On the evolution of sharp Fronts for the Quasi-Geostrophic equation

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Abstract

We consider the problem of the evolution of sharp fronts for the surface quasi-geostrophic (QG) equation. This problem is the analog to the vortex patch problem for the 2-D Euler equation.

The special interest of the quasi-geostrophic equation lies in its strong similarities with the 3-D Euler equation, while being a 2-D model. In particular an analog of the problem considered here, the evolution of sharp fronts for QG, is the evolution of a vortex line for 3-D Euler. The rigorous derivation of an equation for the evolution of a vortex line is still an open problem. The influence of the singularity appearing in the velocity when using the Biot-Savart law still needs to be understood.

We present two derivations for the evolution of a periodic sharp front. The first one, heuristic, shows the presence of a logarithmic singularity in the velocity, while the second, making use of weak solutions, obtains a rigorous equation for the evolution explaining the influence of that term in the evolution of the curve.

Finally, using a Nash-Moser argument as the main tool, we obtain local existence and uniqueness of a solution for the derived equation, in the C^∞ case.

1 Introduction

In this artice we study the surface quasi-geostrophic (QG) equation. The QG system reads

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + u \cdot \nabla\theta = 0 \tag{1}$$

where

$$u = (u_1, u_2) = (-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}) \tag{2}$$

and

$$(-\triangle)^{\frac{1}{2}}\psi = \theta\,, \tag{3}$$

where the variable θ represents an active scalar (potential temperature) convected by the velocity field u and ψ is the stream function.

This equation was originally introduced as a model for atmospheric turbulence (that is the main reason for the terminology "front") and has been studied by numerous people. A derivation of this equation can be found in [Pe], for the evolution of the temperature on the 2-D boundary of a half-space with small Rossby and Ekman numbers and constant potential vorticity.

1.1 QG and 3-D Euler

The main mathematical interest in the quasi-geostrophic system lies in the strong analogies between the quasi-geostrophic equation and the 3-D Euler equation. QG presents a 2-D dimensional equation that contains many of the features of 3-D Euler. We will present briefly the analogies between these two equations.

In its vorticity formulation the Euler equation reads

$$\frac{D\omega}{Dt} = (\nabla u)\omega \tag{4}$$

where $u = (u_1, u_2, u_3)$ is the 3-D velocity satisfying $div\ u = 0$. Also, the vorticity ω is given by $\omega = curl\ u$.

In particular, we observe that by differentiating (1) we obtain the equation [1]

$$\frac{D(\nabla^\perp \theta)}{Dt} = (\nabla u)\nabla^\perp \theta \tag{5}$$

This shows a very strong analogy between the 3-D Euler equation and QG, where $\nabla^\perp \theta$ plays the role of ω.

We list some further analogies:

- The velocity is recovered via the formulas

  $$u(x) = \int_{\mathbb{R}^3} K_3(y)\omega(x+y)dy \qquad\qquad u(x) = \int_{\mathbb{R}^2} K_2(y)\nabla^\perp\theta(x+y)dy$$

  where the kernels K_d, $d = 2, 3$ are homogeneous of degree $1 - d$. Additionally, the strain matrix (the symmetric part of the gradient of the velocity) can be recovered via SIO, given by kernels of degree $-d$.
- Integral curves of ω, and of $\nabla^\perp\theta$ move with the fluid.
- Both systems have conserved energy.
- $|\omega|$ and its analog $|\nabla^\perp\theta|$ evolve according to the same type of equation. ($|\omega|$ measures the infinitesimal length of a vortex line).
- Both systems have analog conditions for a break up of a solution (Beale-Kato-Majda).

These analogies were first noticed by Constantin, Majda and Tabak. We refer the reader to [Co-Ma-Ta1], [Co-Ma-Ta2] and [Ma-Ta] for a complete presentation. Another detailed exposition is found in [Ma-Be].

One of the most active question about QG is the study of the frontogenesis, precisely the formation of a discontinuous temperature front in finite time. There have been both mathematical and numerical analyses of QG concerning this question. See [Co-Ma-Ta1], [Co-Ma-Ta2] ,[Cor], [OhYa], [Co-Ni-So1] and [Co-Ni-So2].

[1] We use the notation $(a, b)^\perp = (-b, a)$ and so $\nabla^\perp = (-\partial_y, \partial_x)$

1.2 Connections with the vortex patch problem: 2-D Euler

We notice here that 2-D Euler, in its vorticity formulation, provides us with a scalar equation that presents a very similar structure to the one we are considering.

Observe that the 2-D Euler equation reads

$$\frac{D\omega}{Dt} = 0 \tag{6}$$

where

$$(u_1, u_2) = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x})$$

and

$$\triangle \psi = \omega$$

Observe that the above system is very similar to (1)-(3) except for the relationship between the stream function and the active scalar. In the case of QG the fractional power of the Laplacian makes the equation more singular.

The question about the global regularity of the vortex patches was positively answered by Chemin [Ch] in 1993 using paradifferential calculus. A simpler proof by Bertozzi and Constantin can be found in [Be-Co] and [Ma-Be].

1.3 Description of the problem

The question about QG that we will be addressing here is the study of the evolution of smooth sharp fronts, in the periodic setting.

We are interested in the evolution of a periodic sharp front. We consider the front originally given by the curve $\varphi_0(x)$ (see figure 1), a smooth periodic function, and assume that the solution to the equation (1) is of the same form (i. e. remains as a sharp front), and is given by $\varphi(x,t)$, a smooth periodic function. This means that the scalar function $\theta(x,y,t)$ is given by

$$\begin{cases} \theta(x,y,t) = 1 & y \geq \varphi(x,t) \\ \theta(x,y,t) = 0 & y < \varphi(x,t) \end{cases} \tag{7}$$

We denote by Ω the fundamental region where $\{y \geq \varphi(x,t)\}$. We will consider $-\frac{1}{2} \leq x \leq \frac{1}{2}$. We denote by Γ the piece of the front in the fundamental region.

The problem that we address in this article is the derivation and solution of an equation for the evolution of such a front. We also prove that the obtained system is locally well-posed.

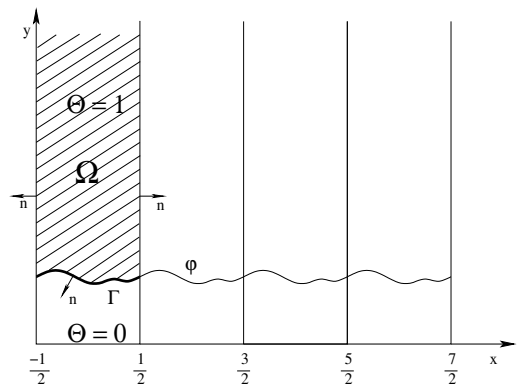

Figure 1: Periodic Sharp Front

2 Evolution of a sharp-front for QG: derivation of the equation

In this section we present two derivations of the equation for the evolution of a sharp front for the surface quasi-geostrophic equation in the periodic setting. In the first section we give a heuristic derivation, that shows interesting features of the velocity as we approach the front. In the second section we provide a rigorous derivation of the equation, that avoids all the difficulties shown in the first approach.

2.1 First approach: redefining the velocity

We will start by eliminating the stream function ψ from the system (1)-(3), by using the second and third equations. The operator $(-\triangle_{x,y})^{-\frac{1}{2}}$ in the cylinder is given by a convolution with the kernel given by the expression

$$K(x,y) = \frac{\chi(x,y)}{(x^2+y^2)^{\frac{1}{2}}} + \eta(x,y), \tag{8}$$

for (x,y) in $[-\frac{1}{2},\frac{1}{2}] \times \mathbf{R}$, and defined in the rest of the plane by extending it periodically in x. χ satisfies

$$\chi(x,y) \,\epsilon\, C_0^\infty \qquad \chi(x,y) = 1 \quad \text{in} \quad |x-y| \leq r \quad \text{and} \quad \textit{supp}\ \chi \subset \{|x-y| \leq R\} \tag{9}$$

where $0 < r < R < \frac{1}{2}$ are positive numbers to be chosen later. Also η^2 satisfies

[2]In order to avoid irrelevant considerations at ∞ we will consider the correcting function η to be compactly supported. This has the effect of modifying $(-\triangle_{x,y})^{-\frac{1}{2}}$ by adding a smoothing operator.

$$\eta(x,y) \,\epsilon\, C_0^\infty \quad \eta(0,0)=0 \tag{10}$$

Observe that both χ and η can be taken to be even functions.

In our later analysis of the equation, we will need a certain degree of control over the support of χ. We will obtain control by modifying the values of both r and R entering in the definition of χ in (9). Notice that changing the value of r and R does not affect the structure of K given by (8), since the difference in the function χ created by changing r and R can be absorbed by the correction-term η.

Under the assumption that the front remains as a front for a small period of time, we proceed with the derivation of the equation for the curve φ. For a point (x,y) not in the front we obtain

$$\psi(x,y,t) = \int_{\mathbb{R}\times\mathbb{R}/_{\mathbb{Z}}} \frac{\theta(\tilde{x},\tilde{y},t)\chi(x-\tilde{x},y-\tilde{y})}{[(x-\tilde{x})^2+(y-\tilde{y})^2]^{\frac{1}{2}}} d\tilde{x}d\tilde{y} + \\ + \int_{\mathbb{R}\times\mathbb{R}/_{\mathbb{Z}}} \theta(\tilde{x},\tilde{y},t)\eta(x-\tilde{x},y-\tilde{y})d\tilde{x}d\tilde{y} \tag{11}$$

Since $u = \nabla^\perp\psi$ and the differential operators $(-\triangle_{x,y})^{-\frac{1}{2}}$ and $\nabla^\perp$ commute, we obtain the equation

$$u(x,y,t) = \int_{\mathbb{R}\times\mathbb{R}/_{\mathbb{Z}}} \frac{\nabla^\perp_{\tilde{x},\tilde{y}}\theta(\tilde{x},\tilde{y},t)\chi(x-\tilde{x},y-\tilde{y})}{[(x-\tilde{x})^2+(y-\tilde{y})^2]^{\frac{1}{2}}} d\tilde{x}d\tilde{y} + \\ + \int_{\mathbb{R}\times\mathbb{R}/_{\mathbb{Z}}} \nabla^\perp_{\tilde{x},\tilde{y}}\theta(\tilde{x},\tilde{y},t)\eta(x-\tilde{x},y-\tilde{y})d\tilde{x}d\tilde{y} \tag{12}$$

We still need to compute $\nabla^\perp\theta$. Notice that the tangent vector to the curve is given by $(1, \frac{\partial\varphi}{\partial x}(x,t))$, and so the unit exterior normal to the region Ω along Γ is given by

$$n = \frac{(\frac{\partial\varphi}{\partial x}(x,t),-1)}{\sqrt{\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2+1}}$$

We obtain $\nabla^\perp\theta$ as a simple application of the Divergence Theorem. We have

$$\nabla^\perp\theta(x,y,t) = (-1, -\frac{\partial\varphi}{\partial x}(x,t))\delta(y-\varphi(x,t)) \tag{13}$$

Plugging this expression in (12), and carrying out the integration with respect to $\tilde{y}$, we obtain

$$u(x,y,t) = -\int_{\mathbb{R}/_{\mathbb{Z}}} (1, \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \frac{\chi(x-\tilde{x},y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2+(y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} d\tilde{x} \tag{14} \\ - \int_{\mathbb{R}/_{\mathbb{Z}}} (1, \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t))\eta(x-\tilde{x},y-\varphi(\tilde{x},t))d\tilde{x}$$

Since we are interested in the evolution of the front, we look at the limit of $u(x,y,t)$ as y approaches the front. Notice that the first integral (14) is divergent as we approach the front, i.e. as $y \to \varphi(x,t)$. We look more closely at the original equation (1) to redefine u as we approach the front.

Recall that the scalar θ is convected by the fluid and so we have the equation

$$(\partial_t + u \cdot \nabla_{x,y})\theta = 0$$

This equation defines the velocity u up to additive factors in the direction of $\nabla^\perp\theta$. Precisely, θ satisfies the equation

$$(\partial_t + [u + h\nabla^\perp\theta] \cdot \nabla_{x,y})\theta = 0$$

for any smooth, periodic function h.

We want to use this observation to correct the singularity of u in the equation (14). We will add and subtract a term in the direction of $\nabla^\perp\theta$. Notice that the direction of $\nabla^\perp\theta$ (see (13)) is the same as the tangent to the curve, given by $(1, \frac{\partial\varphi}{\partial x}(x,t))$. We add and subtract the following terms

$$(1, \frac{\partial\varphi}{\partial x}(x,t))\int_{\mathbb{R}/\mathbb{Z}} \frac{\chi(x-\tilde{x}, y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} d\tilde{x} +$$
$$+(1, \frac{\partial\varphi}{\partial x}(x,t))\int_{\mathbb{R}/\mathbb{Z}} \eta(x-\tilde{x}, y-\varphi(\tilde{x},t)) d\tilde{x}$$

We obtain

$$u(x,y,t) = -(1, \frac{\partial\varphi}{\partial x}(x,t)) \int_{\mathbb{R}/\mathbb{Z}} \frac{\chi(x-\tilde{x}, y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} d\tilde{x} -$$
$$-(1, \frac{\partial\varphi}{\partial x}(x,t)) \int_{\mathbb{R}/\mathbb{Z}} \eta(x-\tilde{x}, y-\varphi(\tilde{x},t)) d\tilde{x} + \tag{15}$$
$$+\int_{\mathbb{R}/\mathbb{Z}} (0, \frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \frac{\chi(x-\tilde{x}, y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} d\tilde{x} +$$
$$+\int_{\mathbb{R}/\mathbb{Z}} (0, \frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \eta(x-\tilde{x}, y-\varphi(\tilde{x},t)) d\tilde{x}$$

Now, the first two integrals are divergent as we approach the front, but they are in the direction of $\nabla^\perp\theta$ and so we can redefine the velocity u to be

$$u(x,y,t) = \int_{\mathbb{R}/\mathbb{Z}} (0, \frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \frac{\chi(x-\tilde{x}, y-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (y-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} d\tilde{x} +$$
$$+\int_{\mathbb{R}/\mathbb{Z}} (0, \frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \eta(x-\tilde{x}, y-\varphi(\tilde{x},t)) d\tilde{x}$$

Notice that now we can pass to the limit when (x, y) approaches the front, i.e. $(x, y) \to (x, \varphi(x,t))$. We obtain

$$u(x,\varphi(x,t),t) = \int_{\mathbb{R}/\mathbb{Z}} (0, \frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \frac{\chi(x-\tilde{x}, \varphi(x,t)-\varphi(\tilde{x},t))}{[(x-\tilde{x})^2 + (\varphi(x,t)-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} d\tilde{x} +$$

$$+ \int_{\mathbb{R}/\mathbb{Z}} (0, \frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)) \eta(x-\tilde{x}, \varphi(x,t)-\varphi(\tilde{x},t)) d\tilde{x}$$

Since u is now purely vertical, the fact that $\Omega = \{y \geq \varphi(x,t)\}$ is convected by u means that

$$\begin{cases} \frac{\partial\varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)}{[(x-\tilde{x})^2 + (\varphi(x,t)-\varphi(\tilde{x},t))^2]^{\frac{1}{2}}} \chi(x-\tilde{x}, \varphi(x,t)-\varphi(\tilde{x},t)) d\tilde{x} + \\ \qquad + \int_{\mathbb{R}/\mathbb{Z}} \Big[\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\tilde{x}}(\tilde{x},t)\Big] \eta(x-\tilde{x}, \varphi(x,t)-\varphi(\tilde{x},t)) d\tilde{x} \\ \varphi(x,0) = \varphi_0(x) \end{cases} \tag{16}$$

which is the equation (17) in Theorem 1. We remark that this derivation of the above equation is only heuristic.

