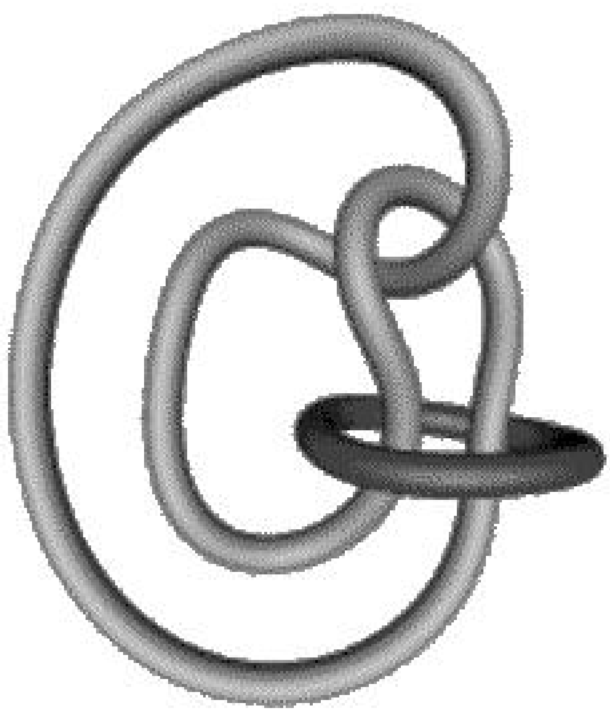
CUTTING AND PASTING



LIBERTÉ, EGALITÉ, HOMOLOGIE!

1. Heegaard splittings
   – Making 3-manifolds from solid handlebodies

2. Surface homeomorphisms
   – Gluing manifolds together along their boundaries

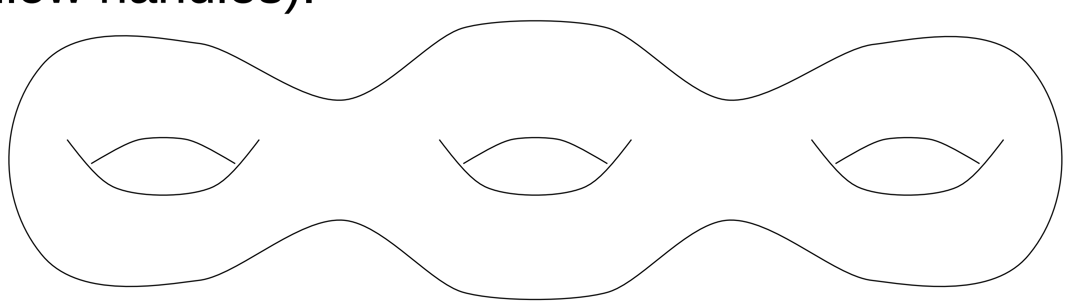
3. Surgery
   – Cutting and pasting

4. Homology spheres
   – If it looks like a sphere. . . it might not be.

1. Heegaard Splittings

n-**manifold**: Compact, connected, Hausdorff topological space M^n, each point of which has a neighbourhood homeomorphic to $\mathbb{R}^n$.
... **with boundary**: Also allow neighbourhoods homeomorphic to $\mathbb{R}^n_+$.
boundary, $\partial\mathrm{M}$: The bit consisting of points with neighbourhoods homeomorphic to $\mathbb{R}^n_+$.

Genus-n handlebody: Compact subset of $\mathbb{R}^3$ bounded by a genus-n surface (a 2-sphere with n hollow handles).



A genus-3 handlebody

Now:

1. Take two identical copies M_1, M_2 of the genus-n handlebody.

2. Choose a homeomorphism $f : \partial M_1 \to \partial M_2$.

3. Form the quotient space $M = M_1 \coprod_f M_2$: Take the disjoint union $M_1 \coprod M_2$ and identify $x \in M_1$ with its image $f(x) \in M_2$.

This is a **Heegaard splitting** (**of genus** n) of the 3-manifold M.

