The vortex patch problem for the surface quasi-geostrophic equation

José Luis Rodrigo*

Department of Mathematics, Princeton University, Washington Road, Princeton, NJ 08544-1000

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In this article, the analog of the Euler vortex patch problem for the surface quasi-geostrophic equation is considered.

The surface quasi-geostrophic (QG) equation was originally introduced as a model for atmospheric turbulence. It represents the evolution of the temperature on the 2D boundary of a half space with small Rossby and Ekman numbers. For a more detailed analysis of the geophysical properties of QG see ref. 1. The QG equation is given by

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + u\cdot\nabla\theta = 0, \qquad [1]$$

where

$$u = (u_1, u_2) = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right) \qquad [2]$$

and

$$(-\Delta)^{1/2}\psi = \theta. \qquad [3]$$

For simplicity we are considering fronts on the cylinder, i.e., we take (x, y) in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. In this setting we define $-\Delta^{-1/2}$ that comes from inverting the third equation by convolution with the kernel.[†]

$$\frac{\chi(u, v)}{(u^2 + v^2)^{1/2}} + \eta(u, v),$$

where $\chi(x, y) \in C_0^\infty$, $\chi(x, y) = 1$ in $|x - y| \leq r$ and $supp\chi$ is contained in $\{|x - y| \leq R\}$ with $0 < r < R < 1/2$. Also $\eta \in C_0^\infty$, $\eta(0, 0) = 0$.

The mathematical interest lies in the strong connections with the 3D Euler equation. Constantin, Majda, and Tabak (2, 3) noticed that the vorticity ω of the Euler equation satisfies the same equation as $\nabla^\perp\theta$, precisely

$$\frac{D\nabla^\perp\theta}{Dt} = (\nabla v)\nabla^\perp\theta.$$

There are many other analogies that have been analyzed in refs. 2 and 3. Other analyses of the equation are found in refs. 4–11.[‡]

The question we are going to explore here is an analog of the vortex patch problem for 2D Euler equation, i.e. the evolution of a solution that consists of two regions, where the function θ takes the values 0 and 1, separated by a smooth curve (see Fig. 1). For 2D Euler the global regularity of the vortex patches has been proved by Chemin (12). A simpler proof has been obtained by Bertozzi and Constantin (13) and Majda and Bertozzi (14). See also refs. 15 and 16.[§]

We notice that the question we are addressing here is the local existence of a solution for the QG-vortex patch problem. Deciding whether there is global regularity of the solutions is a very interesting open problem.

I refer the reader to ref. 17 for a discussion of a possible relationship between the problem studied here and a problem arising from 3D Euler.

We need the following definition:

Definition. A bounded function θ is a weak solution of QG if for any $\phi \in C_0^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R} \times [0, \varepsilon])$ we have

$$\int_{\mathbb{R}^+\times\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \theta(x, y, t)\partial_t\phi(x, y, t)dydxdt + \int_{\mathbb{R}^+\times\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \theta(x, y, t)u(x, y, t)\cdot\nabla\phi(x, y, t)dydxdt = 0, \qquad [4]$$

where u is determined by Eqs. **2** and **3**.

In the case of a sharp front we have the following form for the scalar

$$\theta(x, y, t) = 1 \quad \text{if} \quad y \geq \varphi(x, t) \quad \text{and} \quad \theta = 0 \text{ otherwise.} \qquad [5]$$

For a sharp front for the surface QS equation we have the following results.

Theorem 1. *If θ is a weak solution of the surface QS (see Definition) of the form described in Eq. **5**, then the function φ satisfies the equation,*

$$\frac{\partial\varphi}{\partial t}(x, t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\varphi}{\partial x}(x, t) - \frac{\partial\varphi}{\partial u}(u, t)}{[(x - u)^2 + (\varphi(x, t) - \varphi(u, t))^2]^{1/2}} \cdot\chi(x - u, \varphi(x, t) - \varphi(u, t))du + \int_{\mathbb{R}/\mathbb{Z}} \left[\frac{\partial\varphi}{\partial x}(x, t) - \frac{\partial\varphi}{\partial u}(u, t)\right]\eta(x - u, \varphi(x, t) - \varphi(u, t))du. \qquad [6]$$

Theorem 2. *Given any periodic, smooth function $\varphi_0(x)$ the initial value problem determined by Eq. **6** with initial data $\varphi(x, 0) = \varphi_0(x)$ has a unique smooth solution for a small time, determined by the initial data φ. Moreover the function θ defined by Eq. **5** is a weak solution of the QG equation.*

Abbreviation: QC, quasi-geostrophic.

