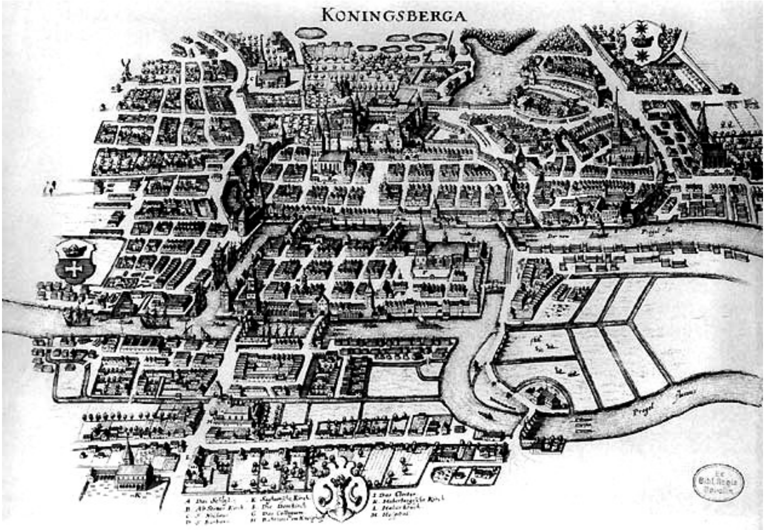
Networks and Small Worlds

Satellite 4

Nicholas Jackson

Easter 2013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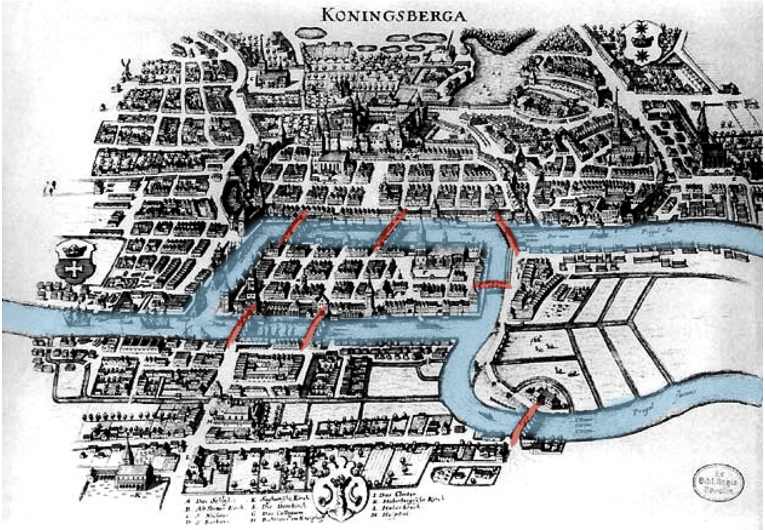

KONINGSBERGA

- Born in Basel, Switzerland
- University of Basel (1720–1723)
- Awarded doctorate in 1726, supervised by Johann Bernoulli
- St Petersburg (1727–1741)
- Prussian Academy of Sciences, Berlin (1741–1766)
- St Petersburg (1766–1783)
- Collected works published from 1911 onwards (76 volumes so far)
- Calculus, graph theory, mechanics, fluid dynamics, optics, astronomy, music, . . .

Thus for any configuration that may arise, the easiest way of determining whether a single crossing of all the bridges is possible is to apply the following rules:

- If there are more than two regions which are approached by an odd number of bridges, no route satisfying the required condition can be found.
- If, however, there are only two regions with an odd number of approach bridges, the required journey can be completed provided it originates in one of the regions.
- If, finally, there is no region with an odd number of approach bridges, the required journey can be effected, no matter where it begins.

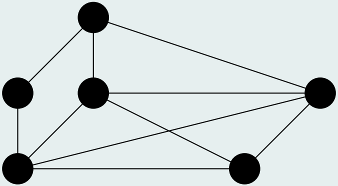
These rules solve completely the problem initially proposed.

— Leonhard Euler,

Solutio problematis ad geometriam situs pertinentis (1735)

Definition

A graph or network consists of a set of nodes or vertices, linked by arcs or edges.



Definition

The degree or valency of a node is the number of incident edges it has.