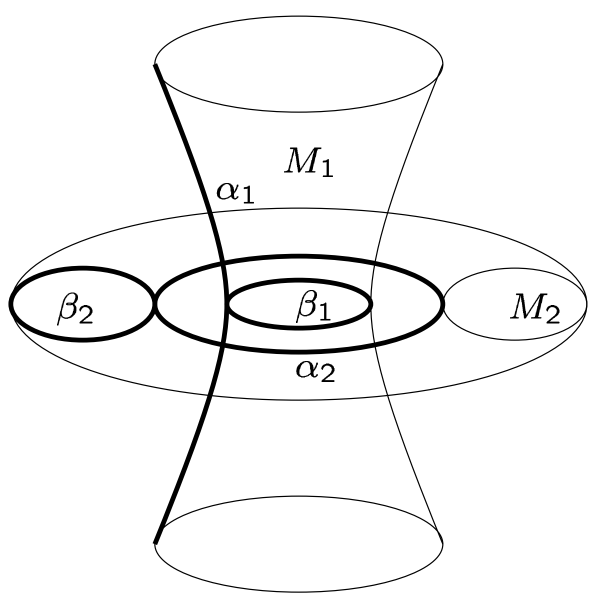
THEOREM

A 3-manifold formed in this way is orientable. Furthermore, any orientable 3-manifold can be presented thusly.

EXAMPLES

1. Genus-0: The 3-sphere $\mathcal{S}^3$.
   Two copies of $\mathcal{B}^3$ glued together along their boundaries ($\cong \mathcal{S}^2$).
   Analogous to $\mathcal{S}^1 = \mathcal{D}^1 \cup \mathcal{D}^1$
   and $\mathcal{S}^2 = \mathcal{D}^2 \cup \mathcal{D}^2$.
   The 'North' and 'South' poles of $\mathcal{S}^3$ are the centres of the $\mathcal{B}^3$s

2. Genus-1: The 3-sphere $\mathcal{S}^3$ (again).
   Can also split $\mathcal{S}^3$ as two solid tori ($\mathcal{D}^2 \times \mathcal{S}^1$):



3. $\mathcal{S}^2 \times \mathcal{S}^1$.
   Glue α_1 to α_2 and β_1 to β_2.

4. Projective space $\mathbb{R}\mathcal{P}^3$.
   Glue each meridian α_i to a $(1, 2)$ torus knot on the surface of the other solid torus.

5. Lens spaces $\mathcal{L}_{p,q}$.
   Glue each meridian α_1 and α_2 to a (q, p) torus knot on the surface of the other torus. This is one construction of the **lens space** $\mathcal{L}_{p,q}$.
   In particular,
   $\mathcal{L}_{1,q} \cong \mathcal{S}^3$,
   $\mathcal{L}_{0,1} \cong \mathcal{S}^2 \times \mathcal{S}^1$,
   and $\mathcal{L}_{2,1} \cong \mathbb{R}\mathcal{P}^3$.
   Also, $\mathcal{L}_{p,q} \cong \mathcal{L}_{p,q'}$ iff $\pm q' \equiv q^{\pm 1}$ (mod p),
   and $\mathcal{L}_{p,q} \approx \mathcal{L}_{p,q'}$ iff $q \equiv m^2 q'$ (mod p).
   For example: $\mathcal{L}_{7,1}$ and $\mathcal{L}_{7,2}$ have the same homotopy type, but are not homeomorphic.

