Homology of racks and quandles[*][†]

Nicholas Jackson

[*]**N J Jackson**, *Extensions of racks and quandles*, Homology Homotopy Appl 7:1 (2005) 151–167

[†]**N J Jackson**, *Rack and quandle homology with heterogeneous coefficients*, to appear in Algebr Geom Topol 6 (2006)

Definitions

A **rack** (or **wrack**) is a set X equipped with a binary operation (sometimes denoted $*$, $\triangleleft$ or $\triangleright$, but here written as exponentiation) such that:

(R1) For every $a, b \in X$ there is a unique $c \in X$ such that $c^b = a$. We write this unique element $a^{\bar{b}}$.

(R2) The **rack identity** $a^{bc} = a^{cb^c}$ holds for every $a, b, c \in X$.

A **quandle** is a rack where

(Q) $a^a = a$ for every $a \in X$.

A rack **homomorphism** is a map $f\colon X \to Y$ such that $f(x^y) = f(x)^{f(y)}$ for all $x, y \in X$.

Examples

1 Conjugation racks

For any group G, let $\mathrm{Conj}\, G$ be the underlying set of G with rack operation $g^h := h^{-1}gh$.

2 Core racks

For any group G, let $\mathrm{Core}\, G$ be the underlying set of G with rack operation $g^h := hg^{-1}h$.

3 Trivial racks

$T_n := \{0, \ldots, n-1\}$ with $a^b := a$.

4 Cyclic racks

$C_n := \{0, \ldots, n-1\}$ with $a^b := a+1$.

5 Dihedral racks

$D_n := \{0, \ldots, n-1\}$ with $a^b := 2b - a \pmod{n}$.

6 Alexander quandles

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$. Then any Λ–module may be equipped with the rack structure

$$a^b := ta + (1 - t)b.$$

The operator group

The rack axioms imply the function $\pi_b\colon\ x \mapsto x^b$ is a permutation of X. The **operator group** of X is the group

$$\operatorname{Op} X := \langle \pi_x : x \in X \rangle.$$

The action of this group splits X into **orbits**. A rack with a single orbit is **transitive**.

$\operatorname{Op}\colon \mathsf{Rack} \to \mathsf{Group}$ is not functorial.

The associated group

The **associated group** of X is the group

$$\operatorname{As} X := \langle x \in X \mid x^y = y^{-1}xy \text{ for } x, y \in X \rangle.$$

This gives a functor $\operatorname{As}\colon \mathsf{Rack} \to \mathsf{Group}$ which is the left adjoint of the conjugation functor $\operatorname{Conj}\colon \mathsf{Group} \to \mathsf{Rack}$

The rack space

The **rack space** BX of a rack X is the analogue of the classifying space BG of a group G.

It defines a functor $B\colon \mathsf{Rack} \to \mathsf{Top}_*$, and consists of:

1. One 0–cell $*$
2. One 1–cell $* \xrightarrow{x} *$ for each element $x \in X$
3. One 2–cell $\begin{array}{ccc} * & \xrightarrow{x^y} & * \\ {\scriptstyle y}\uparrow & & \uparrow{\scriptstyle y} \\ * & \xrightarrow[x]{} & * \end{array}$ for each ordered pair (x, y) of elements of X

$$\vdots$$

n. One n–cell for each ordered n–tuple of elements of X

Homology of racks

We can define

$$\begin{aligned} H_n(X) &:= H_n(BX) \\ H^n(X) &:= H^n(BX). \end{aligned}$$

Specifically,

$$\begin{aligned} C_n(X) &= FA(X^n) \\ \partial_n(x_1, \ldots, x_n) &= \sum_{i=2}^{n} (-1)^i (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &- \sum_{i=2}^{n} (-1)^i (x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n) \end{aligned}$$

Examples

$H_1(BX) = \mathbb{Z}^{|\mathsf{Orb}(X)|}$, and $\pi_1(BX) = \mathsf{As}(X)$

$BT_m \simeq \Omega(\bigvee^m S^2)$ (so $H_n(BT_m) \neq 0$ for all n)

$BF_m \simeq \bigvee^m S^1$ (so $H_n(BF_m) = 0$ for $n > 1$)