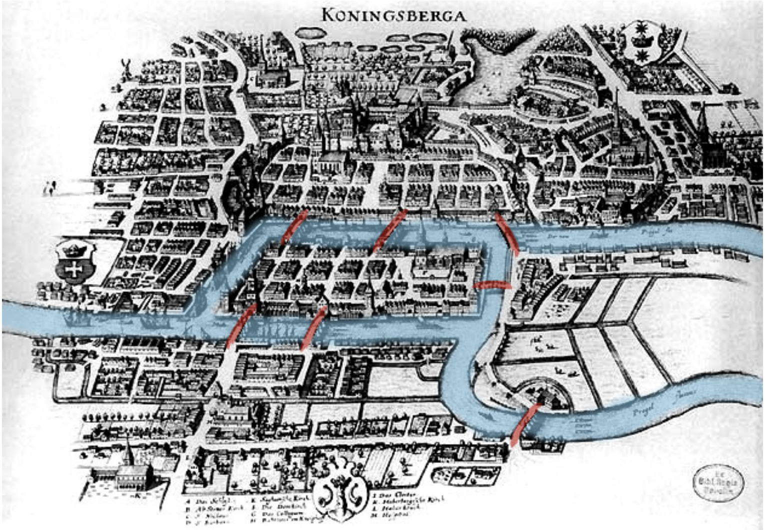
DEFINITION

An Eulerian path in a graph is a route through the graph that passes along each edge exactly once.

THEOREM (EULER 1735)

A graph has an Eulerian path if and only if either:

- *each node has even degree, or*
- *exactly two nodes have odd degree, and all the rest have even degree.*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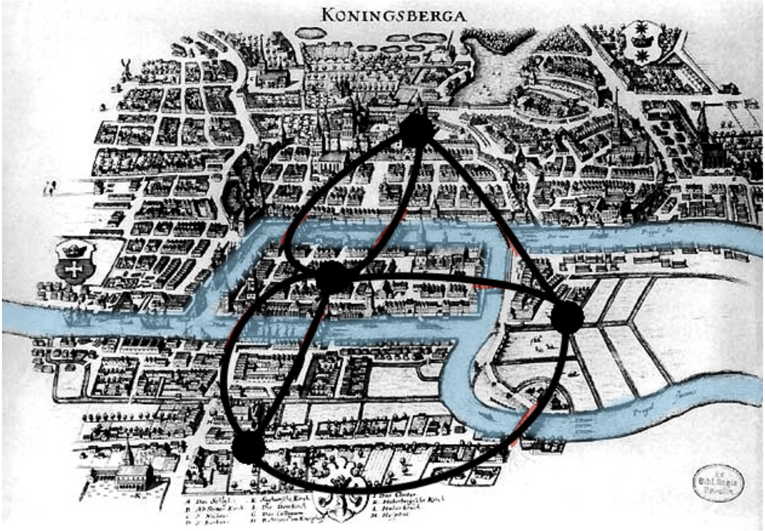