2.2 Rigorous derivation: using weak solutions

Now we will obtain a rigorous derivation of the equation. In particular, we will prove

Theorem 1. *If θ is a weak solution of the surface quasi-geostrophic (see Definition 2.1 below) of the form described in* (7)*, then the function φ satisfies the equation,*

$$\frac{\partial\varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial u}(u,t)}{[(x-u)^2 + (\varphi(x,t)-\varphi(u,t))^2]^{\frac{1}{2}}} \chi(x-u, \varphi(x,t)-\varphi(u,t)) du +$$

$$+ \int_{\mathbb{R}/\mathbb{Z}} \Big[\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial u}(u,t)\Big] \eta(x-u, \varphi(x,t)-\varphi(u,t)) du \tag{17}$$

Moreover, if φ satisfies (17)*, the function θ defined by* (7) *is a weak solution of the QG equation.*

We begin with the definition of weak solution for QG.

Definition 2.1. *A bounded function θ is a weak solution of QG if for any $\phi \in C_0^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R} \times [0,\varepsilon])$ we have*

$$\int_{\mathbb{R}^+\times\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \theta(x,y,t)\, \partial_t\phi\,(x,y,t) dydxdt + \int_{\mathbb{R}^+\times\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \theta\,(x,y,t) u(x,y,t) \cdot \nabla\phi\,(x,y,t) dydxdt = 0 \tag{18}$$

Recall that $u(x,y,t) = \nabla^{\perp}(\triangle^{-\frac{1}{2}}\theta) = \Omega * \theta(x,y,t)$ where Ω looks locally like the orthogonal of the Riesz transform. [3] Since θ is bounded we obtain that u is in BMO. On this see [St2] and [Fe-St] for more details.

In the case we are interested in

$$\theta(x,y,t) = 1 \text{ if } y \geq \varphi(x,t) \quad \text{and} \quad \theta = 0 \text{ otherwise}$$

We substitute the above expression for θ in (18) and try to obtain an equation for the evolution of the curve φ.

$$\int_{\mathbb{R}^+\times\mathbb{R}/_{\mathbb{Z}}\times\mathbb{R}} \theta\, \partial_t\phi\, dydxdt = \int_{y>\varphi(x,t)} \partial_t\phi\, dxdydt =$$

$$= \int_{y=\varphi(x,t)} \phi \frac{\partial_t\varphi}{(1+(\partial_x\varphi)^2+(\partial_t\varphi)^2)^{\frac{1}{2}}}(1+(\partial_x\varphi)^2+(\partial_t\varphi)^2)^{\frac{1}{2}}\, dxdt = \int_{y=\varphi(x,t)} \phi\, \partial_t\varphi\, dxdt$$

As for the other term in (18) (considering only the space integration)

$$\int_{\mathbb{R}\times\mathbb{R}/_{\mathbb{Z}}} \theta\, u\cdot\nabla\phi\, dydx = \lim_{\delta\to 0}\int_{y>\varphi(x,t)+\delta} u\cdot\nabla\phi\, dxdy = \lim_{\delta\to 0}\int_{y=\varphi(x,t)+\delta} u\phi\cdot(\frac{\partial\varphi}{\partial x},-1)dx$$

Now we look more closely to the integrand of the above expression. We have

$$u\phi\cdot(\frac{\partial\varphi}{\partial x},-1) = \phi(x,y,t)\int_{v>\varphi(u,t)} \Omega(x-u,y-v)\cdot(\frac{\partial\varphi}{\partial x},-1)dudv$$

The precise expression of Ω is given by

$$\Omega(u,v) = \nabla^{\perp}\Big\{\frac{\chi(u,v)}{(u^2+v^2)^{\frac{1}{2}}} + \eta(u,v)\Big\}$$

And so [4]

$$u\phi\cdot(\frac{\partial\varphi}{\partial x},-1) =$$

$$= \phi(x,y,t)\int_{v>\varphi(u,t)} -(1,\frac{\partial\varphi}{\partial x})\cdot\nabla_{u,v}\left\{\frac{\chi(x-u,y-v)}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}}+\eta(x-u,y-v)\right\}dudv =$$

$$= \phi(x,y,t)\int_{v>\varphi(u,t)} -div_{u,v}\Bigg(\frac{\chi(x-u,y-v)}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}} + \eta(x-u,y-v),$$

$$,\frac{\partial\varphi}{\partial x}\Big(\frac{\chi(x-u,y-v)}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}} + \eta(x-u,y-v)\Big)\Bigg)dudv =$$

[3] More details about the Riesz transform can be found in [St1] and [St2].

[4] We move the $\perp$ that appears in K to the factor $(\frac{\partial\varphi}{\partial x},-1)$

$$= \phi(x,y,t)\int_{v=\varphi(u,t)} -\left\{ \frac{\chi(x-u,y-v)}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}} + \eta(x-u,y-v)\right\}(1,\frac{\partial\varphi}{\partial x})\cdot(\frac{\partial\varphi}{\partial u},-1)du =$$

$$= \phi(x,y,t)\Bigg\{ \int_{v=\varphi(u,t)} -\frac{\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}}\chi(x-u,y-v) -$$

$$-\Big[\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}\Big]\eta(x-u,y-v)du\Bigg\}$$

Hence we have

$$\lim_{\delta\to 0}\int_{y=\varphi(x,t)+\delta} u\phi\cdot(\frac{\partial\varphi}{\partial x},-1)dx =$$

$$= \int_{y=\varphi(x,t)} -\phi(x,y,t)\int_{v=\varphi(u,t)} \frac{\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}}\chi(x-u,y-v)dudx +$$

$$+\int_{y=\varphi(x,t)} -\phi(x,y,t)\int_{v=\varphi(u,t)} \Big[\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}\Big]\eta(x-u,y-v)dudx$$

Putting these two estimates together we have

$$\int_{y=\varphi(x,t)} \phi(x,y,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt =$$

$$= \int_{y=\varphi(x,t)} -\phi(x,y,t)\int_{v=\varphi(u,t)} \frac{\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}}{((x-u)^2+(y-v)^2)^{\frac{1}{2}}}\chi(x-u,y-v)dudxdt +$$

$$+\int_{y=\varphi(x,t)} -\phi(x,y,t)\int_{v=\varphi(u,t)} \Big[\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial x}\Big]\eta(x-u,y-v)dudxdt$$

From that equality we obtain the equation we were looking for

$$\begin{cases} \frac{\partial\varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\varphi}{\partial x}(x,t)-\frac{\partial\varphi}{\partial u}(u,t)}{[(x-u)^2+(\varphi(x,t)-\varphi(u,t))^2]^{\frac{1}{2}}}\chi(x-u,\varphi(x,t)-\varphi(u,t))du + \\ \qquad + \int_{\mathbb{R}/\mathbb{Z}} \Big[\frac{\partial\varphi}{\partial x}(x,t)-\frac{\partial\varphi}{\partial u}(u,t)\Big]\eta(x-u,\varphi(x,t)-\varphi(u,t))du \\ \varphi(x,0)=\varphi_0(x) \end{cases} \tag{19}$$

Finally, notice that all the step in the proof can be reversed, and so we conclude the proof of Theorem 1.

We will prove that the equation given in Theorem 1 is locally well posed. In particular we have the following result

Theorem 2. *Given any periodic, smooth function $\varphi_0(x)$ the initial value problem determined by the equation* (17) *with initial data $\varphi(x,0) = \varphi_0(x)$. has a unique smooth solution for a small time, determined by the initial data φ.*

3 Discussion of the Nash-Moser argument

Our main tool for proving local existence and uniqueness of a solution for the equation (19) will be an inverse function theorem argument. Since we are interested in the C^∞ result we need to run an inverse function theorem in the category of Nash-Moser. In this chapter we perform a brief introduction to Nash-Moser arguments and present the necessary transformations to the equation in order to be able to employ that argument.

There are many surveys of Nash-Moser in the literature. See [Ha], [Ho], [Ze] and [Al-Ge] for expository articles and [Kl], and [Mo] for some interesting applications of Nash-Moser. Here we follow the approach of Richard Hamilton in [Ha].

We quote here the main result we will be using.

Nash-Moser Thm.[5]

Let F and G be tame spaces and $P : U \subseteq F \to G$ be a smooth tame map. Suppose that the equation

$$DP(f)h = k$$

has a unique solution $h = VP(f)k \quad \forall f \epsilon U$ & $\forall k \epsilon G$. Also assume that $VP : U \times G \to F$ is a smooth tame map.

Then P is locally invertible and each of the local inverses P^{-1} is a smooth tame map.

3.1 Adaptation of the equation

In order to apply the previous theorem, we must first make some necessary modifications. In particular, we need to define the spaces F and G and the operator P.

Since we only want to prove local existence of a solution, i.e. $t\epsilon[0,\varepsilon)$ we perform the change of variables

$$\begin{array}{ll} t = \varepsilon \hat{t} & \longrightarrow \dfrac{\partial}{\partial t} = \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial \hat{t}} \\ 0 \leq \hat{t} \leq 1 & \end{array}$$

So, the equation (16) becomes (after changing the dummy variable $\bar{x}$ into y)

[5]See [Ha] pg. 171

$$
\begin{cases}
\dfrac{\partial\varphi}{\partial\hat{t}}(x,\varepsilon\hat{t}) - \varepsilon\displaystyle\int_{\mathbb{R}/\mathbb{Z}} \dfrac{\dfrac{\partial\varphi}{\partial x}(x,\varepsilon\hat{t}) - \dfrac{\partial\varphi}{\partial y}(y,\varepsilon\hat{t})}{\left[(x-y)^2 + (\varphi(x,\varepsilon\hat{t}) - \varphi(y,\varepsilon\hat{t}))^2\right]^{\frac{1}{2}}} \chi(x-y, \varphi(x,\varepsilon\hat{t}) - \varphi(y,\varepsilon\hat{t}))dy - \\
\qquad\qquad -\varepsilon\displaystyle\int_{\mathbb{R}/\mathbb{Z}} \Big[\dfrac{\partial\varphi}{\partial x}(x,\varepsilon\hat{t}) - \dfrac{\partial\varphi}{\partial y}(y,\varepsilon\hat{t})\Big]\eta(x-y, \varphi(x,\varepsilon\hat{t}) - \varphi(y,\varepsilon\hat{t}))dy = 0 \\
\varphi(x,0) = \varphi_0(x)
\end{cases}
\tag{20}
$$

Since we want to make the space of possible solutions into a vector space, we make the change

$$\varphi(x,\varepsilon\hat{t}) = \varphi_0(x) + f_\varepsilon(x,\hat{t})$$

in order to change the initial data into 0.

The equation becomes

$$
\begin{cases}
\dfrac{\partial f_\varepsilon}{\partial\hat{t}}(x,\hat{t}) - \varepsilon\displaystyle\int_{\mathbb{R}/\mathbb{Z}} \dfrac{\dfrac{\partial\varphi_0}{\partial x}(x) - \dfrac{\partial\varphi_0}{\partial y}(y) + \dfrac{\partial f_\varepsilon}{\partial x}(x,\hat{t}) - \dfrac{\partial f_\varepsilon}{\partial y}(y,\hat{t})}{\left[(x-y)^2 + (\varphi_0(x) - \varphi_0(y) + f_\varepsilon(x,\hat{t}) - f_\varepsilon(y,\hat{t}))^2\right]^{\frac{1}{2}}} \times \\
\qquad\qquad \times\chi(x-y, \varphi_0(x) - \varphi_0(y) + f_\varepsilon(x,\hat{t}) - f_\varepsilon(y,\hat{t}))dy - \\
\quad -\varepsilon\displaystyle\int_{\mathbb{R}/\mathbb{Z}} \Big[\dfrac{\partial\varphi_0}{\partial x}(x) - \dfrac{\partial\varphi_0}{\partial y}(y) + \dfrac{\partial f_\varepsilon}{\partial x}(x,\hat{t}) - \dfrac{\partial f_\varepsilon}{\partial y}(y,\hat{t})\Big] \times \\
\qquad\qquad \times\eta(x-y, \varphi_0(x) - \varphi_0(y) + f_\varepsilon(x,\hat{t}) - f_\varepsilon(y,\hat{t}))dy = 0 \\
f_\varepsilon(x,0) = 0
\end{cases}
\tag{21}
$$

Using the structure of (21), we define the Frechet spaces F and G that appear in the theorem

$$
\begin{aligned}
F &= \{\text{pairs} \quad (v(x,\hat{t}),\varepsilon) \quad | \quad v\epsilon\mathcal{F}, v(x,0) = 0,\ \varepsilon\epsilon\mathbb{R})\} \\
G &= \{\text{pairs} \quad (w(x,\hat{t}),\varepsilon) \quad | \quad w\epsilon\mathcal{G},\ \varepsilon\epsilon\mathbb{R})\}
\end{aligned}
\tag{22}
$$

where $\mathcal{F}$ and $\mathcal{G}$ are tame Frechet spaces of smooth functions in $\mathbb{R}/_{\mathbb{Z}} \times [0,1]$ graded with the seminorms [6]

$$\|v(x,\hat{t})\|_n \overset{def}{=} \sup_{a+b\leq n} \|\partial_x^a\partial_{\hat{t}}^b v(x,\hat{t})\|_{L_x^\infty L_{\hat{t}}^\infty}$$

The seminorms in F and G are given by

$$|||(v(x,\hat{t}),\varepsilon)|||_n \overset{def}{=} |\varepsilon| + \|v(x,\hat{t})\|_n$$

We define the operator P_{φ_0} analogous to P in the above theorem by

[6] For the Sobolev norms we will use the notation $\|\cdot\|_{H^s}$ and $\|\cdot\|_{L^2}$, leaving the notation $\|\cdot\|_s$ only for the seminorms in $\mathcal{F}$ and $\mathcal{G}$

$$P_{\varphi_0} : F \longrightarrow G$$
$$(v, \varepsilon) \longrightarrow (T_{\varphi_0,\varepsilon} v, \varepsilon)$$

where

$$T_{\varphi_0,\varepsilon} v(x,\hat{t}) = \frac{\partial v}{\partial \hat{t}}(x,\hat{t}) - \varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \varphi_0}{\partial x}(x) - \frac{\partial \varphi_0}{\partial y}(y) + \frac{\partial v}{\partial x}(x,\hat{t}) - \frac{\partial v}{\partial y}(y,\hat{t})}{[(x-y)^2 + (\varphi_0(x) - \varphi_0(y) + v(x,\hat{t}) - v(y,\hat{t}))^2]^{\frac{1}{2}}} \times$$

$$\times \chi(x-y, \varphi_0(x) - \varphi_0(y) + v(x,\hat{t}) - v(y,\hat{t}))dy -$$

$$-\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \Big[\frac{\partial \varphi_0}{\partial x}(x) - \frac{\partial \varphi_0}{\partial y}(y) + \frac{\partial v}{\partial x}(x,\hat{t}) - \frac{\partial v}{\partial y}(y,\hat{t}) \Big] \eta(x-y, \varphi_0(x) - \varphi_0(y) + v(x,\hat{t}) - v(y,\hat{t}))dy \tag{23}$$

We devote the next chapters to proving that P_{φ_0} satisfies the hypothesis of the Nash-Moser Theorem presented above. Using that theorem we will then prove the desired result.

3.2 Application of Nash-Moser: proof of Thm 2

Assuming that we are able to prove all the hypothesis of the theorem above, we will now show, how its application provides us with the desired result, the existence and uniqueness of a solution for the evolution of a sharp front (in the periodic setting), proving Theorem 2.