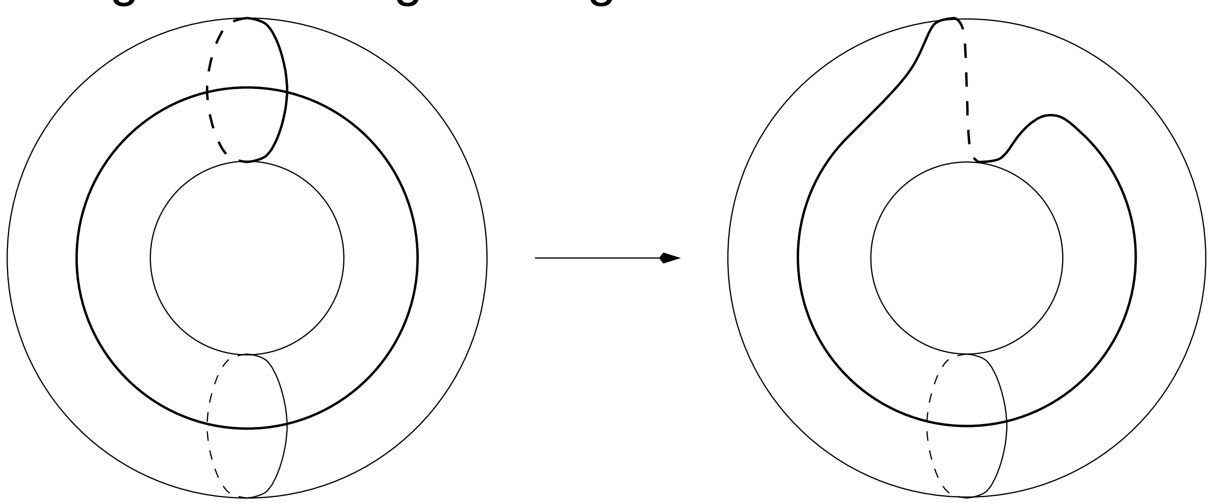
HOMEOMORPHISMS OF SURFACES

Heegaard splittings of a given genus are determined by the gluing homeomorphism.

Want a description of such in terms of suitable elementary operations.

DEHN TWISTS

Cut the surface around a meridional curve, twist, and glue back together again.



THEOREM (DEHN-LICKORISH)

Any orientation-preserving homeomorphism of an oriented 2-manifold (without boundary) is isotopic to a composition of Dehn twists.

COROLLARY

Any orientable 3-manifold can be constructed by cutting out a collection of unknotted solid tori from $\mathcal{S}^3$ and gluing them back in along different boundary homeomorphisms.

COROLLARY (ROKHLIN'S THEOREM)

Every orientable 3-manifold (without boundary) is the boundary of a 4-manifold. That is, $\Omega_3 \cong 0$.

Rational surgery

Take $\mathcal{S}^3$, cut out an unknotted solid torus with meridian α and longitude β. Then glue it back in by identifying α with the curve $p\alpha + q\beta$, where p and q are coprime (this is a (q, p) torus knot).

This surgery is determined completely by the rational number $r = \frac{p}{q}$, which we call the **framing index** of the unknotted torus.

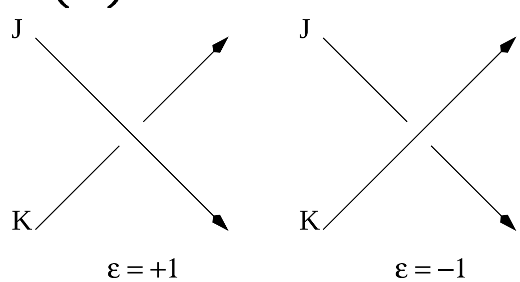
Examples

1. $\bigcirc^0 \cong \mathcal{S}^1 \times \mathcal{S}^2$. This is a **torus switch**
2. $\bigcirc^{\frac{p}{q}} \cong \mathcal{L}_{p,q}$.
3. $\bigcirc^{\pm\frac{1}{n}} \cong \mathcal{S}^3$.
4. $\bigcirc^r = \bigcirc^{\frac{1}{\pm n + \frac{1}{r}}}$

LINKING NUMBERS

To generalise to nontrivial knots, we need to be a bit more careful when choosing the longitude.

Given two curves J and K in $\mathcal{S}^3$, define their **linking number** $\text{lk}(J, K)$ to be the sum, over all the crossings τ, of $\varepsilon(\tau)$:



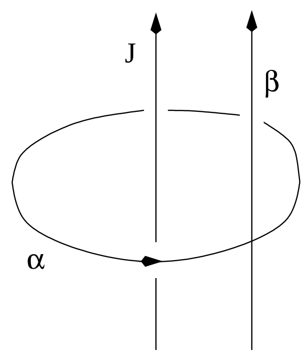
Crossings which don't involve two different components of the link have $\varepsilon = 0$.

$\text{lk}(J, K) = \text{lk}(K, J)$.

If $\text{lk}(J, K) \neq 0$ then J and K are linked. The converse isn't true in general, though.

The linking number is invariant under the Reidemeister moves: It's independent of the isotopy class of the link.

