Chabauty and the Mordell–Weil Sieve—Exercises

(I) For the curve

$$C \; : \; y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1,$$

working in $\mathbb{Q}_3$, calculate the tiny integral

$$\int_{(0,1)}^{(-3,1)} \frac{x dx}{y} \qquad \text{modulo } 3^7.$$

(II) Let

$$C \; : \; Y^2 = 11X^6 - 19.$$

(a) Show that C has points everywhere locally. (**Hint:** Google the Hasse–Weil bounds and Hensel's Lemma).

(b) Let E_1, E_2 be the elliptic curves

$$E_1 \; : \; y^2 = x^3 - 11^2 \cdot 19, \qquad E_2 \; : \; y^2 = x^3 + 11 \cdot 19^2.$$

Define $\phi_i : C \to E_i$ by

$$\phi_1(X, Y) = (11X^2, 11Y), \qquad \phi_2(X, Y) = \left(\frac{-19}{X^2}, \frac{19Y}{X^3} \right).$$

You may suppose that

$$E_1(\mathbb{Q}) = \mathbb{Z} \cdot (995/49, 26732/343), \qquad E_2(\mathbb{Q}) = \mathbb{Z} \cdot (5, 64).$$

Use the commutative diagram

$$\begin{array}{ccccc}
C(\mathbb{Q}) & \xrightarrow{\phi} & E_1(\mathbb{Q}) \times E_2(\mathbb{Q}) & \xleftarrow{\eta} & \mathbb{Z} \times \mathbb{Z} \\
\downarrow \scriptstyle{\text{red}} & & \downarrow \scriptstyle{\text{red}} & \swarrow \scriptstyle{\mu} & \\
C(\mathbb{F}_7) & \xrightarrow{\phi} & E_1(\mathbb{F}_7) \times E_2(\mathbb{F}_7) & &
\end{array}$$

to show that $C(\mathbb{Q}) = \emptyset$.

- $\phi = (\phi_1, \phi_2)$;
- red denotes reduction modulo 7;
- $\eta(m, n) = (mP_1, nP_2)$, where $P_1 = (995/49, 26732/343)$ and $P_2 = (5, 64)$;
- $\mu = \text{red} \circ \eta$.