Once we prove the theorem we will know that the operator P_{φ_0} is invertible in a neighborhood of the origin in G. (Recall (22)). That means that for every smooth function $w(x,\hat{t})$ in $\mathbb{R}/\mathbb{Z} \times [0,1]$ in a certain neighborhood of the origin in the Frechet space $\mathcal{G}$ and every small ε (recall ε is not necessarily positive) the operator P_{φ_0} is invertible. In other words, there exists $v(x,\hat{t})$ such that

$$T_{\varphi_0,\varepsilon} v(x,\hat{t}) = w(x,\hat{t})$$

This means we can fix a certain positive value of ε and take $w(x,\hat{t})$ to be identically 0. We obtain the existence of $v(x,\hat{t})$ satisfying $T_{\varphi_0,\varepsilon} v(x,\hat{t}) = 0$, i.e.

$$\frac{\partial v}{\partial \hat{t}}(x,\hat{t}) - \varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \varphi_0}{\partial x}(x) - \frac{\partial \varphi_0}{\partial y}(y) + \frac{\partial v}{\partial x}(x,\hat{t}) - \frac{\partial v}{\partial y}(y,\hat{t})}{[(x-y)^2 + (\varphi_0(x) - \varphi_0(y) + v(x,\hat{t}) - v(y,\hat{t}))^2]^{\frac{1}{2}}} \times$$

$$\times \chi(x-y, \varphi_0(x) - \varphi_0(y) + v(x,\hat{t}) - v(y,\hat{t}))dy -$$

$$-\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \Big[\frac{\partial \varphi_0}{\partial x}(x) - \frac{\partial \varphi_0}{\partial y}(y) + \frac{\partial v}{\partial x}(x,\hat{t}) - \frac{\partial v}{\partial y}(y,\hat{t}) \Big] \times$$

$$\times\eta(x-y,\varphi_0(x)-\varphi_0(y)+v(x,\hat{t})-v(y,\hat{t}))dy=0$$

Inverting the transformations that we performed in order to reach the above equation, (adding $\varphi_0(x)$ and rescaling $\hat{t}$) we obtain the desired solution to the original equation.

4 P_{φ_0} is smooth and tame

We devote this chapter to proving that the map P_{φ_0} is tame and smooth. Specifically, we need to obtain the following estimate

$$|||P_{\varphi_0}(f,\varepsilon)|||_k \lesssim 1+|||(f,\varepsilon)|||_{k+r} \quad \forall k>k_0 \tag{24}$$

for all (f,ε) in a neighborhood $\mathcal{O}_M$ of $(0,0)$ in F, i.e.

$$\|f\|_M \leq C_0 \text{ for some } M \text{ and } C_0 \qquad \text{and} \qquad |\varepsilon|\leq\varepsilon_0 \tag{25}$$

We will determine M in the next chapters. M might be increased in the next chapters, since some interpolation inequalities will require that we control a certain fixed number of derivatives.

In terms of $T_{\varphi_0,\varepsilon}$ the above tame estimate (24) becomes

$$\|T_{\varphi_0,\varepsilon}f\|_k \lesssim (1+\|f\|_{k+r}) \quad \forall k>k_0 \tag{26}$$

for all (f,ε) in the neighborhood $\mathcal{O}_M$ defined by (25).

Notation

In the previous inequalities we used the symbol $\lesssim$ to indicate the existence of a constant, that depends only on the initial data φ_0, k, and the constants M, C_0 and ε_0 that appear in (25). We will use this notation from now on.

In order to prove (26) we have to obtain the following inequality.

$$\Bigg\|\partial_x^a\partial_{\hat{t}}^b\Bigg\{\frac{\partial f}{\partial\hat{t}}(x,\hat{t})-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial\varphi_0}{\partial x}(x)-\frac{\partial\varphi_0}{\partial y}(y)+\frac{\partial f}{\partial x}(x,\hat{t})-\frac{\partial f}{\partial y}(y,\hat{t})}{[(x-y)^2+(\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))^2]^{\frac{1}{2}}}\times$$

$$\times\chi(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))\,dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\partial\varphi_0}{\partial x}(x)-\frac{\partial\varphi_0}{\partial y}(y)+\frac{\partial f}{\partial x}(x,\hat{t})-\frac{\partial f}{\partial y}(y,\hat{t})\Big]\times$$

$$\times\eta(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))\,dy\Bigg\}\Bigg\|_{L_x^\infty L_{\hat{t}}^\infty}\lesssim$$

$$\lesssim(1+\sup_{c+d\leq a+b+r}\|\partial_x^c\partial_{\hat{t}}^d f\|_{L_x^\infty L_{\hat{t}}^\infty}+|\varepsilon|) \tag{27}$$

Using the triangle inequality we have 3 inequalities to prove. The estimate for the first term is trivial. We show in some detail the estimate for the second term. The third one is completely analog. We have

$$\Bigg\| \partial_x^a \partial_{\hat{t}}^b \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \varphi_0}{\partial x}(x) - \frac{\partial \varphi_0}{\partial y}(y) + \frac{\partial f}{\partial x}(x,\hat{t}) - \frac{\partial f}{\partial y}(y,\hat{t})}{[(x-y)^2 + (\varphi_0(x) - \varphi_0(y) + f(x,\hat{t}) - f(y,\hat{t}))^2]^{\frac{1}{2}}} \times$$

$$\times \chi(x-y, \varphi_0(x) - \varphi_0(y) + f(x,\hat{t}) - f(y,\hat{t}))dy \Bigg\|_{L_x^\infty L_{\hat{t}}^\infty}$$

We will obtain the estimate for the above terms using the lemmas in appendix. We will denote the integrand by $Q(x,y,\hat{t})$. Notice that $Q(x,y,\hat{t})$ is not smooth in x,y but rather $sgn(y-x)$ times a smooth function. For this reason we break up the integral into two.

$$\int_{\mathbb{R}/\mathbb{Z}} Q(x,y,\hat{t})dy = \int_x^{x+\frac{1}{2}} Q(x,y,\hat{t}) + \int_{x-\frac{1}{2}}^x Q(x,y,\hat{t})dy$$

Observe that once we have broken up the domain of integration, we obtain a smooth integrand. It is also clear that both integrals and all their derivatives are estimated in exactly the same way, so we will only deal here with the first one. In addition, notice that

$$Q(x,y) = 0$$

for y in a neighborhood of both $y = x + \frac{1}{2}$ and $y = x - \frac{1}{2}$, due to the presence of the cut-off function χ in Q. (See (9))

Using these observations and lemma 9 in the appendix we have [7]

$$\partial_x^a \partial_{\hat{t}}^b \int_x^{x+\frac{1}{2}} Q(x,y,\hat{t})dy = \partial_{\hat{t}}^b \Big\{ -\sum_{j=0}^{a-1} \frac{d^{a-1-j}}{dx^{a-1-j}} \Big(\frac{\partial^j Q}{\partial u^j}(x,x,\hat{t}) \Big) + \int_x^{x+\frac{1}{2}} \frac{\partial^a Q}{\partial x^a}(x,y,\hat{t})dy \Big\} =$$

$$= \partial_{\hat{t}}^b \Big\{ -\sum_{j=0}^{a-1} \sum_{k=0}^{a-1-j} c_k \frac{\partial^{a-1}}{\partial u^{j+k} \partial v^{a-1-j-k}} Q(x,x,\hat{t}) + \int_x^{x+\frac{1}{2}} \frac{\partial^a Q}{\partial x^a}(x,y,\hat{t})dy \Big\} =$$

$$= -\sum_{j=0}^{a-1} \sum_{k=0}^{a-1-j} c_k \frac{\partial^{a-1}}{\partial u^{j+k} \partial v^{a-1-j-k}} \partial_{\hat{t}}^b Q(x,x,\hat{t}) + \int_x^{x+\frac{1}{2}} \frac{\partial^{a+b} Q}{\partial x^a \partial \hat{t}^b}(x,y,\hat{t})dy$$

Since we are only interested in the $L_x^\infty L_{\hat{t}}^\infty$ norm of the above quantity we note that all the terms in the sum or the integral are of the same form and can be treated using the lemmas in the appendix. We rewrite Q so that we can apply our lemmas. We have the following expression (recall that now $y \geq x + \frac{1}{2}$)

[7] We denote by u the first variable in Q. v will be the second, i.e. we regard Q as $Q(u,v)$.

$$\frac{\dfrac{\dfrac{\partial\varphi_0}{\partial x}(x)-\dfrac{\partial\varphi_0}{\partial y}(y)+\dfrac{\partial f}{\partial x}(x,\hat{t})-\dfrac{\partial f}{\partial y}(y,\hat{t})}{x-y}}{[1+(\dfrac{\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t})}{x-y})^2]^{\frac{1}{2}}}\chi(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))$$

Notice that the integrand is the product of 3 tame functions, since the denominator is bounded below by 1, and hence it is tame. [8]

To prove that the operator P_{φ_0} is smooth we just notice that the decomposition showed above decomposes the function in simple blocks for which is trivial to prove smoothness.

5 Existence and uniqueness of a solution for the linearized equation

In this chapter we prove the existence and uniqueness of a solution for the linearized equation, precisely for the equation

$$DP_{\varphi_0}(f,\varepsilon)(h,\omega)=(k,\sigma) \qquad \forall\ (k,\sigma)\ \text{ in }\ G \tag{28}$$

Recall that

$$\begin{aligned} P_{\varphi_0}: F &\longrightarrow \quad G \\ (f,\varepsilon) &\longrightarrow (T_{\varphi_0,\varepsilon}f,\varepsilon) \end{aligned}$$

where $T_{\varphi_0,\varepsilon}f(x,\hat{t})$ is given by (23).

Now we compute DP_{φ_0}. Observe that

$$DP_{\varphi_0}(f,\varepsilon)(h,\omega)=\lim_{\delta\to 0}\Big(\frac{T_{\varphi_0,\varepsilon+\delta\omega}(f+\delta h)-T_{\varphi_0,\varepsilon}(f)}{\delta},\frac{\varepsilon+\delta\omega-\varepsilon}{\delta}\Big)$$

After a simple, long computation we obtain the following expression for the first component of DP_{φ_0}

$$\frac{\partial h}{\partial t}(x,\hat{t})-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\dfrac{\partial h}{\partial x}(x,\hat{t})-\dfrac{\partial h}{\partial y}(y,\hat{t})}{[(x-y)^2+(\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))^2]^{\frac{1}{2}}}\times$$

$$\times\chi(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{[\dfrac{\partial\varphi_0}{\partial x}(x)-\dfrac{\partial\varphi_0}{\partial y}(y)+\dfrac{\partial f}{\partial x}(x,\hat{t})-\dfrac{\partial f}{\partial y}(y,\hat{t})]\,[\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t})]}{[(x-y)^2+(\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))^2]^{\frac{3}{2}}}\times$$

[8] To see that the numerator and denominator are tame use Lemma 8

$$\times\big[h(x,\hat{t})-h(y,\hat{t})\big]\chi(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\big[h(x,\hat{t})-h(y,\hat{t})\big]\big[\frac{\partial\varphi_0}{\partial x}(x)-\frac{\partial\varphi_0}{\partial y}(y)+\frac{\partial f}{\partial x}(x,\hat{t})-\frac{\partial f}{\partial y}(y,\hat{t})\big]}{[(x-y)^2+(\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))^2]^{\frac{1}{2}}}\times$$

$$\times\chi_v(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\partial\varphi_0}{\partial x}(x)-\frac{\partial\varphi_0}{\partial y}(y)+\frac{\partial f}{\partial x}(x,\hat{t})-\frac{\partial f}{\partial y}(y,\hat{t})\Big]\Big[h(x,\hat{t})-h(y,\hat{t})\Big]\times$$

$$\times\eta_v(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\big[\frac{\partial h}{\partial x}(x,\hat{t})-\frac{\partial h}{\partial y}(y,\hat{t})\big]\eta(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy-$$

$$-\omega\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial\varphi_0}{\partial x}(x)-\frac{\partial\varphi_0}{\partial y}(y)+\frac{\partial f}{\partial x}(x,\hat{t})-\frac{\partial f}{\partial y}(y,\hat{t})}{[(x-y)^2+(\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))^2]^{\frac{1}{2}}}\times$$

$$\times\chi(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy-$$

$$-\omega\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\partial\varphi_0}{\partial x}(x)-\frac{\partial\varphi_0}{\partial y}(y)+\frac{\partial f}{\partial x}(x,\hat{t})-\frac{\partial f}{\partial y}(y,\hat{t})\Big]\eta(x-y,\varphi_0(x)-\varphi_0(y)+f(x,\hat{t})-f(y,\hat{t}))dy$$

The second component of DP_{φ_0} is just ω. Since we have to solve the system

$$DP_{\varphi_0}(f,\varepsilon)(h,\omega)=(k,\sigma)$$

we obtain that $\omega=\sigma$ and are led to solve the first component.

Notice that the initial condition for the equation above comes from the fact that $h\epsilon\mathcal{F}$ and so we have $h(x,0)=0$

We will use the notation $g(x,\hat{t}):=\varphi_0(x)+f(x,\hat{t})$ in order to simplify the formulas. We will also denote $\chi(x-y,g(x,\hat{t})-g(y,\hat{t})))$ and $\eta(x-y,g(x,\hat{t})-g(y,\hat{t})))$ by $\bar{\chi}(x,y,\hat{t})$ and $\bar{\eta}(x,y,\hat{t})$ and $\chi_v(x-y,g(x,\hat{t})-g(y,\hat{t})))$ and $\eta_v(x-y,g(x,\hat{t})-g(y,\hat{t})))$ by $\bar{\chi}_v(x,y,\hat{t})$ and $\bar{\eta}_v(x,y,\hat{t})$. The system we need to solve is

$$\frac{\partial h}{\partial t}(x,\hat{t})-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial h}{\partial x}(x,\hat{t})-\frac{\partial h}{\partial y}(y,\hat{t})}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\big[h(x,\hat{t})-h(y,\hat{t})\big]\big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\big]\big[g(x,\hat{t})-g(y,\hat{t})\big]}{\big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\big]^{\frac{3}{2}}}\bar{\chi}(x,y,\hat{t})dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\big[h(x,\hat{t})-h(y,\hat{t})\big]\big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\big]}{\big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\big]^{\frac{1}{2}}}\bar{\chi}_v(x,y,\hat{t})dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\Big]\Big[h(x,\hat{t})-h(y,\hat{t})\Big]\bar{\eta}_v(x,y,\hat{t})dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\partial h}{\partial x}(x,\hat{t})-\frac{\partial h}{\partial y}(y,\hat{t})\Big]\bar{\eta}(x,y,\hat{t})dy-$$

$$-\omega\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})}{\big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\big]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy-$$

$$-\omega\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\Big]\bar{\eta}(x,y,\hat{t})dy=k(x,\hat{t}) \tag{29}$$

with the initial condition $h(x,0)=0$.

5.1 Simplifying the equation

We perform a detailed analysis of the most singular terms in the equation and perform the necessary transformations to simplify them.

We would like to modify h so that the most singular term in the equation (2nd term) absorbs the following term (3rd term)

$$\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\big[h(x,\hat{t})-h(y,\hat{t})\big]\big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\big]\big[g(x,\hat{t})-g(y,\hat{t})\big]}{\big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\big]^{\frac{3}{2}}}\bar{\chi}(x,y,\hat{t})dy$$

We want to do this since this term is of the type

$$\int_{\mathbb{R}/\mathbb{Z}}\frac{h(x,\hat{t})-h(y,\hat{t})}{|x-y|}\theta(x,y)dy$$

and that will bring up a logarithmic divergence when computing energy estimates.

For this purpose we want to make the change $h(x,\hat{t})=\varphi(x,\hat{t})\bar{h}(x,\hat{t})$ in order to obtain an equation for φ so that we can cancel those two terms.