Now, given a knot J, choose a meridian α on its tubular neighbourhood such that $\mathrm{lk}(\alpha, J) = +1$, and a longitude β which is codirected with J such that $\mathrm{lk}(\beta, J) = 0$.



INTEGER SURGERY

We can now do rational surgery on nontrivial links. It turns out, though, that integer surgery is enough:

THEOREM

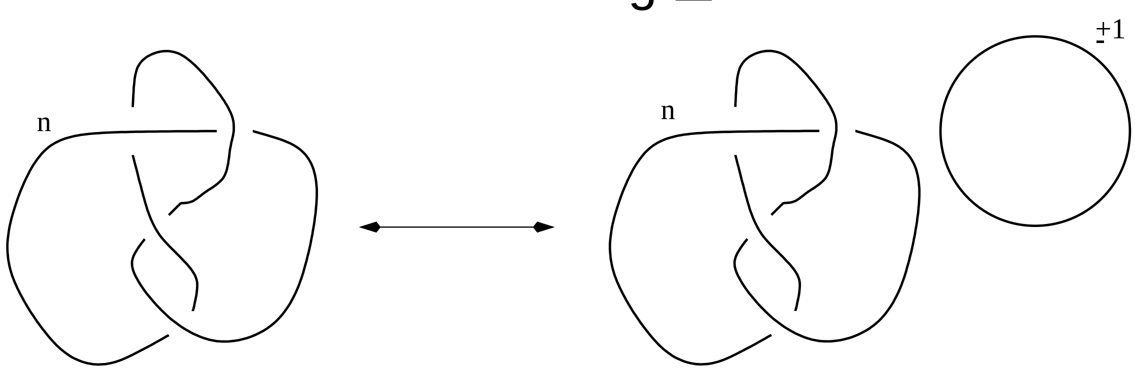
Any compact, orientable 3-manifold without boundary can be obtained by integer surgery on a link in $\mathcal{S}^3$.

Equivalent surgeries

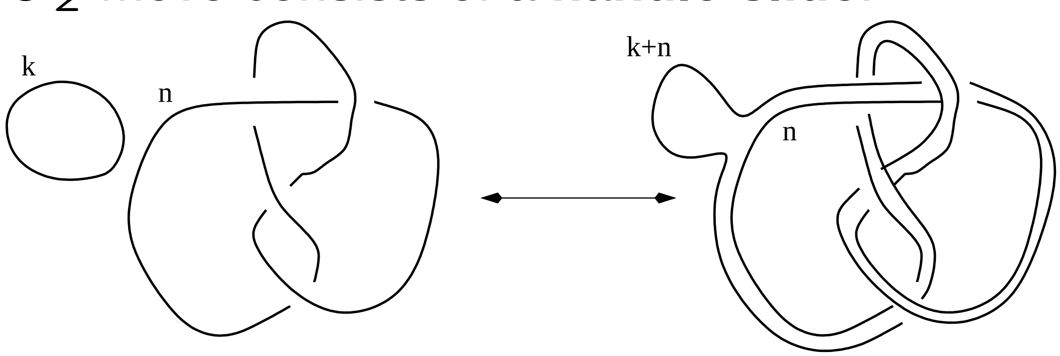
Surgery on $\mathcal{S}^3$ along different framed links can produce homeomorphic manifolds. Two such surgeries are said to be **equivalent**.

The Kirby calculus

An $\mathcal{O}_1$-move consists of adding or deleting an unlinked trivial knot with framing ± 1:

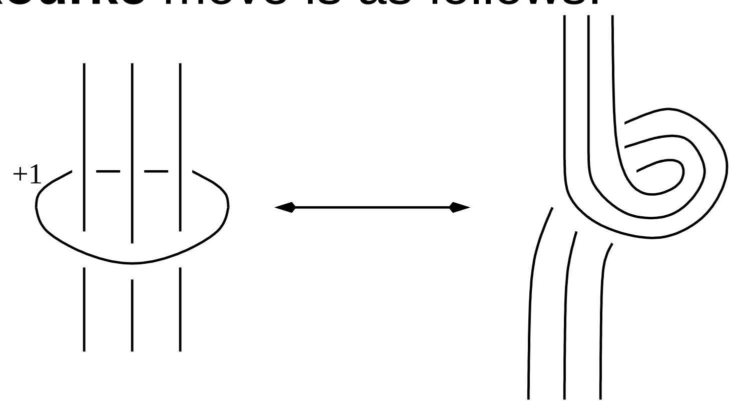


An $\mathcal{O}_2$-move consists of a **handle-slide**:



THEOREM (KIRBY)
Two links in $\mathcal{S}^3$ with integer framings produce the same 3-manifold iff they can be obtained from each other by a finite sequence of Kirby moves and isotopies.

