# Almost sharp fronts for the surface quasi-geostrophic equation 

Diego Córdoba ${ }^{\dagger}$, Charles Fefferman ${ }^{\ddagger}$, and José Luis Rodrigo ${ }^{\ddagger \S}$<br>${ }^{\dagger}$ Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain; and ${ }^{\ddagger}$ Department of Mathematics, Princeton University, Washington Road, Princeton, NJ 08544-1000

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We investigate the evolution of "almost sharp" fronts for the surface quasi-geostrophic equation. This equation was originally introduced in the geophysical context to investigate the formation and evolution of fronts, i.e., discontinuities between masses of hot and cold air. These almost sharp fronts are weak solutions of quasi-geostrophic with large gradient. We relate their evolution to the evolution of sharp fronts.
n this article we study the evolution of "almost sharp" fronts for the surface quasi-geostrophic (QG) equation. This 2D active scalar equation reads

$$
\begin{equation*}
\frac{D \theta}{D t}=\frac{\partial \theta}{\partial t}+u \cdot \nabla \theta=0 \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
u=\left(u_{1}, u_{2}\right)=\left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
(-\Delta)^{1 / 2} \psi=\theta \tag{3}
\end{equation*}
$$

For simplicity we are considering fronts on the cylinder; i.e., we take $(x, y)$ in $\mathbb{R} / \mathbb{Z} \times \mathbb{R}$. In this setting we define $-\Delta^{-1 / 2}$ that comes from inverting the third equation by convolution with the kernel ${ }^{\text {II }}$

$$
\frac{\chi(u, v)}{\left(u^{2}+v^{2}\right)^{1 / 2}}+\eta(u, v)
$$

where $\chi(x, y) \in C_{0}^{\infty}, \chi(x, y)=1$ in $|x-y| \leq r$ and supp $\chi$ is contained in $\{|x-y| \leq R\}$ with $0<r<R<\frac{1}{2}$. Also $\eta \in C_{0}^{\infty}, \eta(0,0)=0$.
The main mathematical interest in the QG equation lies in its strong similarities to the 3D Euler equations. These results were proved by Constantin, Majda, and Tabak (for more details see refs. 1-3). There are several other research lines for this equation, both theoretical and numerical. See refs. $4-12$.\| The question about the regularity of the solutions for QG remains an open problem.
Recently one of the authors obtained the equation for the evolution of sharp fronts (in the periodic setting), proving its local well-posedness for that equation (See ref. 13 for more details.) This is a problem in contour dynamics. Contour dynamics for other fluid equations has been studied extensively (see refs. 14 and $15^{\dagger \dagger}$ ).
We begin our analysis on almost sharp fronts for the QG equation by recalling the notion of weak solution. We have
Definition: A bounded function $\theta$ is a weak solution of QG if for any $\phi \in C_{0}^{\infty}(\mathbb{R} / \mathbb{Z} \times \mathbb{R} \times[0, \varepsilon])$ we have

$$
\begin{equation*}
\int_{\mathbb{R}^{+} \times \mathbb{R} / \mathbb{Z} \times \mathbb{R}} \theta(x, y, t) \partial_{t} \phi(x, y, t) d y d x d t+\int_{\mathbb{R}^{+} \times \mathbb{R} / \mathbb{Z} \times \mathbb{R}} \theta(x, y, t) u(x, y, t) \cdot \nabla \phi(x, y, t) d y d x d t=0, \tag{4}
\end{equation*}
$$

where $u$ is determined by Eqs. 2 and $\mathbf{3}$.
We are interested in studying the evolution of almost sharp fronts for the QG equation. These are weak solutions of the equation with large gradient $[\sim(1 / \delta)$, where $2 \delta$ is the thickness of the transition layer for $\theta]$.