Quandle homology

If X is a quandle, we can define **quandle homology** groups:

$$\begin{aligned} C_n^Q(X) &= FA(X^n)/P_n(X) \\ P_n(X) &= \big\{(x_1,\ldots,x_n)\in X^n : x_i = x_{i+1} \text{ for some } i\big\} \end{aligned}$$

Extensions

Let X be a rack, let A be an Abelian group, and let $f\colon X\times X\to A$ be some function.
We can define a binary operation on $A\times X$:

$$(a,x)^{(b,y)} = (a+f(x,y),x^y)$$

This gives $A\times X$ the structure of a rack iff f satisfies the **2–cocycle condition**

$$f(x^y,z)+f(x,y) = f(x^z,y^z)+f(x,z).$$

Also, any two functions $f_1,f_2\colon X\times X\to A$ give equivalent extensions if

$$f_1 = f_2+g$$

where $g\colon X\times X\to A$ satisfies the **2–coboundary condition**

$$g(x,y) = h(x)+h(x^y)$$

for some $h\colon X\to A$.

In other words, rack extensions of X by the Abelian group A are in bijective correspondence with elements of $H^2(X;A)$. That is,

$$\mathrm{Ext}(X,A) \cong H^2(X;A)$$

In the case where X is a quandle, this process makes $A\times X$ a quandle (rather than just a rack) if

$$(a,x)^{(a,x)} = (a+f(x,x), x^x) = (a,x);$$

that is, if $f(x,x)=0$ for all $x\in X$. But this is just the extra condition on quandle 2–cocycles, so

$$\mathrm{Ext}_Q(X,A) \cong H^2_Q(X;A).$$

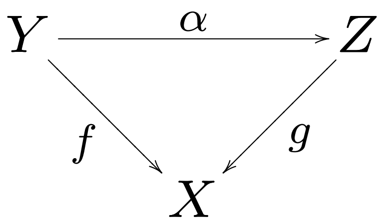
Modules

We consider group homology $H_n(G; A)$ with coefficients in a G–module A, so what is an X–module?

Jon Beck devised a general answer to this question in his 1967 PhD thesis.

Given an object X in a category C, a **Beck module** over X is an Abelian group object in C/X.

The objects of the **slice category** C/X are C–morphisms $f\colon Y \to X$, and the morphisms are commutative triangles



If we do this in the category Group, we find that

$$\mathsf{Ab}(\mathsf{Group}/G) \cong {}_G\mathsf{Mod}$$

Abelian group objects in Group/G correspond to split extensions

$$A \rightarrowtail A \rtimes G \twoheadrightarrow G$$

where A is a G–module.

Following this procedure for the category Rack yields the following:

A (left) **rack module** $\mathcal{A} = (A, \phi, \psi)$ over X consists of

- an Abelian group A_x for each $x \in X$, with
- isomorphisms $\phi_{x,y}\colon A_x \to A_{x^y}$ and
- homomorphisms $\psi_{y,x}\colon A_y \to A_{x^y}$ for each $x, y \in X$

such that

$$
\begin{aligned}
\phi_{x^y,z}\phi_{x,y} &= \phi_{x^z,y^z}\phi_{x,z} \\
\phi_{x^y,z}\psi_{y,x} &= \psi_{y^z,x^z}\phi_{y,z} \\
\psi_{z,x^y} &= \phi_{x^z,y^z}\psi_{z,x} + \psi_{y^z,x^z}\psi_{z,y}
\end{aligned}
$$

for all $x, y, z \in X$.

In particular, this means that $A_x \cong A_y$ if x and y lie in the same orbit of X.
Modules $\mathcal{A}$ where $A_x \cong A_y$ for all $x, y \in X$ are said to be **homogeneous**, otherwise **heterogeneous**.
We may also write $\phi_{x^y,z}\phi_{x,y}$ as $\phi_{x,yz}$, where yz is considered as an element of $\mathsf{As}\, X$.