The second and third term become

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\varphi(x,\hat{t})\frac{\partial\bar{h}}{\partial x}(x,\hat{t})-\varphi(y,\hat{t})\frac{\partial\bar{h}}{\partial y}(y,\hat{t})}{\big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\big]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{h}(x,\hat{t})\frac{\partial\varphi}{\partial x}(x,\hat{t})-\bar{h}(y,\hat{t})\frac{\partial\varphi}{\partial y}(y,\hat{t})}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{[\varphi(x,\hat{t})\bar{h}(x,\hat{t})-\varphi(y,\hat{t})\bar{h}(y,\hat{t})][\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})][g(x,\hat{t})-g(y,\hat{t})]}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{3}{2}}}\bar{\chi}(x,y,\hat{t})dy-$$

We want to find φ so that

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{h}(x,\hat{t})\frac{\partial\varphi}{\partial x}(x,\hat{t})-\bar{h}(y,\hat{t})\frac{\partial\varphi}{\partial y}(y,\hat{t})}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{[\varphi(x,\hat{t})\bar{h}(x,\hat{t})-\varphi(y,\hat{t})\bar{h}(y,\hat{t})][\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})][g(x,\hat{t})-g(y,\hat{t})]}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{3}{2}}}\bar{\chi}(x,y,\hat{t})dy$$

cancel each other, up to smooth terms. Combining the two integrals in one, we can rewrite the integrand in the following form

$$\frac{|x-y|^3}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{3}{2}}}\bar{\chi}(x,y,\hat{t})\times$$

$$\times\left\{\frac{[\varphi(x,\hat{t})\bar{h}(x,\hat{t})-\varphi(y,\hat{t})\bar{h}(y,\hat{t})]\,[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})]\,[g(x,\hat{t})-g(y,\hat{t})]}{|x-y|^3}-\right.$$

$$\left.-\frac{[\bar{h}(x,\hat{t})\frac{\partial\varphi}{\partial x}(x,\hat{t})-\bar{h}(y,\hat{t})\frac{\partial\varphi}{\partial y}(y,\hat{t})]\,[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]}{|x-y|^3}\right\}$$

We want to choose φ so that the function in curly brackets is a smooth function or a smooth function, times $sgn(x-y)$.

We look at the factor multiplying $\bar{h}(y,\hat{t})$ inside the curly brackets. The factor is

$$|x-y|^{-1}\left\{\frac{\partial\varphi}{\partial y}(y,\hat{t})\Big[1+\Big(\frac{g(x,\hat{t})-g(y,\hat{t})}{x-y}\Big)^2\Big]-\right.$$

$$\left.-\varphi(y,\hat{t})\Big[\frac{\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})}{x-y}\Big]\Big[\frac{g(x,\hat{t})-g(y,\hat{t})}{x-y}\Big]\right\}$$

Notice that the factor multiplying $\bar{h}(x,\hat{t})$ is

$$-|x-y|^{-1}\left\{\frac{\partial\varphi}{\partial x}(x,\hat{t})\Big[1+\Big(\frac{g(x,\hat{t})-g(y,\hat{t})}{x-y}\Big)^2\Big]-\right.$$

$$-\varphi(x,\hat{t})\Big[\frac{\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})}{x-y}\Big]\Big[\frac{g(x,\hat{t})-g(y,\hat{t})}{x-y}\Big]\Bigg\}$$

and so the analysis of both terms is completely analogous.

We can take φ so that it solves,

$$\frac{\partial\varphi}{\partial y}(y,\hat{t})[1+(g'(y,\hat{t}))^2]-\varphi(y,\hat{t})[g''(y,\hat{t})][g'(y,\hat{t})]=0$$

where the prime ($'$) stands for the partial derivative with respect to space.

So we have

$$\frac{\partial\varphi}{\partial y}(y,\hat{t})-\varphi(y,\hat{t})\frac{[g''(y,\hat{t})][g'(y,\hat{t})]}{[1+(g'(y,\hat{t}))^2]}=0$$

and so

$$\varphi(y,\hat{t})=c(\hat{t})\times exp\Big(\int_0^y\frac{[g''(\xi,\hat{t})][g'(\xi,\hat{t})]}{[1+(g'(\xi,\hat{t}))^2]}d\xi\Big)$$

Since we have taken

$$h(y,\hat{t})=\varphi(y,\hat{t})\bar{h}(y,\hat{t})$$

we need to make sure that

$$\varphi(y,\hat{t})=\varphi(y+1,\hat{t})$$

in order to preserve the periodicity of h. We must have

$$c(\hat{t})\times exp\Big(\int_y^{y+1}\frac{[g''(\xi,\hat{t})][g'(\xi,\hat{t})]}{[1+(g'(\xi,\hat{t}))^2]}d\xi\Big)=c(\hat{t})$$

and so we must have

$$exp\Big(\int_y^{y+1}\frac{[g''(\xi,\hat{t})][g'(\xi,\hat{t})]}{[1+(g'(\xi,\hat{t}))^2]}d\xi\Big)=1$$

and hence

$$\int_y^{y+1}\frac{[g''(\xi,\hat{t})][g'(\xi,\hat{t})]}{[1+(g'(\xi,\hat{t}))^2]}d\xi=0$$

but this is true since

$$\int_y^{y+1}\frac{[g''(\xi,\hat{t})][g'(\xi,\hat{t})]}{[1+(g'(\xi,\hat{t}))^2]}d\xi=\int_y^{y+1}\frac{1}{2}\frac{d}{d\xi}log[1+(g'(\xi,\hat{t}))^2]d\xi=\frac{1}{2}log[1+(g'(\xi,\hat{t}))^2]\Big|_y^{y+1}=0$$

because g' is periodic.

Since $c(\hat{t})$ does not play any role in the periodicity of h we can take $c(\hat{t})$ to be identically 1.

With this choice of φ the two terms become

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{h}(x,\hat{t})\frac{\partial\varphi}{\partial x}(x,\hat{t})-\bar{h}(y,\hat{t})\frac{\partial\varphi}{\partial y}(y,\hat{t})}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{[\varphi(x,\hat{t})\bar{h}(x,\hat{t})-\varphi(y,\hat{t})\bar{h}(y,\hat{t})][\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})][g(x,\hat{t})-g(y,\hat{t})]}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{3}{2}}}\bar{\chi}(x,y,\hat{t})dy$$

$$=\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\bar{h}(y,\hat{t})sgn(x-y)A_1(x,y,\hat{t})dy-\varepsilon\bar{h}(x,\hat{t})\int_{\mathbb{R}/\mathbb{Z}}sgn(x-y)B(x,y,\hat{t})dy=$$

$$=\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\bar{h}(y,\hat{t})sgn(x-y)A_1(x,y,\hat{t})dy+\varepsilon\bar{h}(x,\hat{t})A_2(x,\hat{t})dy=$$

where

$$A_1(x,y,\hat{t})=sgn(x-y)\Big\{\frac{\partial\varphi}{\partial y}(y,\hat{t})\Big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\Big]-\varphi(y,\hat{t})\Big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\Big]$$

$$\times\Big[g(x,\hat{t})-g(y,\hat{t})\Big]\Big\}[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{-\frac{3}{2}}\bar{\chi}(x,y,\hat{t})$$

$$B(x,y,\hat{t})=sgn(x-y)\Big\{\frac{\partial\varphi}{\partial x}(x,\hat{t})\Big[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2\Big]-\varphi(x,\hat{t})\Big[\frac{\partial g}{\partial x}(x,\hat{t})-\frac{\partial g}{\partial y}(y,\hat{t})\Big]$$

$$\times\Big[g(x,\hat{t})-g(y,\hat{t})\Big]\Big\}[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{-\frac{3}{2}}\bar{\chi}(x,y,\hat{t})$$

and

$$A_2(x,\hat{t})=-\int_{\mathbb{R}/\mathbb{Z}}sgn(x-y)B(x,y,\hat{t})dy$$

Notice that $A_1(x,x,\hat{t})=0$.

With this choice of $\varphi(y,\hat{t})$ the most singular term becomes

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\varphi(x,\hat{t})\frac{\partial\bar{h}}{\partial x}(x,\hat{t})-\varphi(y,\hat{t})\frac{\partial\bar{h}}{\partial y}(y,\hat{t})}{[(x-y)^2+(g(x,\hat{t})-g(y,\hat{t}))^2]^{\frac{1}{2}}}\bar{\chi}(x,y,\hat{t})dy$$

Now we produce several transformations to the equation so that the most singular term becomes

$$\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial h}{\partial x}(x,\hat{t})-\frac{\partial h}{\partial y}(y,\hat{t})}{|x-y|}\theta(x-y)dy$$

Precisely, we write $\varphi(y,\hat{t})$ in the following as $\varphi(y,\hat{t}) = \varphi(x,\hat{t}) + [\varphi(y,\hat{t}) - \varphi(x,\hat{t})]$ in the most singular term and start by dividing the equation by $\varphi(x,\hat{t})$ and then perform a change of variables in the equation. In particular we take

$$\begin{aligned} x &= \phi(\bar{x},\bar{t}) \\ \hat{t} &= \bar{t} \end{aligned}$$

We use the notation

$$\tilde{h}(\bar{x},\bar{t}) := \bar{h}(\phi(\bar{x},\bar{t}),\bar{t}) \qquad \tilde{\varphi}(\bar{x},\bar{t}) := \varphi(\phi(\bar{x},\bar{t}),\bar{t}) \qquad \tilde{g}(\bar{x},\bar{t}) := g(\phi(\bar{x},\bar{t}),\bar{t})$$

$$\tilde{\chi}(\bar{x},\bar{y},\bar{t}) := \bar{\chi}(\phi(\bar{x},\bar{t}),\phi(\bar{y},\bar{t}),\bar{t}) \qquad \tilde{\eta}(\bar{x},\bar{y},\bar{t}) := \bar{\eta}(\phi(\bar{x},\bar{t}),\phi(\bar{y},\bar{t}),\bar{t})$$

$$\tilde{\chi}_v(\bar{x},\bar{y},\bar{t}) := \bar{\chi}_v(\phi(\bar{x},\bar{t}),\phi(\bar{y},\bar{t}),\bar{t}) \qquad \tilde{\eta}_v(\bar{x},\bar{y},\bar{t}) := \bar{\eta}_v(\phi(\bar{x},\bar{t}),\phi(\bar{y},\bar{t}),\bar{t})$$

The most singular term becomes (except for the minus sign):

$$\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\dfrac{1}{\dfrac{\partial\phi}{\partial\bar{x}}(\bar{x},\bar{t})}\dfrac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t}) - \dfrac{1}{\dfrac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})}\dfrac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y},\bar{t})}{\left[(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}))^2 + (\tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))^2\right]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) \frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t}) d\bar{y} =$$

$$= \varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\dfrac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t}) - \dfrac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y},\bar{t})}{\left[(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}))^2 + (\tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))^2\right]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) d\bar{y} +$$

$$+\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\dfrac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t})\left[\dfrac{1}{\dfrac{\partial\phi}{\partial\bar{x}}(\bar{x},\bar{t})} - \dfrac{1}{\dfrac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})}\right]}{\left[(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}))^2 + (\tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))^2\right]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) \frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t}) d\bar{y} =$$

$$= \varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\dfrac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t}) - \dfrac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|\left[(\phi'(\bar{y},\bar{t}))^2 + (g'(\phi(\bar{y},\bar{t}),\bar{t})\phi'(\bar{y},\bar{t}))^2\right]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) d\bar{y} +$$

$$+\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\dfrac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t})\left[\dfrac{1}{\dfrac{\partial\phi}{\partial\bar{x}}(\bar{x},\bar{t})} - \dfrac{1}{\dfrac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})}\right]}{\left[(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}))^2 + (\tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))^2\right]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) \frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t}) d\bar{y} +$$

$$+\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t}) - \frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})}{[(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}))^2 + (\tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))^2]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) d\bar{y} -$$

$$-\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t}) - \frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|[(\phi'(\bar{y},\bar{t}))^2 + (g'(\phi(\bar{y},\bar{t}),\bar{t})\phi'(\bar{y},\bar{t}))^2]^{\frac{1}{2}}} \tilde{\chi}(\bar{x},\bar{y},\bar{t}) d\bar{y}$$

We want to choose the change of variables $x = \phi(\bar{x},\bar{t})$ in such a way that the expression in square brackets in the denominator in the first integral is only a function of time. We want to define ϕ and a in such a way that we have

$$[(\phi^{'}(\bar{y},\bar{t}))^2 + (g^{'}(\phi(\bar{y},\bar{t}),\bar{t})\phi^{'}(\bar{y},\bar{t}))^2]^{\frac{1}{2}} = a(\bar{t})$$

and hence

$$\frac{dy}{d\bar{y}}(1 + (g^{'}(y,\bar{t}))^2)^{\frac{1}{2}} = a(\bar{t}) \tag{30}$$

$$(1 + (g^{'}(y,\bar{t}))^2)^{\frac{1}{2}} dy = a(\bar{t}) d\bar{y}$$

Since we want our change of coordinates to be a diffeomorphism from $\mathbb{R}/_{\mathbb{Z}}$ to $\mathbb{R}/_{\mathbb{Z}}$ we can choose, without loss of generality that $\bar{y} = 0$ is mapped into $y = 0$ and so $\bar{y} = 1$ is mapped into $y = 1$. Therefore we have

$$\int_0^y (1 + (g^{'}(\xi,\bar{t}))^2)^{\frac{1}{2}} d\xi = a(\bar{t})\bar{y}$$

where we define $a(\bar{t})$ in order the preserve the periodicity of the front, i.e.

$$\int_0^1 (1 + (g^{'}(\xi,\bar{t}))^2)^{\frac{1}{2}} d\xi = a(\bar{t}) \tag{31}$$

Some observations about the change of variables are in order. The results needed to apply the Nash-Moser theorem require that we solve the linear equation, but only in a neighborhood $\mathcal{O}_M$ of the origin. In proving the tame estimates for P we have restricted f to a neighborhood of the origin (recall (25)), and so we have

$$\|g\|_{L^\infty} \leq K \quad \|\partial_x g\|_{L^\infty} \leq K \quad \|\partial_x^2 g\|_{L^\infty} \leq K$$

where K depends only on φ_0 and the constant C_0 appearing in (25) (recall that $g = f + \varphi_0$). K is taken to be greater than or equal to

$$max(\|\varphi_0\|_{L^\infty} + \|f\|_{L^\infty}, \|\partial_x\varphi_0\|_{L^\infty} + \|\partial_x f\|_{L^\infty}, \|\partial_x\varphi_0\|_{L^\infty} + \|\partial_x f\|_{L^\infty})$$

Since that gives as a bound on $g^{'}$, looking at (31) we get

$$1 \leq a(\bar{t}) \leq \sqrt{1+K^2}$$

and so using (30) we obtain

$$\frac{1}{\sqrt{1+K^2}} \leq \frac{dx}{d\bar{x}} = \frac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t}) = \frac{a(\hat{t})}{(1+(g^{'}(y,\bar{t}))^2)^{\frac{1}{2}}} \leq \sqrt{1+K^2} \qquad (32)$$

These bounds will be useful in controlling the support of the cutoff functions after the change of variables. The function $\bar{\chi}$ is evaluated at

$$(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}), \tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))$$

and since we have obtained upper bounds for g and $g^{'}$ and lower and upper bounds for ϕ and $\phi^{'}$ we can choose r and R in the original definition of the support of χ (See (9)) sufficiently small so that

$$\chi(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}), \tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))$$

is supported in the strip $|\bar{y}-\bar{x}| \leq \frac{1}{4}$. We still have that $\tilde{\chi} = 1$ if $|\bar{y}-\bar{x}| \leq C(K)$, where we will determine $C(K)$ below.

Since $\frac{1}{\sqrt{1+K^2}} \leq \frac{\partial \phi}{\partial x} \leq \sqrt{1+K^2}$ and $|g'| \leq K$, the norm of $(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}), \tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t}))$ is smaller than $|\bar{y}-\bar{x}|\sqrt{1+K^2+K^2}$ and bigger than $|\bar{y}-\bar{x}|\dfrac{1}{\sqrt{1+K^2}}$ and so we can choose

$$R = \frac{1}{4\sqrt{(1+2K^2}} \quad \text{and} \quad r = \frac{1}{8\sqrt{1+2K^2}}$$

Hence we can take $C(K) = \dfrac{1}{(\sqrt{1+K^2})}\, r$.