FENN-ROURKE MOVES
A **Fenn-Rourke** move is as follows:



(If the circle has framing -1, then the kinks go the other way.)

THEOREM (FENN-ROURKE)
A framed link L_1 can be transformed by Kirby moves into the framed link L_2 iff this can be done by Fenn-Rourke moves.

THE FUNDAMENTAL GROUP π_1

The **fundamental group** of a topological space X, denoted $\pi_1(X)$ is essentially a way of counting the (1-dimensional) holes in X. Its elements are the homotopy classes of based loops (maps $\mathcal{S}^1 \to X$) in X, with the multiplication operation being given by concatenation and the identity being the loop which can be shrunk down to the basepoint.

For example: $\pi_1(\mathcal{B}^3) \cong 0$, because every loop can be shrunk down to the basepoint.
But $\pi_1(\mathcal{S}^1 \times \mathcal{D}^2) \cong \mathbb{Z}$, homotopy classes of loops being determined by the number of times they wind around the central hole.
And $\pi_1(\mathcal{L}_{p,q}) \cong \mathbb{Z}_p$.

HOMOLOGY GROUPS H_n

The **homology groups** $H_n(X)$ are another way of counting the n-dimensional holes in X.

In particular, part of Hurewicz' theorem says that, if X is path-connected:

PROPOSITION

$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$.
That is, the first homology of X is the same as the abelianisation of the fundamental group.

HOMOLOGY 3-SPHERES

A **homology 3-sphere** is a compact, path-connected 3-manifold M^3 (without boundary), which has the same series of homology groups as $\mathcal{S}^3$:

$H_0(M) \cong H_3(M) \cong \mathbb{Z}$
$H_1(M) \cong H_2(M) \cong 0.$

Or (by Poincaré duality and the UCT):
$H_1(M)$ is trivial.

Or (by the above fragment of Hurewicz' theorem):
$\pi_1(M)$ coincides with its commutator subgroup

$$[\pi_1(M), \pi_1(M)] = \{aba^{-1}b^{-1} \mid a, b \in \pi_1(M)\}$$

CONJECTURE (POINCARÉ)

Every homology 3-sphere is homeomorphic to $\mathcal{S}^3$.

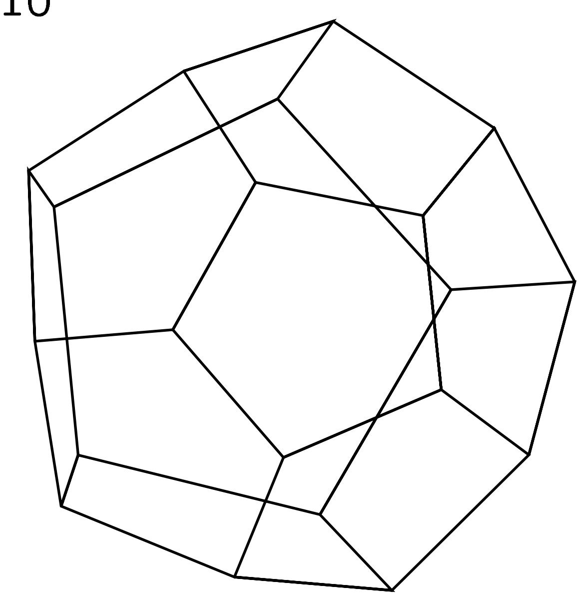
This is false, leading Poincaré to suggest:

CONJECTURE (POINCARÉ)
Every *homotopy* 3-sphere is homeomorphic to $\mathcal{S}^3$.