Given two X–modules $\mathcal{A} = (A, \phi, \psi)$ and $\mathcal{B} = (B, \chi, \omega)$, an X**–map** or X**–module homomorphism** $f\colon \mathcal{A} \to \mathcal{B}$ is a collection of maps $f_x\colon A_x \to B_x$ such that

$$\begin{aligned} f_{x^y}\phi_{x,y} &= \chi_{x,y} f_x \\ f_{x^y}\psi_{x,y} &= \omega_{x,y} f_y \end{aligned}$$

This means we can define a category RMod_X, of rack X–modules and X–maps, which is equivalent to $\mathsf{Ab}(\mathsf{Rack}/X)$.
If X is a quandle, there is also a subcategory $\mathsf{QMod}_X \cong \mathsf{Ab}(\mathsf{Quandle}/X)$.

In particular, these categories are Abelian, and hence suitable environments to define homology theories in.

1 Trivial X–modules

If $\phi_{x,y} = \mathrm{Id}$ and $\psi_{y,x} = 0$, the module is said to be **trivial**. In particular, any Abelian group A can be considered as a trivial homogenous X–module $\mathcal{A}$.

2 As(X)–modules

Let A be an As(X)–module, let $\phi_{x,y}\colon\ a \mapsto y{\cdot}a$, and $\psi_{y,x}\colon\ a \mapsto a{-}x{\cdot}a$. This is a nontrivial homogeneous X–module.

3 Dihedral X–modules

Let $D_x = \mathbb{Z}_n$, $\phi_{x,y}\colon\ k \mapsto -k$ and $\psi_{y,x}\colon\ k \mapsto 2k$. Then $\mathcal{D} = (D, \phi, \psi)$ is a X–module.

4 Alexander X–modules

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$. Then if A_x is a Λ–module, $\phi_{x,y}\colon\ a \mapsto ta$, and $\psi_{y,x}\colon\ a \mapsto (1-t)a$, the triple $\mathcal{A} = (A, \phi, \psi)$ is an X–module.

Extensions revisited

Let X be a rack, let $\mathcal{A} = (A, \phi, \psi)$ be an X–module, and let

$$\sigma = \{\sigma_{x,y} \in A_{x^y} : x, y \in X\}$$

be a collection of elements of the groups A_x. Then we may define a binary operation on the set

$$E[X, \mathcal{A}, \sigma] = \{(a, x) : x \in X, a \in A_x\}$$

given by

$$(a, x)^{(b,y)} := (\phi_{x,y}(a) + \sigma_{x,y} + \psi_{y,x}(b), x^y).$$

This gives $E[X, \mathcal{A}, \sigma]$ the structure of a rack iff

$$\phi_{x^y,z}(\sigma_{x,y}) + \sigma_{x^y,z} = \phi_{x^z,y^z}(\sigma_{x,z}) + \sigma_{x^z,y^z} + \psi_{y^z,x^z}(\sigma_{y,z}).$$

In the case where $\mathcal{A}$ is an Abelian group (considered as a trivial homogeneous X–module), this reduces to

$$\sigma_{x,y} + \sigma_{x^y,z} = \sigma_{x,z} + \sigma_{x^z,y^z}$$

which is just (a slightly rewritten form of) the 2–cocycle condition mentioned earlier.

Similarly, any two extensions $E[X, \mathcal{A}, \sigma]$ and $E[X, \mathcal{A}, \tau]$ are equivalent iff

$$\tau_{x,y} = \sigma_{x,y} + (\phi_{x,y}(\upsilon_x) + \upsilon_{x^y} + \psi_{y,x}(\upsilon_y))$$

where

$$\upsilon = \{\upsilon_x : x \in X, \upsilon_x \in A_x\}$$

is some collection of elements of the groups A_x.

If $\mathcal{A}$ is an Abelian group, this reduces to the 2–coboundary condition

$$\tau_{x,y} = \sigma_{x,y} + (\upsilon_x + \upsilon_{x^y}).$$

So we can define groups Ext$(X, \mathcal{A})$ which classify rack extensions of X by $\mathcal{A}$.

We can also define groups Ext$_Q(X, \mathcal{A})$ which classify quandle extensions, where the **factor set** σ must also satisfy the condition

$$\sigma_{x,x} = 0.$$

Free modules

Given a collection $S = \{S_x : x \in X\}$ of sets, we can define the **free** X**−module** $\mathcal{F} = (F, P, \Lambda)$ with basis S.