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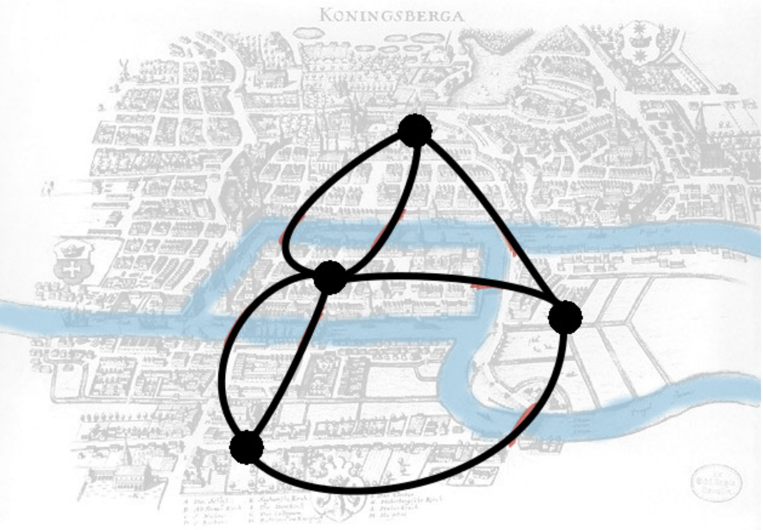
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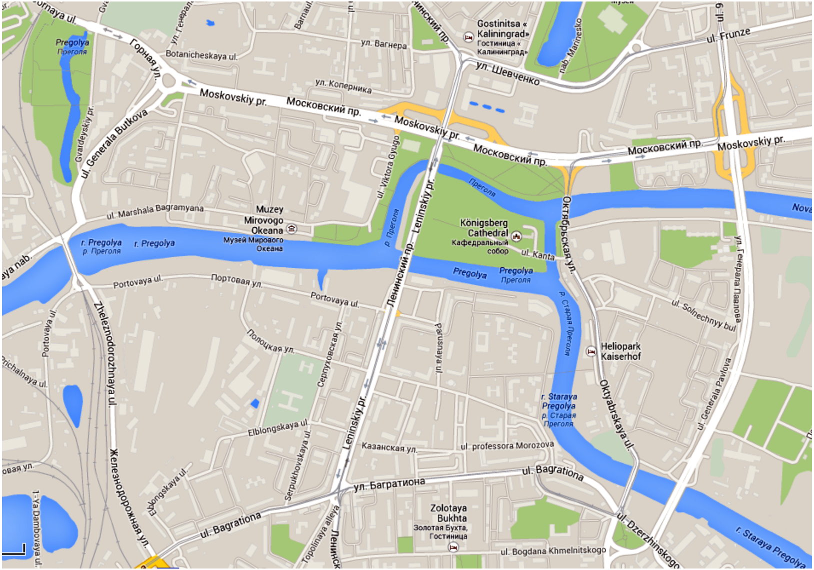
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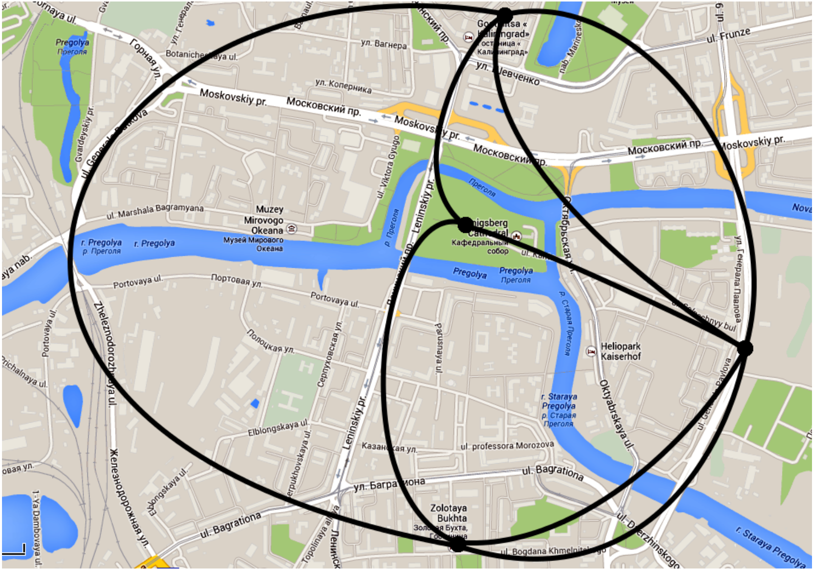
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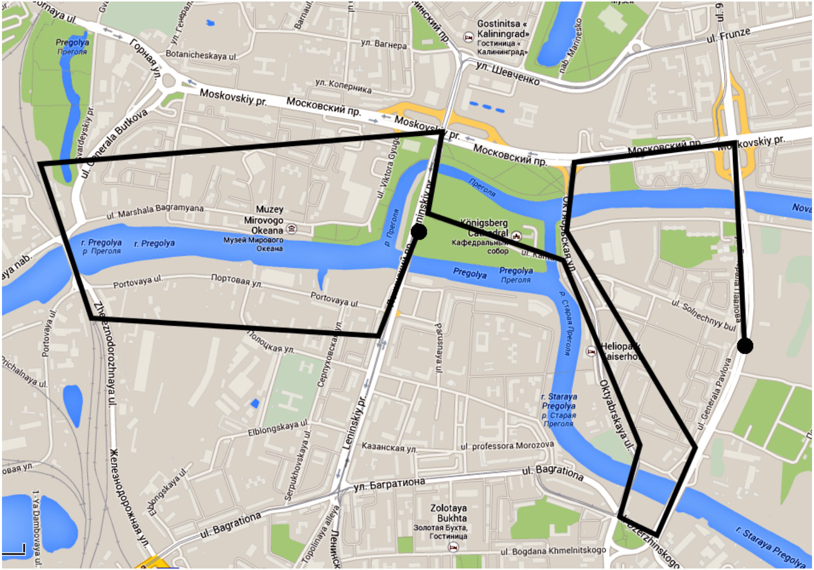