POINCARÉ'S HOMOLOGY 3-SPHERE
This is a 3-manifold P^3 with trivial H_1 but nontrivial π_1 (and is hence not homeomorphic to $\mathcal{S}^3$).

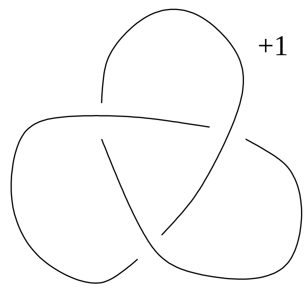
Many different constructions. . .
(cf. Kirby and Scharlemann: *Eight faces of the Poincaré homology 3-sphere*)

DODECAHEDRAL SPACE
Take a solid dodecahedron and identify opposite faces with a $\frac{2\pi}{10}$ twist.



Surgery on the trefoil

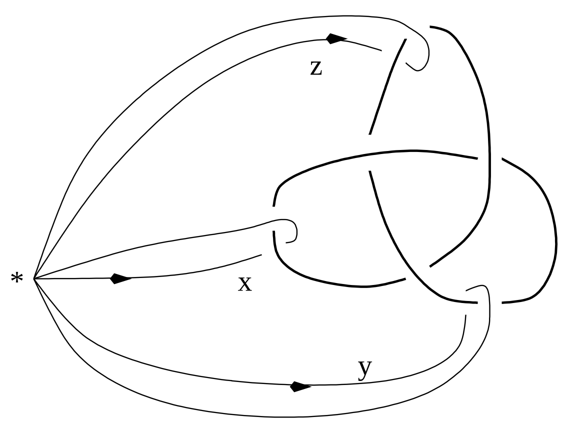
Do surgery on $\mathcal{S}^3$ along the right trefoil K with framing +1.



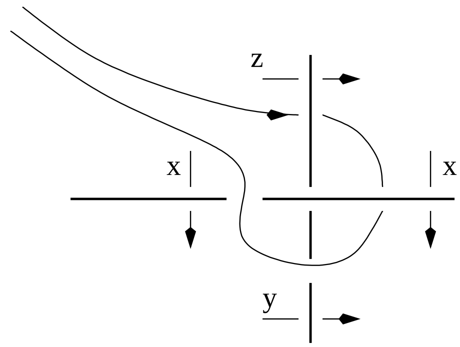
This gives a manifold P which isn't homeomorphic to $\mathcal{S}^3$, because $\pi_1(P)$ is nontrivial.

Calculation of $\pi_1(P)$

First, calculate the fundamental group of the complement $\mathcal{S}^3 \setminus K$. This is generated by three loops x, y, z



subject to the relations $xy = yz = zx$:



giving a presentation

$$\langle x, y \mid xyx = yxy \rangle$$

When we glue in the solid torus, we attach its meridional disk to the longitude of the tubular neighbourhood of the (removed) trefoil, with framing +1.

Thus, $\pi_1(P)$ is the quotient of $\pi_1(\mathcal{S}^3 \setminus K)$ obtained by killing the word $x^{-2}yxz = x^{-2}yx^2yx^{-1}$ corresponding to this longitude:

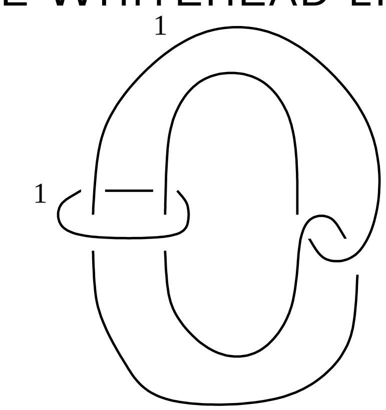
$$\pi_1(P) = I = \langle x, y \mid xyx = yxy, yx^2y = x^3 \rangle$$

By substituting $a = x, b = xy$, we get the neater form:

$$I \cong \langle a, b \mid a^5 = b^3 = (ba)^2 \rangle$$

This group (the **binary icosahedral group**) is non-trivial (it has order 120 and is the isometry group of the icosahedron), hence $P \ncong \mathcal{S}^3$, but it has trivial abelianisation.

Surgery on the Whitehead link



Surgery on the Borromean rings