F_x is the free Abelian group generated by symbols of the form $\rho_{x^{\bar{w}},w}(s)$ and $\rho_{x^{\bar{w}},w}\lambda_{y,x^{\bar{w}\bar{y}}}(t)$, where $s \in S_{x^{\bar{w}}}$ and $t \in S_y$, modulo relations

$$
\begin{array}{lrcl}
(1) & \rho_{x^u,v}\rho_{x,u} & = & \rho_{x^v,u^v}\rho_{x,v} = \rho_{x,uv} \\
(2) & \rho_{x^y,v}\lambda_{y,x} & = & \lambda_{y^v,x^v}\rho_{y,v} \\
(3) & \lambda_{z,x^y} & = & \lambda_{y^z,x^z}\lambda_{z,y} + \rho_{x^z,y^z}\lambda_{z,x} \\
(4) & \rho_{x,w}(p+q) & = & \rho_{x,w}(p) + \rho_{x,w}(q) \\
(5) & \lambda_{y,x}(s+t) & = & \lambda_{y,x}(s) + \lambda_{y,x}(t).
\end{array}
$$

The structure maps P and Λ are given by

$$
\begin{aligned}
P_{x,y} \colon\ F_x \to F_{x^y}; &\qquad a \mapsto \rho_{x,y}a \\
\Lambda_{y,x} \colon\ F_y \to F_{x^y}; &\qquad b \mapsto \lambda_{y,x}b
\end{aligned}
$$

If $S_x = \{*\}$ for each $x \in X$, we obtain the **rack algebra** or **wring**, denoted $\mathbb{Z}X$. This is the analogue of the group ring $\mathbb{Z}G$ in group homology, and the universal enveloping algebra $U\mathfrak{g}$ in Lie algebra homology.

Right X–modules

A right X–module is similar to a left X–module, but the structure maps are reversed. It consists of a collection $A = \{A_x : x \in X\}$ of Abelian groups, isomorphisms $\phi^{x,y}\colon A_{x^y} \to A_x$ and homomorphisms $\psi^{y,x}\colon A_{x^y} \to A_y$ such that

$$\begin{aligned} \phi^{x,y}\phi^{x^y,z} &= \phi^{x,z}\phi^{x^z,y^z} \\ \psi^{y,x}\phi^{x^y,z} &= \phi^{y,z}\psi^{y^z,x^z} \\ \psi^{z,x^y} &= \psi^{z,y}\psi^{y^z,x^z} + \psi^{z,x}\phi^{x^z,y^z} \end{aligned}$$

Given two right X–modules $\mathcal{A} = (A, \phi, \psi)$ and $\mathcal{B} = (B, \chi, \omega)$, an X–map $f\colon \mathcal{A} \to \mathcal{B}$ is a collection of homomorphisms $f_x\colon A_x \to B_x$ such that

$$\begin{aligned} f_x\phi^{x,y} &= \chi^{x,y} f_{x^y} \\ f_y\psi^{x,y} &= \omega^{x,y} f_{x^y} \end{aligned}$$

These objects and maps form an Abelian category RMod^X.

There is an equivalence $\mathsf{RMod}^X \cong \mathsf{RMod}^*_X$, where X^* denotes the **opposite** of X.

Tensor products

Given a right X–module $\mathcal{A} = (A, \phi, \psi)$ and a left X–module $\mathcal{B} = (B, \chi, \omega)$, the **tensor product** $\mathcal{A} \otimes_X \mathcal{B}$ is the Abelian group generated by symbols of the form $a \otimes b$, where $a \in A_x$ and $b \in B_x$ for some $x \in X$, modulo relations

$$
\begin{array}{lrcl}
(1) & (a_1 + a_2) \otimes b & = & a_1 \otimes b + a_2 \otimes b \\
(2) & a \otimes (b_1 + b_2) & = & a \otimes b_1 + a \otimes b_2 \\
(3) & (na) \otimes b & = & a \otimes (nb) = n(a \otimes b) \\
(4) & \phi^{x,y}(c) \otimes b & = & c \otimes \chi_{x,y}(b) \\
(5) & \psi^{y,x}(c) \otimes d & = & c \otimes \omega_{y,x}(d)
\end{array}
$$

This gives a functor $\otimes_X \colon \mathsf{RMod}^X \times \mathsf{RMod}_X \to \mathsf{Ab}$ which is adjoint to the Hom functor:

$$\mathrm{Hom}_{\mathsf{RMod}^X}(\mathcal{A}, \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{B}, C)) \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{A} \otimes_X \mathcal{B}, C)$$

Homology again

We now have all we need to generalise rack and quandle homology to the case where the coefficient object is an X–module.