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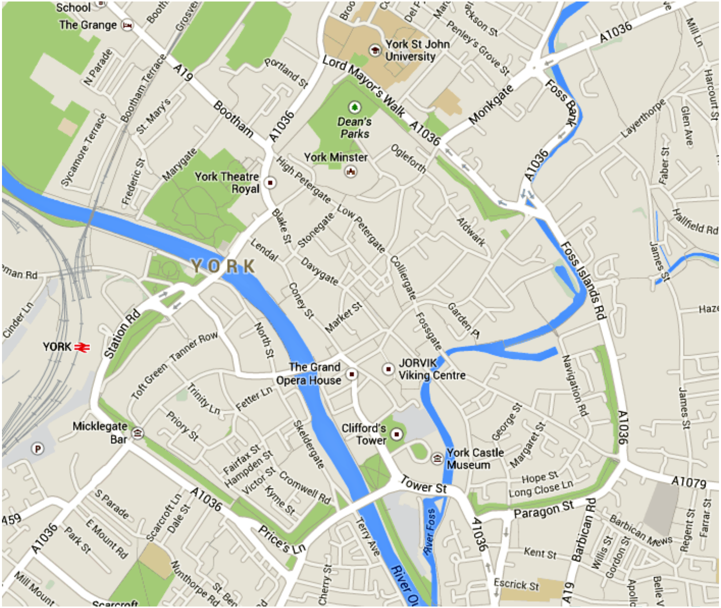
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Frederic St
Marygate
York Theatre Royal
High Petergate
Low Petergate
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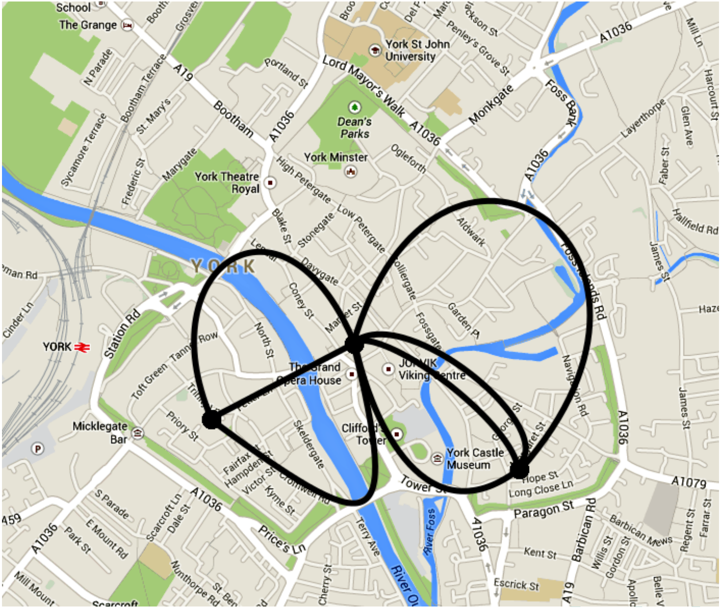
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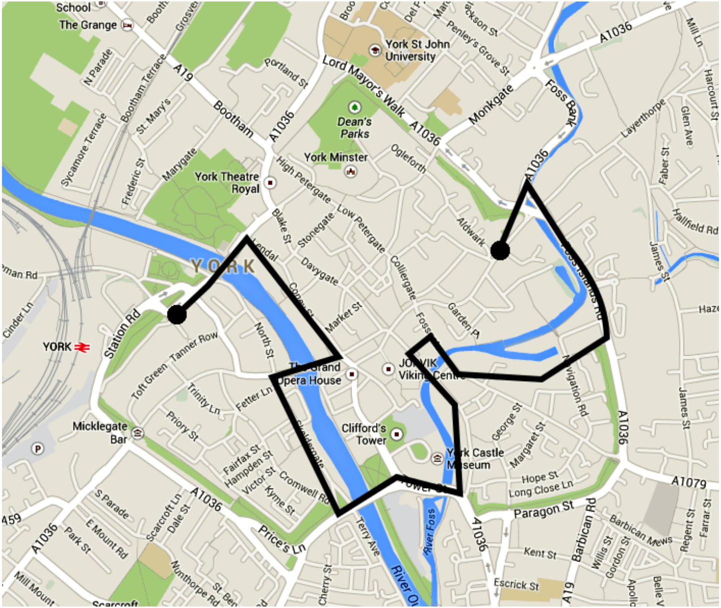
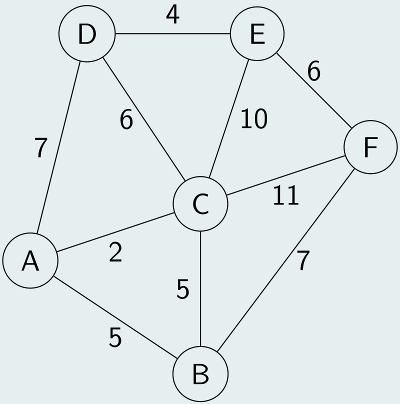