With this choice of r and R we can split χ in

$$\tilde{\chi}(\bar{x},\bar{y},\bar{t}) = \chi(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}), \tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t})) =$$

$$= \theta(\bar{y}-\bar{x}) + \chi(\phi(\bar{x},\bar{t}) - \phi(\bar{y},\bar{t}), \tilde{g}(\bar{x},\bar{t}) - \tilde{g}(\bar{y},\bar{t})) - \theta(\bar{y}-\bar{x}) = \theta(\bar{y}-\bar{x}) + \rho(\bar{x},\bar{y},\bar{t})$$

where

$$\begin{cases} \theta(s) = 1 \quad \text{if} \quad s\epsilon[-C(K), C(K)], \quad \theta \text{ even and} \quad supp\ \theta \subset [-\frac{1}{4}, \frac{1}{4}] \\ supp\ \ \rho \subset \{C(K) < |\bar{y}-\bar{x}| < \frac{1}{4}\} \end{cases}$$

After all this transformations the equation becomes

$$\frac{\partial \tilde{h}}{\partial \bar{t}}(\bar{x},\bar{t}) + \frac{-\dfrac{\partial \phi}{\partial \bar{t}}(\bar{x},\bar{t})}{\dfrac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t})} \frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t}) + \frac{-\dfrac{\partial \phi}{\partial \bar{t}}(\bar{x},\bar{t})}{\dfrac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t})} \frac{\dfrac{\partial \tilde{\varphi}}{\partial \bar{x}}(\bar{x},\bar{t})}{\tilde{\varphi}(\bar{x},\bar{t})} \tilde{h}(\bar{x},\bar{t}) + \frac{\dfrac{\partial \tilde{\varphi}}{\partial \bar{t}}(\bar{x},\bar{t})}{\tilde{\varphi}(\bar{x},\bar{t})} \tilde{h}(\bar{x},\bar{t}) -$$

$$-\varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial h}{\partial \bar{x}}(\bar{x},\bar{t})-\frac{\partial h}{\partial \bar{y}}(\bar{y},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{x}-\bar{y})d\bar{y}-$$

$$-\varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial h}{\partial \bar{x}}(\bar{x},\bar{t})-\frac{\partial h}{\partial \bar{y}}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\rho(\bar{x},\bar{y},\bar{t})d\bar{y}-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})\left[\frac{1}{\frac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t})}-\frac{1}{\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})}\right]}{[(\phi(\bar{x},\bar{t})-\phi(\bar{y},\bar{t}))^2+(\tilde{g}(\bar{x},\bar{t})-\tilde{g}(\bar{y},\bar{t}))^2]^{\frac{1}{2}}}\tilde{\chi}(\bar{x},\bar{y},\bar{t})\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})d\bar{y}-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})-\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})}{[(\phi(\bar{x},\bar{t})-\phi(\bar{y},\bar{t}))^2+(\tilde{g}(\bar{x},\bar{t})-\tilde{g}(\bar{y},\bar{t}))^2]^{\frac{1}{2}}}\tilde{\chi}(\bar{x},\bar{y},\bar{t})d\bar{y}+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})-\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|[(\phi'(\bar{y},\bar{t}))^2+(g'(\bar{y},\bar{t})\phi'(\bar{y},\bar{t}))^2]^{\frac{1}{2}}}\tilde{\chi}(\bar{x},\bar{y},\bar{t})d\bar{y}-$$

$$-\varepsilon\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})\frac{\left[\tilde{\varphi}(\bar{x},\bar{t})-\tilde{\varphi}(\bar{y},\bar{t})\right]}{[(\phi(\bar{x},\bar{t})-\phi(\bar{y},\bar{t}))^2+(\tilde{g}(\bar{x},\bar{t})-\tilde{g}(\bar{y},\bar{t}))^2]^{\frac{1}{2}}}\tilde{\chi}(\bar{x},\bar{y},\bar{t})d\bar{y}+$$

$$+\varepsilon\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\tilde{h}(\bar{y},\bar{t})sgn(\phi(\bar{x},\bar{t})-\phi(\bar{y},\bar{t}))A_1(\phi(\bar{x},\bar{t}),\phi(\bar{y},\bar{t}),\bar{t})\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})d\bar{y}+$$

$$+\varepsilon\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\tilde{h}(\bar{x},\bar{t})A_2(\phi(\bar{x},\bar{t}),\bar{t})-$$

$$-\frac{\varepsilon}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\frac{\left[\tilde{\varphi}(\bar{x},\bar{t})\tilde{h}(\bar{x},\bar{t})-\tilde{\varphi}(\bar{y},\bar{t})\tilde{h}(\bar{y},\bar{t})\right]\left[\frac{\frac{\partial \tilde{g}}{\partial \bar{x}}(\bar{x},\bar{t})}{\frac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t})}-\frac{\frac{\partial \tilde{g}}{\partial \bar{y}}(\bar{y},\bar{t})}{\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})}\right]}{[(\phi(\bar{x},\bar{t})-\phi(\bar{y},\bar{t}))^2+(\tilde{g}(\bar{x},\bar{t})-\tilde{g}(\bar{y},\bar{t}))^2]^{\frac{1}{2}}}\tilde{\chi}_v(\bar{x},\bar{y},\bar{t})\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})d\bar{y}-$$

$$-\frac{\varepsilon}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\left[\tilde{\varphi}(\bar{x},\bar{t})\tilde{h}(\bar{x},\bar{t})-\tilde{\varphi}(\bar{y},\bar{t})\tilde{h}(\bar{y},\bar{t})\right]\left[\frac{\frac{\partial \tilde{g}}{\partial \bar{x}}(\bar{x},\bar{t})}{\frac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t})}-\frac{\frac{\partial \tilde{g}}{\partial \bar{y}}(\bar{y},\bar{t})}{\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})}\right]\tilde{\eta}_v(\bar{x},\bar{y},\bar{t})\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})d\bar{y}-$$

$$-\varepsilon\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\left[\tilde{\varphi}(\bar{x},\bar{t})\frac{1}{\frac{\partial \phi}{\partial \bar{x}}(\bar{x},\bar{t})}\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})-\tilde{\varphi}(\bar{y},\bar{t})\frac{1}{\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})}\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})\right]\tilde{\eta}(\bar{x},\bar{y},\bar{t})\frac{\partial \phi}{\partial \bar{y}}(\bar{y},\bar{t})d\bar{y}-$$

$$-\varepsilon\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\big[\tilde{h}(\bar{x},\bar{t})\frac{1}{\frac{\partial\phi}{\partial\bar{x}}(\bar{x},\bar{t})}\frac{\partial\tilde{\varphi}}{\partial\bar{x}}(\bar{x},\bar{t})-\tilde{h}(\bar{y},\bar{t})\frac{1}{\frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})}\frac{\partial\tilde{\varphi}}{\partial\bar{y}}(\bar{y},\bar{t})\big]\tilde{\eta}(\bar{x},\bar{y},\bar{t})\frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})d\bar{y}-$$

$$-\omega\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{1}{\frac{\partial\phi}{\partial\bar{x}}(\bar{x},\bar{t})}\frac{\partial\tilde{g}}{\partial\bar{x}}(\bar{x},\bar{t})-\frac{1}{\frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})}\frac{\partial\tilde{g}}{\partial\bar{y}}(\bar{y},\bar{t})}{[(\bar{x}-\bar{y})^2+(\tilde{g}(\bar{x},\bar{t})-\tilde{g}(\bar{y},\bar{t}))^2]^{\frac{1}{2}}}\frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})\tilde{\chi}(\bar{x},\bar{y},\bar{t})d\bar{y}-$$

$$-\omega\frac{1}{\tilde{\varphi}(\bar{x},\bar{t})}\int_{\mathbb{R}/\mathbb{Z}}\Big[\frac{\frac{\partial\tilde{g}}{\partial\bar{x}}(\bar{x},\bar{t})}{\frac{\partial\phi}{\partial\bar{x}}(\bar{x},\bar{t})}-\frac{\frac{\partial\tilde{g}}{\partial\bar{y}}(\bar{y},\bar{t})}{\frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})}\Big]\tilde{\eta}(\bar{x},\bar{y},\bar{t})\frac{\partial\phi}{\partial\bar{y}}(\bar{y},\bar{t})d\bar{y}=\frac{k(\phi(\bar{x},\bar{t}),\bar{t})}{\tilde{\varphi}(\bar{x},\bar{t})} \tag{33}$$

The above equation can be rewritten in the following for

$$\frac{\partial\tilde{h}}{\partial\bar{t}}(\bar{x},\bar{t})-\varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial\tilde{h}}{\partial\bar{x}}-\frac{\partial\tilde{h}}{\partial\bar{y}}}{|\bar{x}-\bar{y}|}\theta(\bar{x}-\bar{y})d\bar{y}+\varepsilon T_1(\bar{x},\bar{t})\tilde{h}(\bar{x},\bar{t})+\varepsilon T_2(\bar{x},\bar{t})\frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t})+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\tilde{h}(\bar{y},\bar{t})T_3(\bar{x},\bar{y},\bar{t})d\bar{y}+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\tilde{h}(\bar{y},\bar{t})sgn(\bar{x}-\bar{y})T_4(\bar{x},\bar{y},\bar{t})d\bar{y}+$$

$$+\omega T_5(x,\bar{t})+T_6(\bar{x},\bar{t})\tilde{k}(\bar{x},\bar{t})=0 \tag{34}$$

where T_k smooth functions, that when considered as operators acting on g are tame. We will denote the largest of all the degrees by d_0. That means we have the estimates [9]

$$\|T_k(x,\bar{t})\|_n \stackrel{def}{=} \sup_{a+b\leq n}\|\partial_x^a\partial_t^b T_k(x,\bar{t})\|_{L_x^\infty L_{\bar{t}}^\infty}\lesssim 1+\|g\|_{n+d_0}$$

In the next sections we will also need estimates for the L^2 norms of T_k. Notice that we have

$$\|\partial_x^a\partial_t^b T_k\|_{L_x^2 L_{\bar{t}}^\infty}\lesssim\|\partial_x^a\partial_t^b T_k\|_{L_{x,\bar{t}}^\infty}\lesssim 1+\sup_{\alpha+\beta\leq n+d_0}\|\partial_x^\alpha\partial_{\bar{t}}^\beta g(x,\bar{t})\|_{L_x^\infty L_{\bar{t}}^\infty}$$

[9] We leave the proof to the interested reader. The ideas are the same as in the proof of P being tame.

5.2 Solution to the linearized equation

We make several observations about the most singular term in the above equation (34). First, we will prove that the operator is translation invariant and so is given by a multiplier.

We denote by

$$\Omega\tilde{h}(x,t) = a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y},\bar{t}) - \frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y}$$

and

$$\tau_\delta f(\bar{x}) = f(\bar{x}-\delta)$$

We want to prove that

$$(\tau_\delta\Omega\tilde{h})(\bar{x},\bar{t}) = (\Omega\tau_\delta\tilde{h})(\bar{x},\bar{t})$$

Now

$$(\tau_\delta\Omega\tilde{h})(\bar{x},\bar{t}) = a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y},\bar{t}) - \frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x}-\delta,\bar{t})}{|\bar{y}-\bar{x}+\delta|}\theta(\bar{y}-\bar{x}+\delta)d\bar{y} =$$

$$= a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y}-\delta,\bar{t}) - \frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x}-\delta,\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y}$$

and

$$(\Omega\tau_\delta\tilde{h})(\bar{x},\bar{t}) = a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y}-\delta,\bar{t}) - \frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x}-\delta,\bar{t})}{|y-x|}\theta(\bar{y}-\bar{x})d\bar{y} =$$

$$= a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\tilde{h}}{\partial\bar{y}}(\bar{y}-\delta,\bar{t}) - \frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x}-\delta,\bar{t})}{|y-x|}\theta(\bar{y}-\bar{x})d\bar{y}$$

As a consequence we have

$$\widehat{\Omega\tilde{h}}(k) = \tilde{m}(k)\hat{\tilde{h}}(k)$$

Notice that the same is true for the operator P given by

$$P\tilde{h}(\bar{x},\bar{t}) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\tilde{h}(\bar{y},\bar{t}) - \tilde{h}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y}$$

We will prove that this operator P is symmetric, and since

$$T\tilde{h}(\bar{x},\bar{t}) = \left[P(\frac{\partial}{\partial\bar{x}})(\tilde{h})\right](\bar{x},\bar{t})$$

we obtain that T is skew-symmetric

Now

$$\int_{\mathbb{R}/\mathbb{Z}} P(\tilde{h})(\bar{x},\bar{t})g(\bar{x},\bar{t})d\bar{x} = \int_{\mathbb{R}/\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}} \frac{\theta(\bar{y}-\bar{x})}{|\bar{y}-\bar{x}|}[\tilde{h}(\bar{y},\bar{t})-\tilde{h}(\bar{x},\bar{t})]g(\bar{y},\bar{t})d\bar{y}d\bar{x} =$$

$$= -\int_{\mathbb{R}/\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}} \frac{\theta(\bar{y}-\bar{x})}{|\bar{y}-\bar{x}|}[\tilde{h}(\bar{y},\bar{t})-\tilde{h}(\bar{x},\bar{t})]g(\bar{x},\bar{t})d\bar{y}d\bar{x} =$$

$$= \frac{1}{2}\int_{\mathbb{R}/\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}} \frac{\theta(\bar{y}-\bar{x})}{|\bar{y}-\bar{x}|}[\tilde{h}(\bar{y},\bar{t})-\tilde{h}(\bar{x},\bar{t})][g(\bar{y},\bar{t})-g(\bar{x},\bar{t})]d\bar{y}d\bar{x}$$

where in the second line we have interchanged the dummy variables x and y and used the fact that θ is an even function. Since we have proven that T is skew-symmetric and we know that T is given by a multiplier $\tilde{m}$, we obtain that the multiplier is purely imaginary. So we have

$$\widehat{T\tilde{h}}(k) = i \cdot m(k)\widehat{\tilde{h}}(k)$$

where $m(k)$ is real.

We want to prove local existence of a solution for the equation (34), using energy methods. That is, we intend to prove the existence of a solution for a regularized evolution equation in certain Banach space and obtain energy estimates that allow us to pass to the limit.

We choose a smooth function ζ , even and compactly supported. We define the operator S_R by

$$S_R f(x) = \frac{1}{R}\int_{\mathbb{R}/\mathbb{Z}} \zeta\Big(\frac{x-y}{R}\Big)f(y)dy$$

where $0 < R < 1$.

In particular we want to solve

$$\frac{\partial h_R}{\partial \bar{t}} = \varepsilon S_R(a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial h_R}{\partial \bar{y}}(\bar{y},\bar{t}) - \frac{\partial h_R}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y}) - \varepsilon S_R(T_1(\bar{x},\bar{t})h_R(\bar{x},\bar{t})) -$$

$$-\varepsilon S_R(T_2(\bar{x},\bar{t})\frac{\partial h_R}{\partial \bar{x}}(\bar{x},\bar{t})) - \varepsilon S_R(\int_{\mathbb{R}/\mathbb{Z}} h_R(\bar{y},\bar{t})T_3(\bar{x},\bar{y},\bar{t})d\bar{y}) -$$

$$-\varepsilon S_R(\int_{\mathbb{R}/\mathbb{Z}} h_R(\bar{y},\bar{t})sgn(\bar{y}-\bar{x})T_4(\bar{x},\bar{y},\bar{t})d\bar{y}) - \omega S_R(T_5(\bar{x},\bar{t})) - S_R(T_6(\bar{x},\bar{t})\tilde{k}(\bar{x},\bar{t})) \tag{35}$$

with the initial condition $h_R(x,0) = 0$, in some Banach Space, and obtain energy estimates independent of R so that we can pass to the limit when R tends to 0.