The standard complex

Let $z \in X$ be some fixed rack element (which will only be relevant in dimensions 0 and 1). Then the (z–)**standard complex** is

$$\mathbf{R}^z = \cdots R_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathcal{R}_1 \xrightarrow{d_1^z} \mathcal{R}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where R_n denotes the free X–module with basis S_n where $(S_n)_x = \{(x_1, \ldots, x_n) : x_1^{x_2 \ldots x_n} = x\}$.
(In particular, $R_1 \cong R_0 \cong \mathbb{Z}X$.)
The boundary maps are

$$d_n = \sum_{i=1}^{n} (-1)^{i+1} d_n^i$$

where, for $1 \leqslant i \leqslant n-1$,

$$(d_n^i)_{x_1^{x_2 \ldots x_n}}(x_1, \ldots, x_n) = \rho_{x_1^{x_2 \ldots \widehat{x}_{i+1} \ldots x_n}, x_{i+1}^{x_{i+2} \ldots x_n}}(x_1, \ldots, \widehat{x}_{i+1}, \ldots, x_n) - (x_1^{x_{i+1}}, \ldots, x_i^{x_{i+1}}, x_{i+2}, \ldots, x_n);$$

for $n > 1$,

$$(d_n^n)_{x_1^{x_2 \ldots x_n}}(x_1, \ldots, x_n) = (-1)^{n+1} \lambda_{x_2^{x_3 \ldots x_n}, x_1^{x_3 \ldots x_n}}(x_2, \ldots, x_n);$$

$$(d_1^z)_x\colon\ (x) \mapsto \lambda_{z, x^{\bar{z}}}(*);$$

and $\varepsilon\colon \mathbb{Z}X \to \mathbb{Z}$ is the **augmentation map** given by

$$\varepsilon_x\colon\ \begin{cases} \rho_{x^{\bar{w}}, w}(s) & \mapsto 1 \\ \rho_{x^{\bar{w}}, w} \lambda_{y, x^{\bar{w}\bar{y}}}(t) & \mapsto 0 \end{cases}$$

(We denote $\ker \varepsilon$ by $\mathcal{I}X$ and call it the **augmentation module**.)

A routine calculation confirms that

$$\varepsilon d_1^z = d_1^z d_2 = \cdots = d_{n-1} d_n = \cdots = 0$$

and so $\mathbf{R}^z$ is a chain complex of X–modules.

We can now define homology and cohomology groups by applying $- \otimes_X \mathcal{A}$ and $\mathrm{Hom}_X(-, \mathcal{A})$ to this complex:

$$\begin{aligned} H_n(X; \mathcal{A}) &:= H_n(R^z \otimes_X \mathcal{A}) \\ H^n(X; \mathcal{A}) &:= H^n(\mathrm{Hom}_X(R^z, \mathcal{A})) \end{aligned}$$

If $\mathcal{A}$ is trivial homogeneous, this reduces to the (topological) (co)homology of the rack space BX with coefficients in an Abelian group A.
If $\mathcal{A}$ is homogeneous, then we recover the more general (co)homology theory described by Andruskiewitsch and Graña*. If the ψ–maps are all zero, this further reduces to the theory studied by Etingof and Graña†.

Quandle homology

To generalise the quandle homology groups, let R_n denote the free *quandle* X–module with basis

$$(S_n)_x = \{(x_1, \ldots, x_n) \in X^n : x_1^{x_2 \ldots x_n} = x\}$$

and let P_n denote the free quandle X–module with basis

$$(T_n)_x = \{(x_1, \ldots, x_n) \in (S_n)_x : x_i = x_{i+1} \text{ for some } i\}$$

Then define $Q_n = R_n / P_n$ to get a complex

$$\mathbf{Q}^z = \cdots Q_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathcal{Q}_1 \xrightarrow{d_1^z} \mathcal{Q}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

*__N Andruskiewitsch__, __M Graña__, *From racks to pointed Hopf algebras*, Adv Math 178 (2003) 177–243.