Sometimes we care about distances or travel times between nodes. Model this by attaching a weight (a positive integer) to each edge.



What is the shortest path from A to E?

- Born in Rotterdam, The Netherlands
- Cowrote the first ALGOL 60 compiler
- Received the 1972 Turing Award
- Shunting yard algorithm for parsing mathematical expressions
- Banker's algorithm for shared resource allocation and deadlock avoidance
- *Go To Statement Considered Harmful*, Communications of the ACM 11:3 (1968) 147–148
- Strongly opposed teaching BASIC
- "[Computer science] is like referring to surgery as 'knife science'"

Algorithm

1. Give each node a estimated distance or cost: 0 for the start node and ∞ for everything else.

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ALGORITHM

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4. For any nodes whose estimated distances were updated, set their predecessor node to be the current node.

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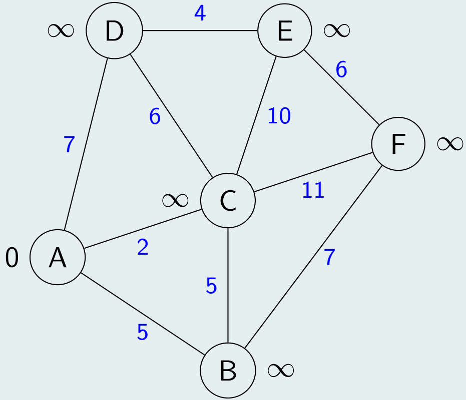
3. Consider all unvisited neighbours of the current node, recalculate their estimated distances, and update any whose new estimated distance is less than the old value.
4. For any nodes whose estimated distances were updated, set their predecessor node to be the current node.
5. If we're not yet at the destination node, move to the unvisited node with the smallest estimated distance and go to step 2.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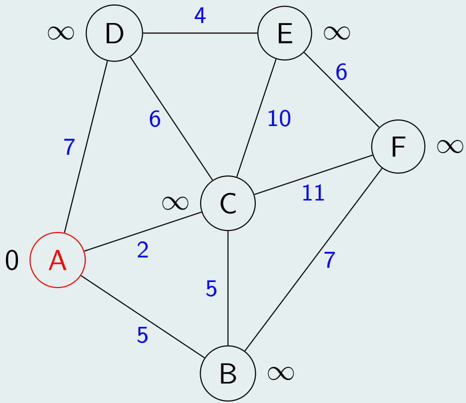
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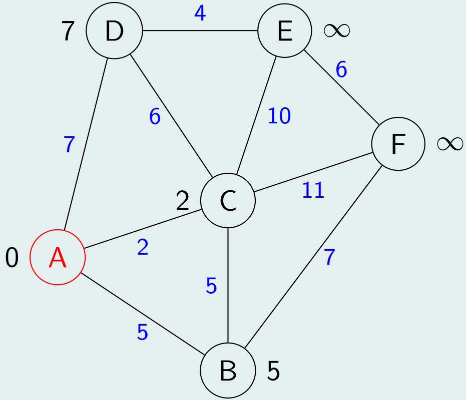
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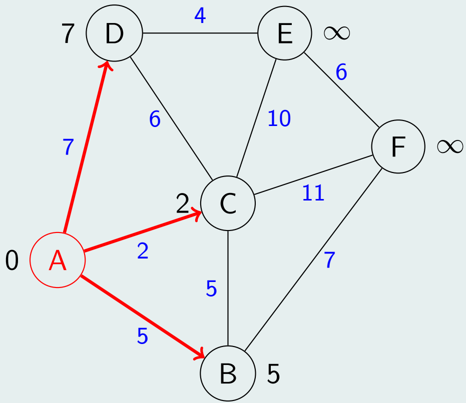
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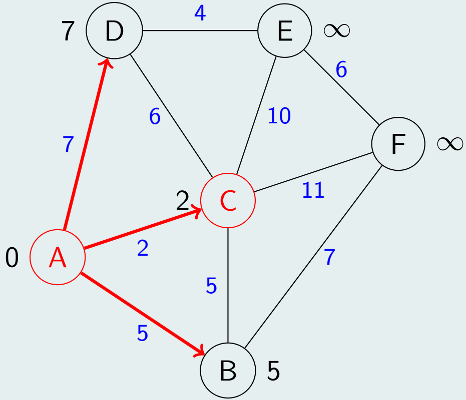