First we want to obtain the estimate

$$\|h_R\|_{L^2_{\bar{x}}} \leq c$$

for all $\bar{t}$ in $[0,1]$ with c independent of R.

We multiply (35) by h and integrate with respect to $\bar{x}$. We have

$$\frac{1}{2}\frac{d}{d\bar{t}}\int_{\mathbb{R}/\mathbb{Z}}|h_R|^2 d\bar{x} = \varepsilon\int_{\mathbb{R}/\mathbb{Z}} S_R(a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial h_R}{\partial \bar{y}}(\bar{y},\bar{t}) - \frac{\partial h_R}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y})h_R(\bar{x},\bar{t})d\bar{x} -$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}} S_R(T_1(\bar{x},\bar{t})h_R(\bar{x},\bar{t}))h_R(\bar{x},\bar{t})d\bar{x} - \varepsilon\int_{\mathbb{R}/\mathbb{Z}} S_R(T_2(\bar{x},\bar{t})\frac{\partial h_R}{\partial \bar{x}}(\bar{x},\bar{t}))h_R(\bar{x},\bar{t})d\bar{x} -$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}} S_R(\int_{\mathbb{R}/\mathbb{Z}} h_R(\bar{y},\bar{t})T_3(\bar{x},\bar{y},\bar{t})d\bar{y})h_R(\bar{x},\bar{t})d\bar{x} -$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}} S_R(\int_{\mathbb{R}/\mathbb{Z}} h_R(\bar{y},\bar{t})sgn(\bar{y}-\bar{x})T_4(\bar{x},\bar{y},\bar{t})d\bar{y})h_R(\bar{x},\bar{t})d\bar{x} -$$

$$-\omega\int_{\mathbb{R}/\mathbb{Z}} S_R(T_5(\bar{x},\bar{t}))h_R(\bar{x},\bar{t})d\bar{x} - \int_{\mathbb{R}/\mathbb{Z}} S_R(T_6(\bar{x},\bar{t})\tilde{k}(\bar{x},\bar{t}))h_R(\bar{x},\bar{t})d\bar{x}$$

The ideas involve in proving this estimate are very standard. We just outline the main ideas needed.

Since $\Omega(\cdot)$ is skew-symmetric we have

$$\varepsilon\int_{\mathbb{R}/\mathbb{Z}} S_R(a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial h_R}{\partial \bar{y}}(\bar{y},\bar{t}) - \frac{\partial h_R}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y})h_R(\bar{x},\bar{t})d\bar{x} = 0$$

We also need to prove that the following commutator

$$[T_2(\bar{x},\bar{t}), S_R]\frac{\partial}{\partial \bar{x}}$$

is bounded in L^2 uniformly in R. Now

$$[T_2(\bar{x},\bar{t}), S_R]\frac{\partial f}{\partial \bar{x}} = T_2(\bar{x},\bar{t})S_R(\frac{\partial f}{\partial \bar{x}}) - S_R(T_2(\bar{x},\bar{t})\frac{\partial f}{\partial \bar{x}}) =$$

$$= \int_{\mathbb{R}/\mathbb{Z}} T_2(\bar{x},\bar{t})\frac{1}{R}\zeta(\frac{\bar{x}-\bar{y}}{R})\frac{\partial f}{\partial \bar{y}}d\bar{y} - \int_{\mathbb{R}/\mathbb{Z}}\frac{1}{R}\zeta(\frac{\bar{x}-\bar{y}}{R})T_2(\bar{y},\bar{t})\frac{\partial f}{\partial \bar{y}}d\bar{y} =$$

$$= \int_{\mathbb{R}/\mathbb{Z}} T_2(\bar{x},\bar{t})\frac{1}{R^2}\zeta'(\frac{\bar{x}-\bar{y}}{R})f(\bar{y},\bar{t})d\bar{y} - \int_{\mathbb{R}/\mathbb{Z}}\frac{1}{R^2}\zeta'(\frac{\bar{x}-\bar{y}}{R})T_2(\bar{y},\bar{t})f(\bar{y},\bar{t})d\bar{y} +$$

$$+\int_{\mathbb{R}/\mathbb{Z}}\frac{1}{R}\zeta(\frac{\bar{x}-\bar{y}}{R})T_2'(\bar{y},\bar{t})f(\bar{y},\bar{t})d\bar{y} =$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \frac{T_2(\bar{x},\bar{t}) - T_2(\bar{y},\bar{t})}{\bar{x}-\bar{y}} \frac{\bar{x}-\bar{y}}{R} \frac{1}{R}\zeta'(\frac{\bar{x}-\bar{y}}{R}) f(\bar{y},\bar{t}) d\bar{y} + \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{R}\zeta(\frac{\bar{x}-y}{R}) T_2'(\bar{y},\bar{t}) f(\bar{y},\bar{t}) d\bar{y}$$

From this equality we obtain

$$\|[T_2(\bar{x},\bar{t}), S_R]\frac{\partial f}{\partial \bar{x}}\|_{L^2_{\bar{x}}} \lesssim \|T_2'\|_{L^\infty_{\bar{x}}} \|f\|_{L^2_{\bar{x}}}$$

where we assume that ζ satisfies

$$\int_{\mathbb{R}/\mathbb{Z}} \zeta^2(\bar{x}) d\bar{x} < \infty \qquad \int_{\mathbb{R}/\mathbb{Z}} (\zeta')^2(\bar{x}) d\bar{x} < \infty$$

The same ideas allow us to obtain the general estimate

$$\int_{\mathbb{R}/\mathbb{Z}} \Big|\frac{\partial^k h_R}{\partial^k \bar{x}}(\bar{x},\bar{t})\Big|^2 d\bar{x} \leq c$$

with c independent of R.

We still need to prove that the system

$$\frac{\partial h_R}{\partial \bar{t}} = \varepsilon S_R(a^{-1}(\bar{t}) \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial h_R}{\partial \bar{y}} - \frac{\partial h_R}{\partial \bar{x}}}{|\bar{y}-\bar{x}|} \theta(\bar{y}-\bar{x}) d\bar{y}) - \varepsilon S_R(T_1(\bar{x},\bar{t}) h_R(\bar{x},\bar{t})) -$$

$$-\varepsilon S_R(T_2(\bar{x},\bar{t}) \frac{\partial h_R}{\partial \bar{x}}(\bar{x},\bar{t})) - \varepsilon S_R(\int_{\mathbb{R}/\mathbb{Z}} h_R(\bar{y},\bar{t}) T_3(\bar{x},\bar{y},\bar{t}) d\bar{y}) -$$

$$-\varepsilon S_R(\int_{\mathbb{R}/\mathbb{Z}} h_R(\bar{y},\bar{t}) sgn(\bar{y}-\bar{x}) T_4(\bar{x},\bar{y},\bar{t}) d\bar{y}) + \omega S_R(T_5(\bar{x},\bar{t})) + S_R(T_6(\bar{x},\bar{t})\tilde{k}(\bar{x},\bar{t}))$$

$$h_R(\bar{x},0) = S_R(h_0(\bar{x}))$$

has a solution in some Banach space.

We will use Picard's Theorem:

Let $O \subset \mathbf{B}$ be an open subset of a Banach space $\mathbf{B}$ and let $F : O \to \mathbf{B}$ be a mapping that satisfies the following:

1. *$F(X)$ maps O to $\mathbf{B}$*
2. *F is locally Lipshitz continuous, i.e. for any $X \epsilon O$ there exists $L > 0$ and an open neighborhood $U_X \subset O$ of X such that*

$$\|F(\hat{X}) - F(\hat{Y})\|_{\mathbf{B}} \leq L\|\hat{X} - \hat{Y}\|_{\mathbf{B}} \text{ for all } \hat{X}, \hat{Y} \epsilon U_X.$$

Then for any $X_0 \epsilon O$, there exists a time T such that that ODE

$$\frac{dX}{dt} = F(X), \qquad X|_{t=0} = X_0 \epsilon O,$$

has a unique (local) solution $X \epsilon C^1[(-T,T); O]$.

Denote the right hand side of the above equation by $F(\bar{t}, h)$

We need to prove

$$\|F(t,h_1)-F(t,h_2)\|_B \le L\|h_1(\bar{t})-h_2(\bar{t})\|_B \tag{36}$$

where B is some Banach space. We take $B=H^k(\mathbb{R}/_{\mathbb{Z}})$ for some large k.

It is clear that to prove that F is Lipshitz we have to prove that each of the terms in the right hand side is Lipshitz. We have

$$\|\varepsilon S_R(a^{-1}(\bar{t})\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\frac{\partial h_{1,R}}{\partial \bar{y}}(\bar{y},\bar{t})-\frac{\partial h_{1,R}}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y})-$$

$$-\varepsilon S_R(a^{-1}(\bar{t})\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\frac{\partial h_{2,R}}{\partial \bar{y}}(\bar{y}-\bar{t})-\frac{\partial h_{2,R}}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y})\|_{H^k_x} \le$$

$$\le C\|\frac{1}{a(\bar{t})}\|_{L^\infty}\times\|S_R(h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t}))\|_{H^{k+2+\delta}} \le \qquad \text{for any } \delta>0$$

$$\le \frac{1}{R^{2+\delta}}\|h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t})\|_{H^k}$$

2nd term

$$\|\varepsilon S_R(T_1(\bar{x},\bar{t})h_{1,R}(\bar{x},\bar{t}))-\varepsilon S_R(T_1(\bar{x},\bar{t})h_{2,R}(\bar{x},\bar{t}))\|_{H^k} \le$$

$$\le C(T_1)\|h_{1,R}(\bar{x},\bar{t}))-h_{2,R}(\bar{x},\bar{t}))\|_{H^k}$$

3rd term

$$\|\varepsilon S_R(T_2(\bar{x},\bar{t})\frac{\partial h_{1,R}}{\partial \bar{x}}(\bar{x},\bar{t}))-\varepsilon S_R(T_2(\bar{x},\bar{t})\frac{\partial h_{2,R}}{\partial \bar{x}}(\bar{x},\bar{t}))\|_{H^k} \le$$

$$\le C(T_2)\|S_R(\frac{\partial h_{1,R}}{\partial \bar{x}}(\bar{x},\bar{t})-\frac{\partial h_{2,R}}{\partial \bar{x}}(\bar{x},\bar{t}))\|_{H^k} \le C(T_2)\|S_R(h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t}))\|_{H^{k+1}}$$

$$\le C(T_2)\frac{1}{R}\|S_R(h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t}))\|_{H^k} \le C(T_2)\frac{1}{R}\|h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t})\|_{H^k}$$

4th term

$$\|\varepsilon S_R(\int_{\mathbb{R}/_{\mathbb{Z}}}h_{1,R}(\bar{y},\bar{t})T_3(\bar{x},\bar{y},\bar{t})d\bar{y})-\varepsilon S_R(\int_{\mathbb{R}/_{\mathbb{Z}}}h_{2,R}(\bar{y},\bar{t})T_3(\bar{x},\bar{y},\bar{t})d\bar{y})\|_{H^k} \le$$

$$\le C(T_3)\|h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t})\|_{H^k}$$

5th term

$$\|\varepsilon S_R(\int_{\mathbb{R}/\mathbb{Z}} h_{1,R}(\bar{y},\bar{t})sgn(\bar{y}-\bar{x})T_4(\bar{x},\bar{y},\bar{t})d\bar{y})-$$

$$-\varepsilon S_R(\int_{\mathbb{R}/\mathbb{Z}} h_{2,R}(\bar{y},\bar{t})sgn(\bar{y}-\bar{x})T_4(\bar{x},\bar{y},\bar{t})d\bar{y})\|_{H^k} \leq$$

$$\leq C(T_4)\|h_{1,R}(\bar{x},\bar{t})-h_{2,R}(\bar{x},\bar{t})\|_{H^k}$$

Observe that the terms involving k cancel each other out.

In order to apply Picard's Theorem we still need to define the neighborhood O and check the remaining hypothesis about F. Since we have taken $B = H^k(\mathbb{R}/_{\mathbb{Z}})$ and F considered as an operator in h loses $1+\delta$ derivatives, we take O to be $\{h\epsilon H^{k+2}|\ \|h\|_{H^{k+2}} \leq C\}$ where C is the constant appearing in the energy estimates obtained above.

That way we can assure that $F : O \to B$ is satisfied. Since we have proven above that F is Lipshitz for h in O, we can conclude that the mollified equation has a solution for $t\epsilon[0,T(R)]$.

Notice that we can make T independent of R using the a priori energy estimates that we have obtained above. Precisely, since we know that the solution that we have obtained for $[0,T(R)]$ still remains in O, we can iterate Picards theorem until $T(R)$ reaches 1. Recall that we have re-scaled our original equation using the parameter ε and so we are only considering $t\epsilon[0,1]$. In addition we can not (a priori) extend the solution past that point since we have obtained the energy estimates only for that time.

In order to obtain a solution to the linearized equation (34), we observe that since we have the bounds

$$\|h_R\|_{H^k} \leq C$$

we can use the Banach-Alaoglu Theorem to obtain the existence of a function h and a sequence of functions h_{R_j} such that

$$h_{R_j} \rightharpoonup h \qquad \text{in} \quad H^k$$

and hence using Rellich's Theorem we can conclude that

$$h_{R_j} \to h \qquad \text{in} \quad H^s \quad \text{for} \quad s<k$$

Observation 3. *The problem about the uniqueness of the limiting function h and the possible existence of other limits coming from the Banach-Alaoglu theorem gets ruled out, once we prove that h satisfies the linearized equation and that the solution to the linearized equation is unique.*

Observe that since $h_R \to h$ in H^s for $s<k$ and $h_R \to h$ in $C([0,T],H^s)$ we also obtain that $h_R \to h$ in $C([0,T],C^{s'})$ using Sobolev embeddings. (Notice that the equation provides us with derivatives in time for the solution).

Recall that the final form of the linearized equation is

$$\frac{\partial \tilde{h}}{\partial t}(\bar{x},\bar{t})-\varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}}\frac{\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t})-\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{y}-\bar{x}|}\theta(\bar{y}-\bar{x})d\bar{y}+\varepsilon T_1(\bar{x},\bar{t})\tilde{h}(\bar{x},\bar{t})+\varepsilon T_2(\bar{x},\bar{t})\frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})+$$

$$+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\tilde{h}(\bar{y},\bar{t})T_3(\bar{x},\bar{y},\bar{t})d\bar{y}+\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\tilde{h}(\bar{y},\bar{t})sgn(\bar{y}-\bar{x})T_4(\bar{x},\bar{y},\bar{t})d\bar{y}+$$

$$+\omega T_5(\bar{x},\bar{t})+T_6(\bar{x},\bar{t})\tilde{k}(\bar{x},\bar{t})=0 \tag{37}$$

Uniqueness follows as an application of Gronwall's inequality to the difference of two solutions h_1-h_2.

Observation 4. *Notice that the time of existence of the solution to the mollified equation is independent of k and hence we obtain the existence of a C^∞ solution.*

6 DP_{φ_0}'s inverse is smooth and tame

We still need to prove that the inverse of the map DP_{φ_0} is smooth and tame. Recall that

$$DP_{\varphi_0}(f,\varepsilon)(h,\omega)=(k,\sigma) \qquad \forall\ (k,\sigma)\ \text{ in }\ G$$

and hence the inverse VP_{φ_0} [10] is given by

$$VP((f,\varepsilon))(k,\sigma)=(h,\omega)$$

In this notation the tame estimate we need to prove is

$$|||(h,\omega)|||_n \lesssim 1+|||(f,\varepsilon)|||_{n+s}+|||(k,\sigma)|||_{n+s} \tag{38}$$

Recall that in the previous chapter we have proved that $\sigma=\omega$, and so the in terms of h the above inequality is equivalent to

$$\|h\|_n \lesssim 1+|\varepsilon|+|\omega|+\|f\|_{n+s}+\|k\|_{n+s} \tag{39}$$

In order to prove this estimate we are going to use the equation obtained in chapter 5, for $\tilde{h}$ and $\tilde{k}$, (34)

We will prove the tame estimate in terms of $\tilde{h}$

$$\|\tilde{h}\|_n \lesssim 1+|\varepsilon|+|\omega|+\|g\|_{n+s}+\|k\|_{n+s} \tag{40}$$

and use that

$$\tilde{h}(\bar{x},\bar{t})=\bar{h}(\phi(\bar{x},\bar{t}),\bar{t})=\frac{h(\phi(\bar{x},\bar{t}),\bar{t})}{\varphi(\phi(\bar{x},\bar{t}),\bar{t})}$$

to conclude the estimate, once we proved that the change of coordinates ϕ and the auxiliary function φ are tame maps with respect to g.