We are going to consider the cylindrical case here. We consider a transition layer of thickness $<2 \delta$ in which $\theta$ changes from 0 to 1 (Fig. 1). That means we are considering $\theta$ of the form

$$
\begin{array}{cl}
\theta=1 \text { if } & y \geq \varphi(x, t)+\delta \\
\theta \text { bounded if } & |\varphi(x, t)-y| \leq \delta  \tag{5}\\
\theta=0 \text { if } & y \leq \varphi(x, t)-\delta
\end{array}
$$

[^0]

Fig. 1. Almost sharp front.
where $\varphi$ is a smooth periodic function and $0<\delta<\frac{1}{2}$.
For these solutions we have the following Theorem.
Theorem. If the active scalar $\theta$ is as in Eq. 5 and satisfies Eq. 4, then $\varphi$ satisfies the equation

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t}(x, t)= & \int_{\mathbb{R} / \mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x, t)-\frac{\partial \varphi}{\partial u}(u, t)}{\left[(x-u)^{2}+(\varphi(x, t)-\varphi(u, t))^{2}\right]^{1 / 2}} \chi(x-u, \varphi(x, t)-\varphi(u, t)) d u \\
& +\int_{\mathbb{R} / \mathbb{Z}}\left[\frac{\partial \varphi}{\partial x}(x, t)-\frac{\partial \varphi}{\partial u}(u, t)\right] \eta(x-u, \varphi(x, t)-\varphi(u, t)) d u+\text { Error }
\end{aligned}
$$

with $\mid$ Error $|\leq C \delta| \log \delta \mid$ where $C$ depends only on $\|\theta\|_{L_{\infty}}$ and $\|\nabla \varphi\|_{L_{\infty}}$.
Remark: Note that Eq. 5 specifies the function $\varphi$ up to an error of order $\delta$. Theorem provides an evolution equation for the function $\varphi$ up to an error of order $\delta|\log \delta|$.

To analyze the evolution of the almost sharp front, we substitute the above expression for $\theta$ in the definition of a weak solution (4). We use the notation $X=O(Y)$ to indicate that $|X| \leq C|Y|$ where the constant $C$ depends only on $\|\theta\|_{L^{\infty},}\|\nabla \varphi\|_{L^{\infty}}$ and $\|\phi\|_{C^{1}}$, where $\phi$ is a test function appearing in Definition.

We consider the three different regions defined by the form on $\theta$. Because $\theta=0$ in region I , the contribution from that region is 0 ; i.e.,

$$
\int_{I \times \mathbb{R}} \theta(x, y, t) \partial_{t} \phi(x, y, t) d y d x d t+\int_{I \times \mathbb{R}} \theta(x, y, t) u(x, y, t) \cdot \nabla \phi(x, y, t) d y d x d t=0
$$

As for region II,

$$
\int_{I I \times \mathbb{R}} \theta(x, y, t) \partial_{t} \phi(x, y, t) d x d y d t=O(\delta),
$$

because $\theta$ is bounded and hence $O(1)$, and the area of region II is $O(\delta)$. As for the second term,

$$
\int_{I I \times \mathbb{R}} u \theta \nabla \phi d x d y d t=O(\delta \log (\delta))
$$

To see this, we fix $t$. We must estimate

$$
\int_{\mathbb{R}^{2}} u \cdot\left(\mathbb{1}_{I I} \theta \nabla \phi\right) d x d y .
$$

Using Eqs. 2 and 3, we obtain

$$
u=\nabla^{\perp}\left(\Delta^{-1 / 2} \theta\right)
$$

and hence $u(x, y, t)=K * \theta(x, y, t)$ where $K$ looks locally like the orthogonal of the Riesz transform, precisely

$$
\begin{equation*}
K(u, v)=\nabla^{\perp}\left\{\frac{\chi(u, v)}{\left(u^{2}+v^{2}\right)^{1 / 2}}+\eta(u, v)\right\} \tag{6}
\end{equation*}
$$

Because $\theta$ is bounded, from the above expression for $K$ we obtain that $u$ is of exponential class (16), and hence

$$
\int_{\mathbb{R}^{2}} u \cdot\left(\mathbb{1}_{I I} \theta \nabla \phi\right) d x d y
$$

is the integral of a function of exponential class over a set of measure $\delta$, set $I I$. Recalling that the dual of the set of functions of exponential class is $L \log L$, we obtain

$$
\int_{\mathbb{R}^{2}} u \cdot\left(\mathbb{1}_{I I} \theta \nabla \phi\right) d x d y=O(\delta \ln \delta)
$$