*E-mail: jrodrigo@math.princeton.edu.

[†]To avoid irrelevant considerations at ∞, we will take η to be compactly supported.

[‡]Constantin, P., Proceedings of the Workshop on Earth Climate as a Dynamical System, September 25–26, 1992, Argonne, IL; ANL/MCSTM-170.

[§]Krasny, R., Proceedings of the International Congress of Mathematics, August 21–29, 1990, Kyoto, Japan.

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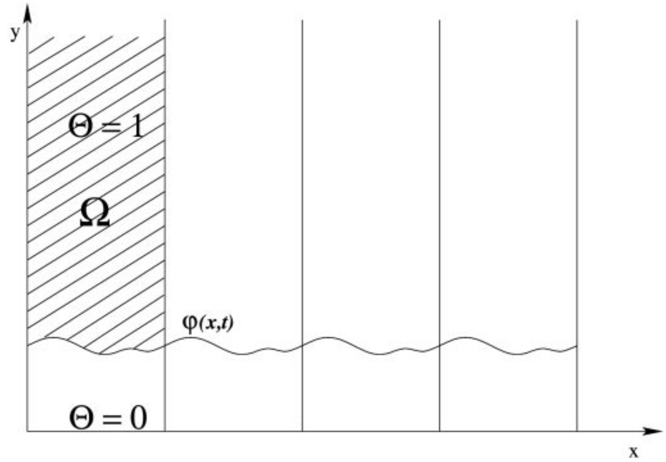

Fig. 1. Sharp front.

To prove *Theorem 1* we substitute the above expression for θ in Eq. **4** and try to obtain an equation for the evolution of the curve φ.

Notice that by using Eqs. **2** and **3** we obtain $u(x, y, t) = K * \theta(x, y, t)$, where K looks locally like the orthogonal of the Riesz transform. More details about the Riesz transform can be found in refs. 18 and 19.

$$u = \nabla^{\perp}(\Delta^{-1/2}\theta),$$

we obtain that u is in BMO (see refs. 19 and 20 for more details) and hence the integrals below will make perfect sense.

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \theta\partial_t\phi\, dydxdt &= \int_{y>\varphi(x,t)} \partial_t\phi\, dxdydt \\ &= \int_{y=\varphi(x,t)} \phi\, \frac{\partial_t\varphi}{(1+(\partial_x\varphi)^2+(\partial_t\varphi)^2)^{1/2}} \\ &\qquad \cdot(1+(\partial_x\varphi)^2+(\partial_t\varphi)^2)^{1/2}\, dxdt \\ &= \int_{y=\varphi(x,t)} \phi\partial_t\varphi\, dxdt.\end{aligned}$$

As for the other term in Eq. **4** (considering only the space integration)

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}/\mathbb{Z}} \theta u\cdot\nabla\phi\, dydx &= \lim_{\delta\to 0}\int_{y>\varphi(x,t)+\delta} u\cdot\nabla\phi\, dxdy \\ &= \lim_{\delta\to 0}\int_{y=\varphi(x,t)+\delta} u\phi\cdot\left(\frac{\partial\varphi}{\partial x'}{-1}\right)dx.\end{aligned}$$

Now we look more closely to the integrand of the above expression. We have

$$\begin{aligned}u\phi\cdot\left(\frac{\partial\varphi}{\partial x'}{-1}\right) = \phi(x,y,t)\int_{u>\varphi(u,t)} K(x-u, y-v) \\ \cdot\left(\frac{\partial\varphi}{\partial x'}{-1}\right)dudv.\end{aligned}$$

The precise expression of K is given by

$$K(u, v) = \nabla^{\perp}\left\{\frac{\chi(u, v)}{(u^2+v^2)^{1/2}} + \eta(u, v)\right\},$$

where $\chi(u, v)$ is supported in $|u - v| \leq 1/4$ and η is compactly supported.