[10] We follow Hamilton's notation for the inverse.

6.1 Change of coordinates $\mathbf{x} = \phi(\bar{\mathbf{x}}, \bar{\mathbf{t}})$

We prove that the change of coordinates $x = \phi(\bar{x}, \bar{t})$ and its inverse $\bar{x} = \psi(x, t)$ are both tame.

Recall that

$$\phi'(\bar{y}, \bar{t})[1 + (g'(\bar{y}, \bar{t}))^2]^{\frac{1}{2}} = a(\bar{t})$$

and so

$$\bar{y} = \frac{1}{a(\bar{t})} \int_0^y [1 + (g'(\xi, t))^2]^{\frac{1}{2}} d\xi := \psi(y, t)$$

where

$$a(t) = \int_0^1 [1 + (g'(\xi, t))^2]^{\frac{1}{2}} d\xi$$

First notice that since the function $w(r) = [1 + r^2]^{\frac{1}{2}}$ has all its derivatives bounded independently of r we have the following tame estimate

$$\|\partial_t^k a(t)\|_{L_t^\infty} \lesssim 1 + \|\partial_t^{k+2} g\|_{L_t^\infty}$$

We still need to prove that ψ is tame. Since we know that $1 \leq a(t) \leq \sqrt{1 + K^2}$. we can use the lemmas in the appendix to obtain that $\dfrac{1}{a(t)}$ is tame. We will denote the largest of the degrees of tamenes of a and a^{-1} by d_1.

As for the second term $\displaystyle\int_0^y [1 + (g'(\xi, t))^2]^{\frac{1}{2}} d\xi$, using the same remark about w, we have

$$\|\partial_y^a \partial_t^b \int_0^y [1 + (g'(\xi, t))^2]^{\frac{1}{2}} d\xi\|_{L_x^\infty L_t^\infty} \leq \|\partial_x^{a+1} \partial_t^{b+2} g\|_{L_x^\infty L_t^\infty}$$

Using the lemmas in the appendix we have that ψ is tame, precisely

$$\|\partial_x^a \partial_t^b \psi(x, t)\|_{L_x^\infty L_t^\infty} \leq 1 + \|\partial_x^{a+1} \partial_t^{b+2} g\|_{L_x^\infty L_t^\infty}$$

In order to prove that the inverse map (ϕ) is tame we use the identity

$$\psi(\phi(\bar{x}, \bar{t}), \bar{t}) = Id(\bar{x})$$

We have

$$\partial_x \psi(\phi(\bar{x}, \bar{t}), \bar{t}) \partial_{\bar{x}} \phi(\bar{x}, \bar{t}) = 1$$

and so

$$\partial_x \psi(\phi(\bar{x}, \bar{t}), \bar{t}) \partial_{\bar{x}}^2 \phi(\bar{x}, \bar{t}) + \partial_x^2 \psi(\phi(\bar{x}, \bar{t}), \bar{t}) (\partial_{\bar{x}} \phi(\bar{x}, \bar{t}))^2 = 0$$

$$\vdots$$

$$\partial_x\psi(\phi(\bar{x},\bar{t}),\bar{t})(\partial_{\bar{x}}^k\phi(\bar{x},\bar{t}))+\sum_{j=2}^{k}\sum_{\substack{a_1+\ldots+a_j=k\\ a_i>0}} c_{k,a_1,\ldots,a_j}\partial_x^j\psi(\phi(\bar{x},\bar{t}),\bar{t})\partial_{\bar{x}}^{a_1}\phi(\bar{x},\bar{t})\ldots\partial_{\bar{x}}^{a_s}\phi(\bar{x},\bar{t})=0$$

so we have

$$\partial_{\bar{x}}^k\phi(\bar{x},\bar{t})=\frac{-1}{\partial_x\psi(\phi(\bar{x},\bar{t}),\bar{t})}\Big\{\sum_{j=2}^{k}\sum_{\substack{a_1+\ldots+a_j=k\\ a_i\geq 1}} c_{k,a_1,\ldots,a_j}\partial_x^j\psi(\phi(\bar{x},\bar{t}),\bar{t})\partial_{\bar{x}}^{a_1}\phi(\bar{x},\bar{t})\ldots\partial_{\bar{x}}^{a_s}\phi(\bar{x},\bar{t})\Big\}$$

we will use the following interpolation inequalities

$$\|\psi\|_{C^j}\lesssim\|\psi\|_{C^1}^{\frac{k-j}{k-1}}\|\psi\|_{C^k}^{\frac{j-1}{k-1}}$$

$$\|\phi\|_{C^a}\lesssim\|\phi\|_{C^1}^{\frac{k-a-1}{k-2}}\|\phi\|_{C^{k-1}}^{\frac{a-1}{k-2}}$$

Since $\frac{1}{\sqrt{1+K^2}}\leq\partial_x\psi(x,t)\leq\sqrt{1+K^2}$ using the interpolation inequalities we obtain

$$\|\partial_{\bar{x}}^k\phi(\bar{x},\bar{t})\|_{L_{\bar{x}}^\infty L_{\bar{t}}^\infty}\leq C\Big\{\sum_{j=2}^{k}\|\psi\|_{C^k}^{\frac{j-1}{k-1}}\|\phi\|_{C^{k-1}}^{\frac{k-j}{k-2}}\Big\}$$

Suppose that we know by the induction hypothesis that

$$\|\phi\|_{C^{k-1}}\lesssim\|\psi\|_{C^{k-1}}+1$$

By interpolation we have

$$\|\psi\|_{C^{k-1}}+1\lesssim(\|\psi\|_{C^k}+1)^{\frac{k-2}{k-1}}$$

when $\|\psi\|_{C^1}\lesssim 1$.

Inserting this in the above inequality produces

$$\|\partial_{\bar{x}}^k\phi(\bar{x},\bar{t})\|_{L_{\bar{x}}^\infty L_{\bar{t}}^\infty}\lesssim\|\partial^k\psi\|_{L_{\bar{x}}^\infty L_{\bar{t}}^\infty}$$

and since the right hand side is tame with respect to g we have just proved the same property for ϕ. The above argument can be trivially extended to time derivatives, and so we have proved that ϕ is tame.

We will denote the degree of tameness of ϕ by d_2.

6.2 Auxiliary function φ

Recall the definition of φ

$$\varphi(y,t)=exp\Big(\int_0^y\frac{[g''(\xi,t)][g'(\xi,t)]}{[1+(g'(\xi,t))^2]}d\xi\Big)$$

The first observation is that since we have defined the neighborhood of the origin we have $\|g\|_2 \leq C$ and hence we have

$$\frac{1}{e^{C^2}} \leq \|exp\Big(\int_0^y \frac{[g''(\xi,t)][g'(\xi,t)]}{[1+(g'(\xi,t))^2]}d\xi\Big)\| \leq e^{C^2}$$

and so

$$\frac{1}{e^{C^2}} \leq |\varphi| \leq e^{C^2}$$

Notice that for a function of the form

$$h = e^f$$

we have

$$h^{(k)} = h \sum_{\substack{a_1+\ldots+a_s=k \\ a_j \geq 1}} c_{a_1,\ldots,a_s} f^{(a_1)} \times \ldots \times f^{(a_s)}$$

In our case f is

$$f = \int_0^y \frac{[g''(\xi,t)][g'(\xi,t)]}{[1+(g'(\xi,t))^2]}d\xi$$

As we have noticed before

$$\frac{d^k}{dr^k}\frac{1}{1+r^2} \leq c_k$$

independent of r. Since f is the product of 3 tame functions, we have that f itself is tame. Together with the interpolation inequalities in the appendix, that completes the proof of the fact that φ is tame. We will denote the degree by d_3.

In order to simplify the presentation we notice here that since $\tilde{k}(\bar{x},\bar{t}) = k(\phi(\bar{x},\bar{t}),\bar{t})$ and we have proved that ϕ is tame with respect to g we obtain the tame estimate (we denote the degree by d_4.

$$\|\tilde{k}\|_n \lesssim 1 + \|k\|_{n+d_4} + \|g\|_{n+d_4}$$

In the next section we will use the tame estimates that we have proved for T_i, a, ϕ, φ and $\tilde{k}$. We will assume that all of them have the same degree that we will denote by d, i.e. $d = max(d_0, d_1, d_2, d_3, d_4)$.

6.3 Tame estimates for the inverse: modifying the definition of energy

Now we try to obtain tame estimates for $\tilde{h}$. Recall

$$\frac{\partial\tilde{h}}{\partial\bar{t}}(\bar{x},\bar{t}) - \varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\tilde{h}}{\partial\bar{y}} - \frac{\partial\tilde{h}}{\partial\bar{x}}}{|\bar{x}-\bar{y}|}\theta(\bar{x}-\bar{y})d\bar{y} + \varepsilon T_1(\bar{x},\bar{t})\tilde{h}(\bar{x},\bar{t}) + \varepsilon T_2(\bar{x},\bar{t})\frac{\partial\tilde{h}}{\partial\bar{x}}(\bar{x},\bar{t}) +$$

$$+\varepsilon \int_{\mathbb{R}/_{\mathbb{Z}}} \tilde{h}(\bar{y},\bar{t}) T_3(\bar{x},\bar{y},\bar{t}) d\bar{y} + \varepsilon \int_{\mathbb{R}/_{\mathbb{Z}}} \tilde{h}(\bar{y},\bar{t}) sgn(\bar{x}-\bar{y}) T_4(\bar{x},\bar{y},\bar{t}) d\bar{y} +$$

$$+\omega T_5(\bar{x},\bar{t}) + T_6(\bar{x},\bar{t})\tilde{k}(\bar{x},\bar{t}) = 0 \tag{41}$$

We will obtain L^2 estimates for $\tilde{h}$ and its derivatives and then use Sobolev embedding theorems to obtain the L^∞ results.

First we define the energy

$$E_{a,b}(\bar{t}) = \frac{1}{2} \int_{\mathbb{R}/_{\mathbb{Z}}} \left(\frac{\partial^{a+b} \tilde{h}}{\partial \bar{x}^a \partial \bar{t}^b} (\bar{x},\bar{t}) \right)^2 d\bar{x}$$

We need to prove the estimates

$$E_{a,b}(\bar{t}) \lesssim 1 + \|k\|^2_{a+b+s} + \|g\|^2_{a+b+s} \tag{42}$$

for some s independent of a and b.

Due to technical problems controlling some of the terms involved in $E_{a,b}$ we need to introduce the following auxiliary quantity.

$$E_{a,b,l}(\bar{t}) = \frac{1}{2} \int_{\mathbb{R}/_{\mathbb{Z}}} \left(\frac{\partial^{a+b}}{\partial \bar{x}^a \partial \bar{t}^b} Q^l(\tilde{h})(\bar{x},\bar{t}) \right)^2 d\bar{x}$$

where Q is an operator given by multiplication by a function that behaves like $log|k|$ for large k in Fourier space. We take

$$Q(f)(x) = \int_{\mathbb{R}/_{\mathbb{Z}}} \frac{f(u)-f(x)}{|u-x|} \varrho(u-x) du$$

where ϱ is a smooth even function, supported in $[-\frac{1}{4}, \frac{1}{4}]$. Moreover ϱ is identically 1 on a fixed neighborhood around the origin. We need to include this cut-off function, since otherwise $\dfrac{1}{|x-u|}$ is not well defined for u, x in $\mathbb{R}/_{\mathbb{Z}}$.

We want to obtain the estimate

$$E_{a,b,l}(\bar{t}) \lesssim C(l)(1 + \|k\|^2_{a+b+s} + \|g\|^2_{a+b+s}) \tag{43}$$

We will obtain this estimate in 3 steps.

1. Obtain $E_{0,0,l}(\bar{t}) \lesssim C(l)(1 + \|g\|^2_s + \|k\|^2_s)$.
2. Obtain $E_{a,0,l}(\bar{t}) \lesssim C(l)(1 + \|g\|^2_{a+s} + \|k\|^2_{a+s})$ using induction on a.
3. Incorporate time derivatives and run induction on b. We will prove the estimate

   $$E_{a,b,l}(\bar{t}) \lesssim 1 + \|g\|^2_{a+b+s} + \|k\|^2_{a+b+s} + E_{a+1,b-1,l+1}(\bar{t})$$

   and use the induction hypothesis

   $$E_{\alpha,\beta}(\bar{t}) \leq C(l)(1 + \|g\|^2_{\alpha+\beta+s} + \|k\|^2_{\alpha+\beta+s})$$

   for any α and $\beta < b$ to conclude the argument.

In order to obtain 1 and 2 we prove the tame estimates for $E_{0,0,0}(\bar{t})$ and $E_{a,0,0}(\bar{t})$ using induction (notice that we take $l=0$) and use the fact that

$$E_{a,0,l}(\bar{t}) \leq C(l) E_{a+1,0,0}(\bar{t})$$

to obtain the desired estimates.

In order to obtain the estimate for $E_{0,0,0}(\bar{t})$ we multiply the equation (41) by $\tilde{h}$ and integrate with respect to $\bar{x}$. We have

$$E'_{0,0,0}(\bar{t}) = \varepsilon \frac{1}{a(\bar{t})} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t}) - \frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{x}-\bar{y}|} \theta(\bar{x}-\bar{y}) d\bar{y} \tilde{h}(\bar{x},\bar{t}) d\bar{x} -$$

$$-\varepsilon \int_{\mathbb{R}/\mathbb{Z}} T_1(\bar{x},t) \tilde{h}^2(\bar{x},\bar{t}) d\bar{x} - \varepsilon \int_{\mathbb{R}/\mathbb{Z}} T_2(\bar{x},\bar{t}) \frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t}) \tilde{h}(\bar{x},\bar{t}) d\bar{x} -$$

$$-\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \tilde{h}(\bar{y},\bar{t}) T_3(\bar{x},\bar{y},\bar{t}) d\bar{y} \tilde{h}(\bar{x},\bar{t}) d\bar{x} -$$

$$-\varepsilon \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \tilde{h}(\bar{y},\bar{t}) sgn(\bar{x}-\bar{y}) T_4(\bar{x},\bar{y},\bar{t}) d\bar{y} \tilde{h}(\bar{x},\bar{t}) d\bar{x} -$$

$$-\int_{\mathbb{R}/\mathbb{Z}} T_5(\bar{x},\bar{t}) \tilde{h}(\bar{x},\bar{t}) d\bar{x} - \int_{\mathbb{R}/\mathbb{Z}} T_6(\bar{x},\bar{t}) \tilde{k}(\bar{x},\bar{t}) \tilde{h}(\bar{x},\bar{t}) d\bar{x}$$

Using the skew-symmetry of the most singular term and integrating by parts in the term containing T_2 we obtain

$$E'_{0,0,0}(\bar{t}) \lesssim (\|T_1\|_{L^\infty} + \|\frac{\partial T_2}{\partial \bar{x}}\|_{L^\infty} + \|T_3\|_{L^\infty} + \|T_4\|_{L^\infty}) E_{0,0,0}(\bar{t}) +$$

$$+\|T_5\|^2_{L^\infty} + \|T_6\|^2_{L^\infty} \|\tilde{k}\|^2_{L^2} + E_{0,0,0}(\bar{t}) \lesssim$$

$$\lesssim 1 + \|g\|^2_d + \|g\|_{d+1} E_{0,0,0}(\bar{t}) + \|k\|^2_d$$

and hence using Gronwall's inequality we obtain

$$E_{0,0,0}(\bar{t}) \lesssim e^{\|g\|_{d+1}} [E_{0,0,0}(0) + \int \|g\|^2_d + \|k\|^2_0] \lesssim 1 + \|g\|^2_d + \|k\|^2_0$$

since we are proving this estimates in a neighborhood of the origin and we know that the exponent of e is smaller than C. This proves the estimate (1) for any $s > d$, say $s = d+1$ and $M = d+1$. We will still need to increase M in the next estimates.