As for region III, we proceed as follows.
We decompose $u=u_{f}+u_{b}$, where $u_{f}=K * \mathbb{1}_{I I}$ and $u_{b}=K * \theta \mathbb{1}_{I I}$. Notice, then, that both $u_{f}$ and $u_{b}$ are divergence-free.
For a fixed $t$ we have

$$
\int_{I I I} u_{b} \cdot \nabla \phi d x=\int_{\mathbb{R}^{2}}\left[K *\left(\theta \mathbb{1}_{I I}\right)\right] \cdot\left[\mathbb{1}_{I I} \nabla \phi\right] d x d y=\int_{\mathbb{R}^{2}} \theta \mathbb{1}_{I I} K * \cdot\left[\mathbb{1}_{I I} \nabla \phi\right] d x d y=O(\delta \log \delta)
$$

Because the area where we are integrating is $O(\delta)$ and the function $K *\left[\mathbb{1}_{I I I} \nabla \phi\right]$ is of exponential class, we have used the notation $K * \cdot$ to denote the sum of the application of the operator to each component.

We are left to estimate the terms

$$
\int_{I I I \times \mathbb{R}} \theta \partial_{t} \phi d x d y d t+\int_{I I I \times \mathbb{R}} \theta u_{f} \cdot \nabla \phi d x d y d t=: A+B .
$$

The first term gives

$$
A=\int_{I I I \times \mathbb{R}} \theta \partial_{t} \phi d y d x d t=\int_{y>\varphi(x, t)+\delta} \partial_{t} \phi d x d y d t=\int_{y=\varphi(x, t)+\delta} \phi \partial_{t} \varphi d x d t
$$

To analyze $B$, fix $t$ and consider only the space integration

$$
\int_{I I I} \theta u_{f} \cdot \nabla \phi d y d x=\lim _{\varepsilon \rightarrow 0^{+}} \int_{y>\varphi(x, t)+\delta+\varepsilon} u_{f} \cdot \nabla \phi d x d y=\lim _{\varepsilon \rightarrow 0^{+}} \int_{y=\varphi(x, t)+\delta+\varepsilon} u_{f} \phi \cdot\left(\frac{\partial \varphi}{\partial x},-1\right) d x .
$$

Now we look more closely to the integrand of the above expression. We have

$$
u_{f} \phi \cdot\left(\frac{\partial \varphi}{\partial x},-1\right)=\phi(x, y, t) \int_{v>\varphi(u, t)+\delta} K(x-u, y-v) \cdot\left(\frac{\partial \varphi}{\partial x},-1\right) d u d v
$$

Recall the expression for $K$ obtained in Eq. 6. Therefore, ${ }^{\text {\#\# }}$

$$
\begin{aligned}
u_{f} \phi \cdot\left(\frac{\partial \varphi}{\partial x},-1\right)= & \phi(x, y, t) \int_{v>\varphi(u, t)+\delta}-\left(1, \frac{\partial \varphi}{\partial x}\right) \cdot \nabla_{u, v}\left\{\frac{\chi(x-u, y-v)}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}}+\eta(x-u, y-v)\right\} d u d v \\
= & \phi(x, y, t) \int_{v>\varphi(u, t)+\delta}-\operatorname{div}_{u, v}\left(\frac{\chi(x-u, y-v)}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}}+\eta(x-u, y-v), \frac{\partial \varphi}{\partial x}\left(\frac{\chi(x-u, y-v)}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}}\right.\right. \\
& +\eta(x-u, y-v))) d u d v
\end{aligned}
$$

[^1]\[

$$
\begin{aligned}
& =\phi(x, y, t) \int_{v=\varphi(u, t)+\delta}-\left(1, \frac{\partial \varphi}{\partial x}\right) \cdot\left(\frac{\partial \varphi}{\partial u},-1\right)\left\{\frac{\chi(x-u, y-v)}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}}+\eta(x-u, y-v)\right\} d u \\
& =\phi(x, y, t)\left\{\int_{v=\varphi(u, t)+\delta}-\frac{\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}} \chi(x-u, y-v)-\left[\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}\right] \eta(x-u, y-v) d u\right\}
\end{aligned}
$$
\]

Hence we have

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0^{+}} \int_{y=\varphi(x, t)+\varepsilon+\delta} u_{f} \phi \cdot\left(\frac{\partial \varphi}{\partial x},-1\right) d x= & \int_{y=\varphi(x, t)+\delta}-\phi(x, y, t) \int_{v=\varphi(u, t)+\delta} \frac{\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}} \chi(x-u, y-v) d u d x \\
& +\int_{y=\varphi(x, t)+\delta}-\phi(x, y, t) \int_{v=\varphi(u, t)+\delta}\left[\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}\right] \eta(x-u, y-v) d u d x
\end{aligned}
$$

and hence we obtain the following expression for $B$.