And so[¶]

$$\begin{aligned}&u\phi\cdot\left(\frac{\partial\varphi}{\partial x'}{-1}\right) \\ &= \phi(x,y,t)\int_{v>\varphi(u,t)} -\left(1, \frac{\partial\varphi}{\partial x}\right) \\ &\quad \cdot\nabla_{u,v}\left\{\frac{\chi(x-u, y-v)}{((x-u)^2+(y-v)^2)^{1/2}} + \eta(x-u, y-v)\right\}dudv \\ &= \phi(x,y,t)\int_{v>\varphi(u,t)} -\, div_{u,v}\left(\frac{\chi(x-u, y-v)}{((x-u)^2+(y-v)^2)^{1/2}}\right. \\ &\quad + \eta(x-u, y-v), \\ &\quad \left.\frac{\partial\varphi}{\partial x}\left(\frac{\chi(x-u, y-v)}{((x-u)^2+(y-v)^2)^{1/2}} + \eta(x-u, y-v)\right)\right)dudv \\ &= \phi(x,y,t)\int_{v=\varphi(u,t)} -\left\{\frac{\chi(x-u, y-v)}{((x-u)^2+(y-v)^2)^{1/2}}\right. \\ &\quad \left. + \eta(x-u, y-v)\right\}\left(1, \frac{\partial\varphi}{\partial x}\right)\cdot\left(\frac{\partial\varphi}{\partial u'}{-1}\right)du \\ &= \phi(x,y,t)\left\{\int_{v=\varphi(u,t)} -\frac{\dfrac{\partial\varphi}{\partial u} - \dfrac{\partial\varphi}{\partial x}}{((x-u)^2+(y-v)^2)^{1/2}}\right. \\ &\quad \chi(x-u, y-v) \\ &\quad \left. -\left[\frac{\partial\varphi}{\partial u} - \frac{\partial\varphi}{\partial x}\right]\eta(x-u, y-v)du\right\}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\lim_{\delta\to 0}\int_{y=\varphi(x,t)+\delta} u\phi\cdot\left(\frac{\partial\varphi}{\partial x'}{-1}\right)dx \\ &= \int_{y=\varphi(x,t)} -\,\phi(x,y,t)\int_{v=\varphi(u,t)} \frac{\dfrac{\partial\varphi}{\partial u} - \dfrac{\partial\varphi}{\partial x}}{((x-u)^2+(y-v)^2)^{1/2}} \\ &\quad \chi(x-u, y-v)dudx \\ &\quad + \int_{y=\varphi(x,t)} -\,\phi(x,y,t)\int_{v=\varphi(u,t)}\left[\frac{\partial\varphi}{\partial u} - \frac{\partial\varphi}{\partial x}\right] \\ &\quad \eta(x-u, y-v)dudx.\end{aligned}$$

Putting these two equalities together we have

¶We move the $\perp$ that appears in K to the factor $(\partial\varphi/\partial x, -1)$.

$$\int_{y=\varphi(x,t)} \phi(x,y,t)\,\frac{\partial\varphi}{\partial t}(x,t)dxdt$$

$$= \int_{y=\varphi(x,t)} -\,\phi(x,y,t)\int_{v=\varphi(u,t)} \frac{\dfrac{\partial\varphi}{\partial u}-\dfrac{\partial\varphi}{\partial x}}{((x-u)^2+(y-v)^2)^{1/2}}$$
$$\chi(x-u,y-v)dudxdt$$
$$+\int_{y=\varphi(x,t)} -\,\phi(x,y,t)\int_{v=\varphi(u,t)}\left[\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}\right]$$
$$\eta(x-u,y-v)dudxdt.$$

From that equality we obtain the equation we were looking for,

$$\frac{\partial\varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\dfrac{\partial\varphi}{\partial x}(x,t)-\dfrac{\partial\varphi}{\partial u}(u,t)}{[(x-u)^2+(\varphi(x,t)-\varphi(u,t))^2]^{1/2}}$$
$$\chi(x-u,\varphi(x,t)-\varphi(u,t))du$$
$$+\int_{\mathbb{R}/\mathbb{Z}}\left[\frac{\partial\varphi}{\partial x}(x,t)-\frac{\partial\varphi}{\partial u}(u,t)\right]\eta(x-u,\varphi(x,t)-\varphi(u,t))du. \quad [7]$$

This completes the proof of *Theorem 1*.

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