†__P Etingof__, __M Graña__, *On rack cohomology*, J Pure Appl Alg 177 (2003) 49–59.

This gives us a generalised quandle homology theory:

$$\begin{aligned} H_n^Q(X;\mathcal{A}) &:= H_n(Q^z \otimes_X \mathcal{A}) \\ H^n_Q(X;\mathcal{A}) &:= H^n(\mathrm{Hom}_X(Q^z,\mathcal{A})) \end{aligned}$$

In this case where $\mathcal{A}$ is trivial homogeneous, this recovers the quandle homology theory of Carter, Saito, *et al*[*]. If $\mathcal{A}$ is a homogeneous Alexander module, then we get the 'twisted' quandle homology of Carter, Elhamdadi and Saito[†].

[*] **J S Carter**, **D Jelsovsky**, **S Kamada**, **L Langford**, **M Saito**, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans AMS 355 (2003) 3947–3989.

[†] **J S Carter**, **M Elhamdadi**, **M Saito**, *Twisted quandle homology theory and cocycle knot invariants*, Algebr Geom Topol 2 (2002) 95–135.

Cartan–Eilenberg homology

We now have an Abelian category, and well-defined notions of free modules, $\otimes_X$ and Hom_X. Is there a derived-functor interpretation of rack or quandle homology?

Yes and no. RMod_X and QMod_X both have enough projectives and enough flats, so we can define (co)homology theories in terms of derived functors of $\mathrm{Hom}_X(-,\mathcal{A})$ and $-\otimes_X \mathcal{A}$. However,

$$0 \to \mathcal{I}T_m \xrightarrow{i} \mathbb{Z}X \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a T_m–free resolution, so

$$\overline{H}_n(T_m;\mathcal{A}) = \overline{H}^n(T_m;\mathcal{A}) = 0$$

if $n > 1$. This is the sort of behaviour we tend to expect from 'trivial' objects.

Conjecture

$$\overline{H}_n(X;\mathcal{A}) = \overline{H}^n(X;\mathcal{A}) = 0$$

if $\mathrm{As}\,X$ is free, or free Abelian.

Applications

State-sum invariants of links

Carter, Saito, *et al.* defined a powerful family of link invariants.

1. Let L be an oriented link.
1. Choose a (small) finite quandle X.
2. Choose an Abelian group A and write it multiplicatively.
3. Choose a 2–cocycle $f \in H^2_Q(X;A)$.
4. Colour the (arcs of the) link consistently with elements of X.
5. The **weight** $w(\chi)$ of a crossing χ is

$$f(x,y)^{\varepsilon(\chi)}$$

where x, y are the labels of the incoming strands, and $\varepsilon(\chi)$ is the sign of χ.

6. The **state sum**

$$\Phi_f(L) = \sum_{\text{colourings}} \prod_{\text{crossings } \chi} w(\chi)$$

is an ambient isotopy invariant of L.

More generally, let $L^n \hookrightarrow M^{n+2}$ be a codimension–2 embedded manifold. Given an $(n+1)$–cocycle $f \in H_Q^{n+1}(X; A)$, the state sum $\Phi_f(L)$ is an ambient isotopy invariant of L.

This all generalises to the case $f \in H^{n+1}(X; \mathcal{A})$, where $\mathcal{A}$ is an arbitrary (possibly heterogeneous, possibly nontrivial) quandle X–module, and f is an (almost) arbitrary $(n+1)$–cocycle.

2–racks

A 2–group is a 'categorification' of an ordinary group, essentially a category equipped with certain endofunctors which make it behave like a group. A **strict** 2–group is one where all the group identities hold exactly; in a **coherent** 2–group they hold only up to isomorphism. 2–groups are classified by the third group cohomology $H^3(-;-)$. A similar definition and classification exist for Lie 2–algebras.

What are 2–racks and 2–quandles, and can they be classified by either of the cohomology theories described today?

Representation theory

Group representations correspond to modules over the complex group ring $\mathbb{C}G$. It is possible to define an analogous notion of a rack or quandle representation, in terms of a module over $\mathbb{C}X$.

Is this (a) interesting, or (b) useful?