$$
\begin{aligned}
B= & \int_{I I I \times \mathbb{R}} \theta u_{f} \cdot \nabla \phi d y d x d t=\int_{y=\varphi(x, t)+\delta}-\phi(x, y, t) \int_{v=\varphi(u, t)+\delta} \frac{\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}} \chi(x-u, y-v) d u d x d t \\
& +\int_{y=\varphi(x, t)+\delta}-\phi(x, y, t) \int_{v=\varphi(u, t)+\delta}\left[\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}\right] \eta(x-u, y-v) d u d x d t .
\end{aligned}
$$

Now, considering all contributions from all regions, we obtain that Eq. 4 is equivalent to

$$
\begin{aligned}
& \int_{y=\varphi(x, t)+\delta} \phi(x, y, t) \frac{\partial \varphi}{\partial t}(x, t) d x d t+\int_{y=\varphi(x, t)+\delta}-\phi(x, y, t) \int_{v=\varphi(u, t)+\delta} \frac{\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}}{\left((x-u)^{2}+(y-v)^{2}\right)^{1 / 2}} \chi(x-u, y-v) d u d x d t \\
& +\int_{y=\varphi(x, t)+\delta}-\phi(x, y, t) \int_{v=\varphi(u, t)+\delta}\left[\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial x}\right] \eta(x-u, y-v) d u d x d t+O(\delta|\log \delta|)=0
\end{aligned}
$$

and hence we obtain the equation

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t}(x, t)= & \int_{\mathbb{R} / \mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x, t)-\frac{\partial \varphi}{\partial u}(u, t)}{\left[(x-u)^{2}+(\varphi(x, t)-\varphi(u, t))^{2}\right]^{1 / 2}} \chi(x-u, \varphi(x, t)-\varphi(u, t)) d u \\
& +\int_{\mathbb{R} / \mathbb{Z}}\left[\frac{\partial \varphi}{\partial x}(x, t)-\frac{\partial \varphi}{\partial u}(u, t)\right] \eta(x-u, \varphi(x, t)-\varphi(u, t)) d u+O(\delta|\log \delta|),
\end{aligned}
$$

which proves Theorem.
It would be interesting to give a rigorous construction of an almost sharp front solving the surface QG equation for given initial data $\varphi$ and arbitrarily small thickness $\delta$.

The problem of the evolution of almost sharp fronts considered here could be a simple model for a very interesting and hard problem of justifying rigorously the equation for the evolution of a vortex line.

If one imagines the vorticity as a $\delta$ function supported along a curve, the attempt of recovering the velocity by using the Biot-Savart law shows a singularity proportional to the inverse of the distance to the curve. The heuristic derivations of an equation for the evolution of the curve simply remove that singularity from the equation. The main problem faced in a rigorous derivation is that a vortex line, as described above, fails to be a weak solution of the Euler equation.

Modifying the definition of weak solution may introduce objects unrelated to physically meaningful solutions of the 3D Euler equation (17). Instead one could try to consider solutions supported on a very small neighborhood of the "vortex line" and obtain an equation for the evolution of such a solution based on the core line and the thickness, hoping that some limit could be found when sending the thickness to 0 .

The analysis we have presented here is the analog of the proposed strategy for the surface QG equation. This equation also contains a singularity in the velocity as we approach the front. The singularity is only logarithmic, weaker than the one in the 3D Euler equation. Nevertheless, we hope this result might provide some insights on the vortex line problem.

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[^0]:    Abbreviation: QG, quasi-geostrophic.
    §To whom correspondence should be addressed. E-mail: jrodrigo@math.princeton.edu.
    "To avoid irrelevant considerations at $\infty$, we will take $\eta$ to be compactly supported.
    |Constantin, P., Proceedings to the Workshop on the Earth Climate as a Dynamical System, September 25-26, 1992, Argonne, IL; ANL/MCSTM-170.
    ${ }^{\dagger+}$ Krasny, R., Proceedings of the International Congress of Mathematics, August 21-29, 1990, Kyoto, Japan.
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    ( 2004 by The National Academy of Sciences of the USA

[^1]:    ${ }^{\ddagger}$ We move the $\perp$ that appears in $K$ to the factor $((\partial \varphi / \partial x),-1)$.