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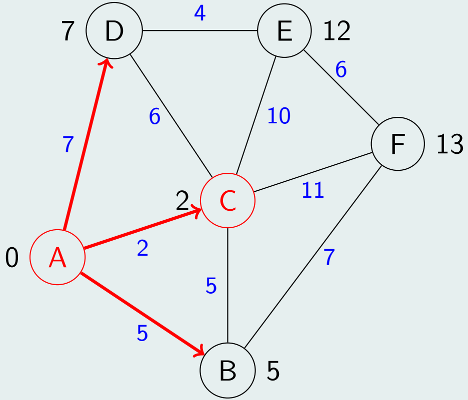
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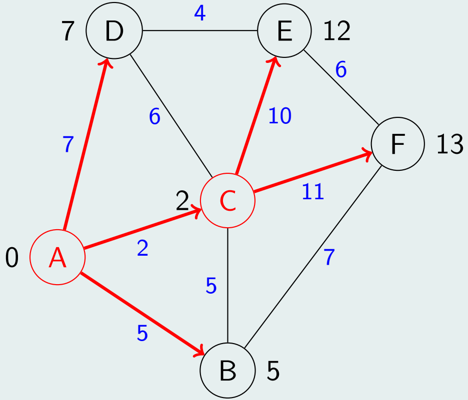
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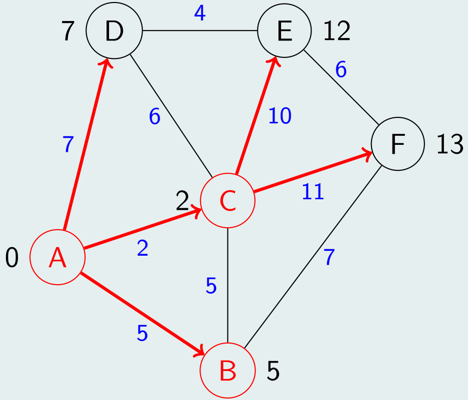
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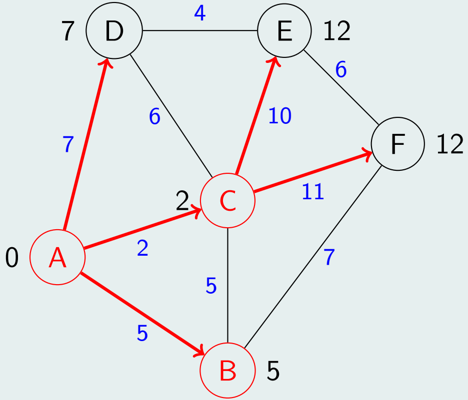
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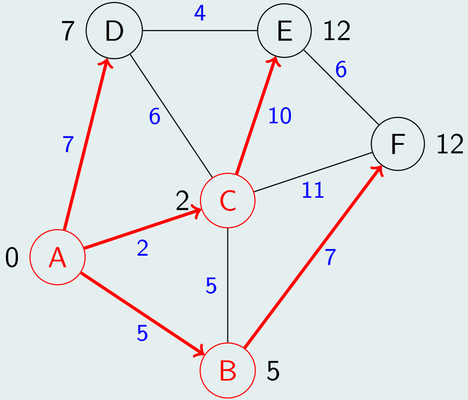
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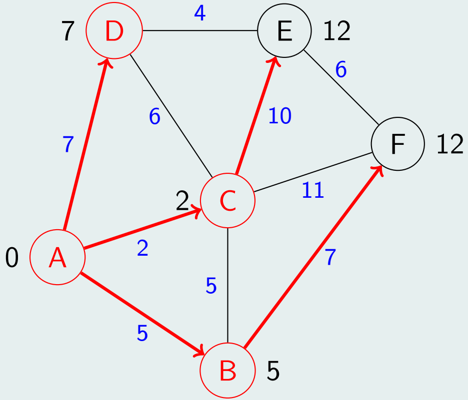
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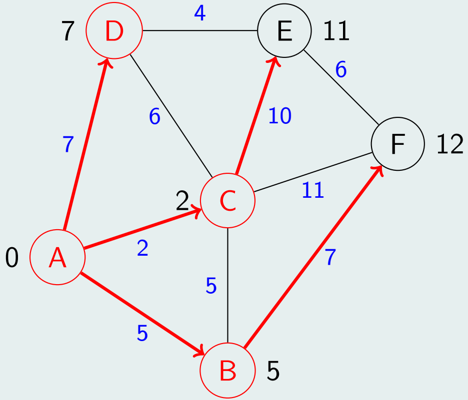

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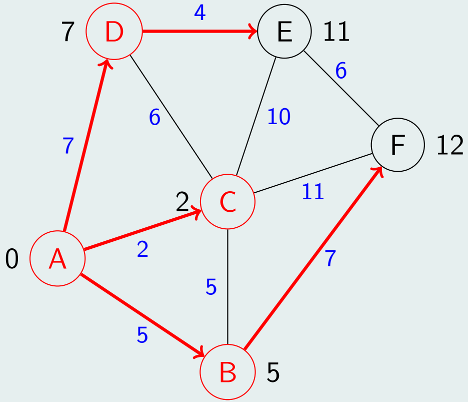
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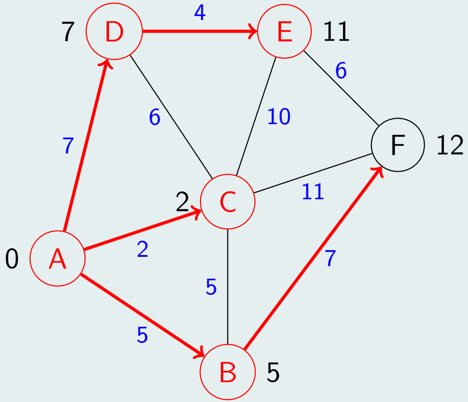

