5), (B1)–(B4). Your supervisor will mark some, but not all, of these questions. Solutions to your supervisor's pigeon-loft by **2pm Thursday, week 10.**

(A1) Let ρ and τ be the following permutations:

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}.$$

(i) Compute ρ^{-1}, $\rho\tau$, τ^2.
(ii) Write ρ and τ as products of disjoint cycles.
(iii) Write ρ and τ as products of transpositions and state if they're even or odd.

(A2) Which of the following pairs of permutations are equal elements of S_6?
(i) $(1, 2, 3)(4, 6)$ and $(6, 4)(2, 3, 1)(5)$.
(ii) $(4, 5, 6)(1, 2, 3)$ and $(3, 1, 2)(5, 4, 6)$.

(A3) Let $\rho = (1, 2, 3)(4, 5)$ and $\tau = (1, 2, 3, 4)$. Write the following in cycle notation (i.e. as a product of disjoint cycles): ρ^{-1}, τ^{-1}, $\rho\tau$, $\tau\rho^2$.

(A4) Recall that we may interpret elements of D_4 as bijections from $\{1, 2, 3, 4\}$ to itself (page 21 of the printed notes). Thus every element of D_4 can be thought of as an element of S_4. Write down the eight elements of D_4 in cycle notation (e.g. $\rho_2 = (1, 3)(2, 4)$).

(A5) Let f be a polynomial in variables $x_1, x_2, \ldots, x_n$. Let σ be a permutation in S_n. We define $\sigma(f)$ to be the polynomial $f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n})$. For example, if $f = x_1 + x_2^2 + x_3x_4$ and $\sigma = (1, 4)(2, 3)$ then σ swaps x_1 and x_4, and swaps x_2 and x_3; thus $\sigma(f) = x_4 + x_3^2 + x_2x_1$. Compute $\sigma(f)$ for the following pairs f, σ:
(i) $f = x_1^2 - x_2x_3$, $\sigma = (1, 2, 3)$.
(ii) $f = x_1x_2 + x_3x_4$, $\sigma = (1, 3)(2, 4)$.

(B1) Let ρ and $\tau \in S_n$. Show that τ is even if and only if $\rho^{-1}\tau\rho$ is even.

(B2) Write down the elements of A_3 and check that it is cyclic. Show that S_n is non-abelian for $n \geq 3$. Show that A_n is non-abelian for $n \geq 4$.

(B3) (a) Use Lagrange's Theorem to show that S_4 does not have an element of order 5.
(b) Let $\sigma = (a_1, a_2, \ldots, a_m)$ be a cycle of length m in S_n. Explain why σ has order m.

(c) Now let $\rho = \sigma_1\sigma_2 \dots \sigma_k$ where the σ_i are disjoint cycles of lengths m_i in S_n. Explain carefully why ρ has order $\operatorname{lcm}(m_1, m_2, \dots, m_k)$.

(d) Show that S_4 does not have elements of order 6. Could you have shown this using Lagrange's Theorem?

(B4) Let f be a polynomial in variables $x_1, \dots, x_n$.

(a) Let H be a subgroup of S_n. We say that f is *H-invariant* if it satisfies the property that $\sigma(f) = f$ for all $\sigma \in H$. We say that f is *symmetric* if it is S_n-invariant. Find a polynomial in x_1, x_2, x_3, x_4 that is D_4-invariant but not symmetric.

(b) Define $\operatorname{Stab}(f) = \{\sigma \in S_n : \sigma(f) = f\}$. Show that $\operatorname{Stab}(f)$ is a subgroup of S_n. Write down $\operatorname{Stab}(f)$ for the following polynomials in $x_1, \dots, x_4$:

(i) $x_4^2 + x_1x_2x_3$.

(ii) $x_1x_2 + x_3x_4$.

(C1) This exercise concerns the 15-tile puzzle. The puzzle consists of 15 square tiles (numbered $1, 2, \dots, 15$) arranged in a 4×4 square with one position blank. The initial arrangement of the tiles is as follows:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

You can slide any tile adjacent to the blank into the position of the blank. Show that it is impossible to reach

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

You might want to follow these hints and tips:

(i) Think of the blank as a tile numbered 16. This way every rearrangement is a permutation on $1, \dots, 16$ and so an element of S_{16}.

(ii) Observe that every move is a transposition involving 16.

(iii) Observe that to go from the initial arrangement to the desired final arrangement, tile 16 must make the same number moves down as up, and the same number of moves right as left.