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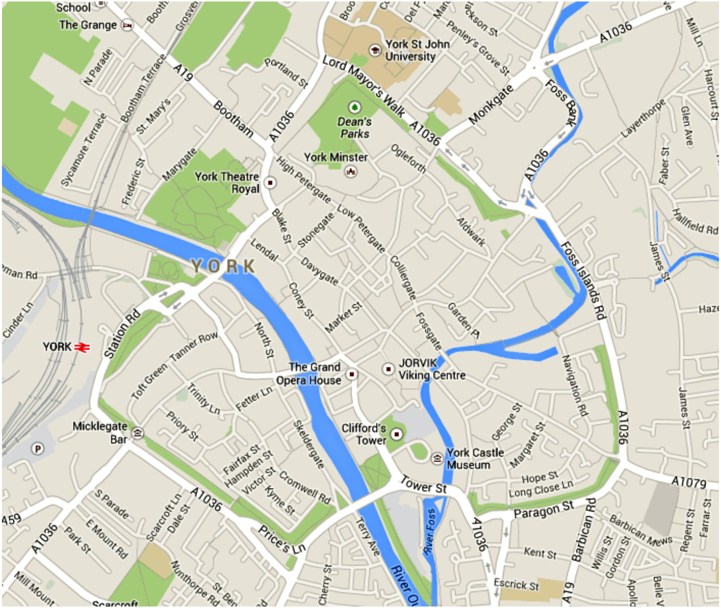
Sometimes we want to study traffic or flow through networks, especially road or telecommunications networks.

Each edge $e = (u, v)$ between two vertices u and v has a capacity $c(u, v)$ and a (variable) flow $f(u, v)$, such that:

- $f(u, v) \leqslant c(u, v)$: flow cannot exceed capacity
- $f(v, u) = -f(u, v)$: net flow from u to v must be the opposite of the net flow from v to u
- $\sum_v f(u, v) = 0$ unless u is a source or sink: total flow through a node u is conserved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Sycamore Terrace
Frederic St
Marygate

8–14 September 2000: Fuel protests and blockades

4 November 2000: Floods

4 November 2000: Floods

- Born in Budapest, Hungary
- Itinerant mathematician: would turn up at a department and announce "my brain is open"
- A machine for turning coffee (and amphetamines) into theorems
- Very prolific and collaborative: $\sim$ 1525 published articles with 511 coauthors
- Combinatorics, graph theory, number theory, analysis, probability theory, set theory, . . .
- Died while attending a mathematics conference in Warsaw
- Paul Hoffman, *The Man Who Loved Only Numbers* (1998)

Erdős number: minimal path-length in the collaboration graph.

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- **P Erdős**, **M Ram Murty**, **V Kumar Murty**, *On the enumeration of finite groups*, Journal of Number Theory 25 (1987) 360–378

Erdős number: minimal path-length in the collaboration graph.

- **N M Dunfield**, **S K Friedl**, **N J Jackson**, *Twisted Alexander polynomials of hyperbolic knots*, Experimental Mathematics 21 (2012) 329–352
- **N M Dunfield**, **D Ramakrishnan**, *Increasing the number of fibered faces of arithmetic hyperbolic 3-manifolds*, American Journal of Mathematics 132 (2010) 53–97
- **V Kumar Murty**, **D Ramakrishnan**, *Period relations and the Tate conjecture for Hilbert modular surfaces*, Inventiones Mathematicæ 89 (1987) 319–345
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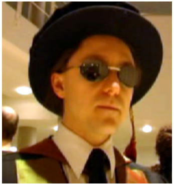

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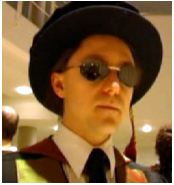 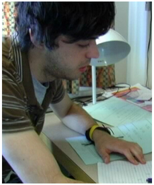

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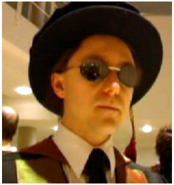
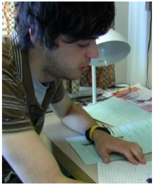


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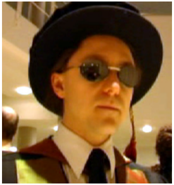
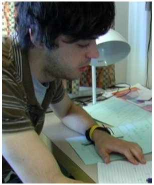

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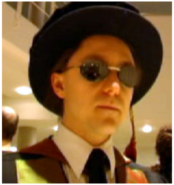 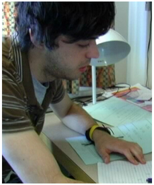   

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Bacon numbers

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Definition

The closeness centrality of a node v measures the average length of shortest paths from v to every other node:

$$C_C(v) = \left(\frac{\sum_w d(v,w)}{N-1} \right)^{-1}$$

If $C_C(v) = 1$ then v is connected to every other node by one step.

As of 2004,

$$C_D(\text{Erdős}) = 511 \qquad C_C(\text{Erdős}) = 0.215$$

Erdős is the most central node in the main component of the mathematical collaboration network.

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But

$$C_D(\text{Keitel}) = 4259 \qquad C_C(\text{Keitel}) = 0.352$$

Kevin Bacon is the 370th most central film actor (out of 2.6 million people listed in the IMDb).

- First posited by Frigyes Karinthy (1887–1938), a Hungarian playright and author, in 1929 short story *Láncszemek* (*Chains*)
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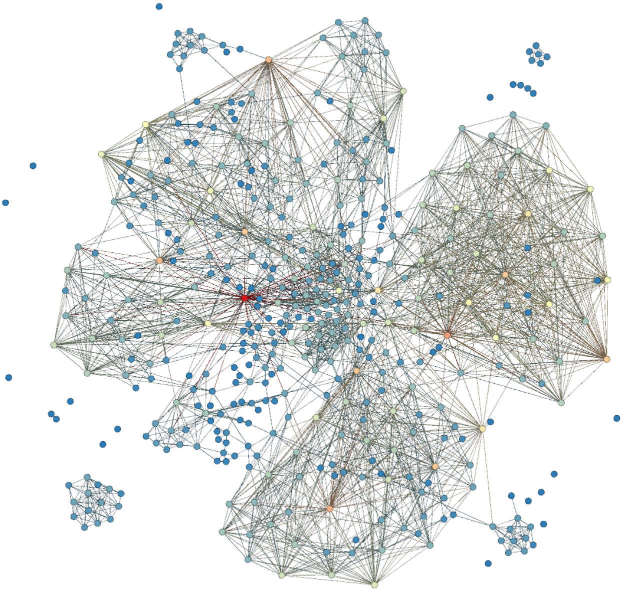
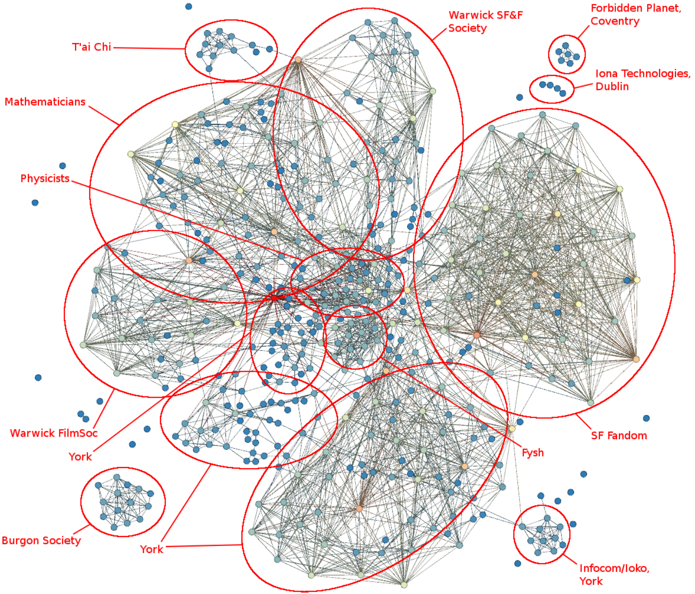
- Problems with methodology: bias due to low rate of response (only 64 of 296 letters got to their destination).
- Small world network: mean shortest path length grows slowly ($\propto \log(N)$), high clustering coefficient, many hubs (high-degree nodes).
- The mathematical and film collaboration graphs are small world networks. (Clustering coefficient for mathematics is 0.14.)

Definition

If a node v has k_v neighbours, there can be at most $\frac{1}{2}k_v(k_v - 1)$ edges between them all (the complete graph with k_v vertices). The clustering coefficient C_v is the proportion of these edges that exist. The clustering coefficient C is the average of all the C_v.




T'ai Chi
Warwick SF&F Society
Forbidden Planet, Coventry
Iona Technologies, Dublin
Mathematicians
Physicists
SF Fandom
Warwick FilmSoc
York
Fysh
Burgon Society
York
Infocom/Ioko, York

Definition

The betweenness centrality of a node v measures how many shortest paths go through v:

$$C_B(v) = \sum_{u,w} \frac{g_{u,v,w}}{g_{u,w}} \Big/ \tfrac{1}{2}(N-1)(N-2)$$

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Can also define the betweenness of an edge.

Betweenness clustering algorithm

While (betweenness of any edge) $>$ (fixed threshold value),

- Remove the edge with the highest betweenness
- Recalculate betweenness for all edges

This isn't very efficient (scales as $O(N^3)$), but mostly works.

Can use these techniques to understand the spread of epidemics.

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- Measles has a $\sim 90\%$ infectivity amongst susceptible contacts.
- The basic reproduction number (R_0) varies between 12 and 18 depending on environmental factors.
- Vaccination breaks links in the infectivity network.
- Critical vaccination coverage is $1 - 1/R_0$; for measles this is between $\sim 91.7\%$ and $\sim 94.4\%$.

- The Oracle of Bacon: `http://oracleofbacon.org/`
- The Erdős Number Project: `http://www.oakland.edu/enp/`
- **Duncan Watts**, *Small Worlds*, Princeton University Press (1999)
- **Paul Hoffman**, *The Man Who Loved Only Numbers*, Fourth Estate (1998)
- Coursera: *Social Network Analysis*