Now we try to obtain the estimate for $E_{a,0,0}(\bar{t})$. We differentiate the equation (41) a times with respect to $\bar{x}$ and then multiply the equation by $\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})$ and integrate with respect to $\bar{x}$. We obtain

$$\frac{1}{2}E'_{a,0,0}(\bar{t}) = \varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^a}{\partial \bar{x}^a}\left\{\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \tilde{h}}{\partial \bar{y}}(\bar{y},\bar{t}) - \frac{\partial \tilde{h}}{\partial \bar{x}}(\bar{x},\bar{t})}{|\bar{x}-\bar{y}|}\theta(\bar{x}-\bar{y})d\bar{y}\right\}\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{x}-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\sum_{\alpha+\beta=a} c_{\alpha,\beta}\frac{\partial^\alpha T_1}{\partial \bar{x}^\alpha}(\bar{x},\bar{t})\frac{\partial^\beta \tilde{h}}{\partial \bar{x}^\beta}(\bar{x},\bar{t})\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{x}-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\sum_{\alpha+\beta=a} c_{\alpha,\beta}\frac{\partial^\alpha T_2}{\partial \bar{x}^\alpha}(\bar{x},\bar{t})\frac{\partial^{\beta+1} \tilde{h}}{\partial \bar{x}^{\beta+1}}(\bar{x},\bar{t})\frac{\partial^a h}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{x}-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}} \tilde{h}(\bar{y},\bar{t})\frac{\partial^a T_3}{\partial \bar{x}^a}(\bar{x},\bar{y},\bar{t})\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{y}d\bar{x}-$$

$$-\varepsilon\int_{\mathbb{R}/\mathbb{Z}}\frac{\partial^a}{\partial \bar{x}^a}\Big(\int_{\mathbb{R}/\mathbb{Z}} \tilde{h}(\bar{y},\bar{t})sgn(\bar{x}-\bar{y})T_4(\bar{x},\bar{y},\bar{t})d\bar{y}\Big)\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{x}-$$

$$-\omega\int_{\mathbb{R}/\mathbb{Z}}\frac{\partial^a T_5}{\partial \bar{x}^a}(\bar{x},\bar{t})\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{x}-$$

$$-\int_{\mathbb{R}/\mathbb{Z}}\sum_{\alpha+\beta=a} c_{\alpha,\beta}\frac{\partial^\alpha T_6}{\partial \bar{x}^\alpha}(\bar{x},\bar{t})\frac{\partial^\beta \tilde{k}}{\partial \bar{x}^\beta}(\bar{x},\bar{t})\frac{\partial^a \tilde{h}}{\partial \bar{x}^a}(\bar{x},\bar{t})d\bar{x}$$

Obtaining the required estimate is a simple exercise using the skew-symmetry of the main term, and the interpolation inequalities described in the appendix combined with Gronwall's inequality.

We have

$$E_{a,0,0}(\bar{t}) \lesssim 1 + \|g\|^2_{a+s} + \|k\|^2_{a+s}$$

and hence

$$E_{a,0,l}(\bar{t}) \lesssim 1 + \|g\|^2_{a+s+1} + \|k\|^2_{a+s+1}$$

In order to obtain the estimate 3 we need to commute the operator Q past the different operators in the equation.

The following formula for Q^l can be proved by induction.

$$\frac{\partial Q^l(\tilde{h})}{\partial \bar{t}}(\bar{x},\bar{t}) = \varepsilon a^{-1}(\bar{t})\int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial Q^l(\tilde{h})}{\partial \bar{x}}(\bar{x},\bar{t}) - \frac{\partial Q^l(\tilde{h})}{\partial \bar{y}}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\theta(\bar{x}-\bar{y})d\bar{y}-$$

$$-\varepsilon T_1(\bar{x},\bar{t})Q^l(\tilde{h})(\bar{x},\bar{t})-\varepsilon\sum_{m+n=l-1}\int_{\mathbb{R}/\mathbb{Z}} Q^m(\tilde{h})(\bar{y},\bar{t})Q^n\Big(\frac{T_1(\bar{x},\bar{t})-T_1(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\varrho(\bar{x}-\bar{y})\Big)d\bar{y}-$$

$$-\varepsilon T_2(\bar{x},\bar{t})\frac{\partial Q^l(\tilde{h})}{\partial \bar{x}}(\bar{x},\bar{t})-\varepsilon\sum_{m+n=l-1}\int_{\mathbb{R}/\mathbb{Z}} Q^m\Big(\frac{\partial \tilde{h}}{\partial \bar{y}}\Big)(\bar{y},\bar{t})Q^n\Big(\frac{T_2(\bar{x},\bar{t})-T_2(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\varrho(\bar{x}-\bar{y})\Big)d\bar{y}-$$

$$-\varepsilon\int_{\mathbb{R}/_{\mathbb{Z}}}\tilde{h}(\bar{y},\bar{t})Q^l(T_3)(\bar{x},\bar{y},\bar{t})d\bar{y}-\varepsilon Q^l(\int_{\mathbb{R}/_{\mathbb{Z}}}\tilde{h}(\bar{y},\bar{t})sgn(\bar{x}-\bar{y})T_4(\bar{x},\bar{y},\bar{t})d\bar{y})-\omega Q^l(T_5(\bar{x},\bar{t}))-$$

$$-\int_{\mathbb{R}/_{\mathbb{Z}}}T_6(\bar{y},\bar{t})Q^{l-1}(\frac{\tilde{k}(\bar{x},\bar{t})-\tilde{k}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\varrho(\bar{x}-\bar{y}))d\bar{y}-$$

$$-\int_{\mathbb{R}/_{\mathbb{Z}}}\tilde{k}(\bar{y},\bar{t})Q^{l-1}(\frac{T_6(\bar{x},\bar{t})-T_6(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\varrho(\bar{x}-\bar{y}))d\bar{y} \qquad (44)$$

In order to prove the estimate for $E_{a,b,l}(\bar{t})$ we differentiate the equation (44) a-times with respect to $\bar{x}$ and b-times with respect to $\bar{t}$, and multiply the resulting equation by

$$\frac{\partial^{a+b}}{\partial\bar{x}^a\partial\bar{t}^b}Q^l(\tilde{h})(\bar{x},\bar{t})$$

and integrate with respect to $\bar{x}$. We have

$$\frac{1}{2}E'_{a,b,l}(\bar{t})=\varepsilon\sum_{i+j=b}c_{i,j}\frac{\partial^i}{\partial\bar{t}^i}(a^{-1}(\bar{t}))\times$$

$$\times\int_{\mathbb{R}/_{\mathbb{Z}}}\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\frac{\partial^{a+1+j}Q^l(\tilde{h})}{\partial\bar{x}^{a+1}\partial\bar{t}^j}(\bar{x},\bar{t})-\frac{\partial^{a+1+j}Q^l(\tilde{h})}{\partial\bar{y}^{a+1}\partial\bar{t}^j}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\theta(\bar{x}-\bar{y})d\bar{y}\frac{\partial^{a+b}Q^l(\tilde{h})}{\partial\bar{x}^a\partial\bar{t}^b}(\bar{x},\bar{t})d\bar{x}-$$

$$-\sum_{i+j=b}\sum_{\alpha+\beta=a}c_{i,j,\alpha,\beta}\,\varepsilon\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\partial^{\alpha+i}T_1}{\partial\bar{x}^\alpha\partial\bar{t}^i}(\bar{x},\bar{t})\frac{\partial^{\beta+j}Q^l(\tilde{h})}{\partial\bar{x}^\beta\partial\bar{t}^j}(\bar{x},\bar{t})\frac{\partial^{a+b}Q^l(\tilde{h})}{\partial\bar{x}^a\partial\bar{t}^b}(\bar{x},\bar{t})d\bar{x}-$$

$$-\varepsilon\sum_{i+j=b}c_{i,j}\sum_{m+n=l-1}\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\partial^a}{\partial\bar{x}^a}\Big\{\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\partial^iQ^m(\tilde{h})}{\partial\bar{t}^i}(\bar{y},\bar{t})\frac{\partial^j}{\partial\bar{t}^j}Q^n(\frac{T_1(\bar{x},\bar{t})-T_1(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\times$$

$$\times\varrho(\bar{x}-\bar{y}))d\bar{y}\Big\}\frac{\partial^{a+b}Q^l(\tilde{h})}{\partial\bar{x}^a\partial\bar{t}^b}(\bar{x},\bar{t})d\bar{x}-$$

$$-\sum_{\alpha+\beta=a}\sum_{i+j=b}c_{i,j,\alpha,\beta}\,\varepsilon\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\partial^{\alpha+i}T_2}{\partial\bar{x}^\alpha\partial\bar{t}^i}(\bar{x},\bar{t})\frac{\partial^{\beta+1+j}Q^l(\tilde{h})}{\partial\bar{x}^{\beta+1}\partial\bar{t}^j}(\bar{x},\bar{t})\frac{\partial^{a+b}Q^l(\tilde{h})}{\partial\bar{x}^a\partial\bar{t}^b}(\bar{x},\bar{t})d\bar{x}-$$

$$-\sum_{i+j=b}c_{i,j}\sum_{m+n=l-1}\varepsilon\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\partial^a}{\partial\bar{x}^a}\Big\{\int_{\mathbb{R}/_{\mathbb{Z}}}\frac{\partial^iQ^m(\frac{\partial\tilde{h}}{\partial\bar{y}})}{\partial\bar{t}^i}(\bar{y},\bar{t})\frac{\partial^j}{\partial\bar{t}^j}Q^n(\frac{T_2(\bar{x},\bar{t})-T_2(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|}\times$$

$$\times\varrho(\bar{x}-\bar{y}))d\bar{y}\Big\}\frac{\partial^{a+b}Q^l(\tilde{h})}{\partial\bar{x}^a\partial\bar{t}^b}(\bar{x},\bar{t})d\bar{x}-$$

$$-\varepsilon \sum_{i+j=b} c_{i,j} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^i \tilde{h}}{\partial \bar{t}^i}(\bar{y},\bar{t}) \frac{\partial^{a+j}}{\partial \bar{x}^a \bar{t}^j} Q^l(T_3)(\bar{x},\bar{y},\bar{t}) d\bar{y} \frac{\partial^{a+b} Q^l(\tilde{h})}{\partial \bar{x}^a \partial \bar{t}^b}(\bar{x},\bar{t}) d\bar{x} -$$

$$-\varepsilon \sum_{i+j=b} c_{i,j} \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^a}{\partial \bar{x}^a} \Big\{ Q^l \big(\int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^i \tilde{h}}{\partial \bar{t}^i}(\bar{y},\bar{t}) sgn(\bar{x}-\bar{y}) \frac{\partial^j}{\partial \bar{t}^j} T_4(\bar{x},\bar{y},\bar{t}) d\bar{y} \big) \Big\} \frac{\partial^{a+b} Q^l(\tilde{h})}{\partial \bar{x}^a \partial \bar{t}^b}(\bar{x},\bar{t}) d\bar{x} -$$

$$-\omega \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^{a+b}}{\partial \bar{x}^a \bar{t}^b} Q^l(T_5(\bar{x},\bar{t})) \frac{\partial^{a+b} Q^l(\tilde{h})}{\partial \bar{x}^a \partial \bar{t}^b}(\bar{x},\bar{t}) d\bar{x} -$$

$$- \sum_{i+j=b} c_{i,j} \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^a}{\partial \bar{x}^a} \Big\{ \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^i T_6}{\partial \bar{t}^i}(\bar{y},\bar{t}) \frac{\partial^j}{\partial \bar{t}^j} Q^{l-1} \big(\frac{\tilde{k}(\bar{x},\bar{t}) - \tilde{k}(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|} \varrho(\bar{x}-\bar{y}) \big) d\bar{y} \Big\} \times$$

$$\times \frac{\partial^{a+b} Q^l(\tilde{h})}{\partial \bar{x}^a \partial \bar{t}^b}(\bar{x},\bar{t}) d\bar{x} -$$

$$- \sum_{i+j=b} c_{i,j} \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^a}{\partial \bar{x}^a} \Big\{ \int_{\mathbb{R}/\mathbb{Z}} \frac{\partial^i \tilde{k}}{\partial \bar{t}^i}(\bar{y},\bar{t}) \frac{\partial^j}{\partial \bar{t}^j} Q^{l-1} \big(\frac{T_6(\bar{x},\bar{t}) - T_6(\bar{y},\bar{t})}{|\bar{x}-\bar{y}|} \varrho(\bar{x}-\bar{y}) \big) d\bar{y} \Big\} \times$$

$$\times \frac{\partial^{a+b} Q^l(\tilde{h})}{\partial \bar{x}^a \partial \bar{t}^b}(\bar{x},\bar{t}) d\bar{x}$$

As before, a careful analysis of each terms, using integration by parts, interpolation inequalities and the skew-symmetry we obtained the desired result. We leave the details to the interested reader.

To conclude the proof of the tame estimate we use Sobolev embeddings. Recall that we are interested in the L^∞ norms, not the L^2.

We have

$$\|\tilde{h}\|_{a+b} \lesssim \sup_{i+j=a+b} \|\partial_{\bar{t}}^j \tilde{h}\|_{H^{i+1}_{\bar{x}} L^\infty_{\bar{t}}} = \sup_{i+j=a+b} E_{i+1,j,0}(\bar{t}) \lesssim 1 + \|g\|^2_{a+b+s+1} + \|k\|^2_{a+b+s+1}$$

and so we obtain the desired estimate.

In order to conclude the argument we still need to prove that the inverse is smooth. Recall that the inverse map is $h = VP(f)k$ where $VP : U \times G \longrightarrow F$.

In order to prove the smoothness we will use the following

Theorem 5. *Let $L : (U \subseteq F) \times H \longrightarrow K$ be a family of invertible linear maps of Frechet spaces and let $V : (U \subseteq F) \times K \longrightarrow H$ be the family of inverses. If L is smooth and V is continuous then V is smooth and we also have*

$$DV(f)k, g = -V(f)DL(f)V(f)k, g$$

A proof can be found in [Ha].

We will apply the above theorem taking DP as the operator L. We have already proved in chapter 4 that such operator is smooth and tame. We have just proved in this chapter that the inverse VP is tame. In order to complete the proof we just need to prove that VP is continuous. Recall that we have produced the following transformations to h

$$h(x,\hat{t}) = \varphi(x,\hat{t})\bar{h}(x,\hat{t})$$

and

$$\bar{h}(\phi(\bar{x},\bar{t}),\bar{t}) = \tilde{h}(\bar{x},\bar{t})$$

Recall that both φ and ϕ have been proved to be smooth when considered as operators acting on g

Finally, we look at the equation for $\tilde{h}$, (41). The dependence of $\tilde{h}$ on k and g come from the terms T_k and k, which have been proven to be continuous when consider as operators acting on g.

Putting all this results together we conclude the continuity of our map. This concludes the argument.

7 Appendix A: Auxiliary lemmas

In this appendix we compile some lemmas that we have used throughout the thesis.

Lemma 1. *Let f and g be smooth functions defined on $\mathbb{R}/_{\mathbb{Z}}$. Then if $i+j=a$ we have*

$$\|\partial^i f \partial^j g\|_{L^2} \lesssim \|f\|_{H^a}\|g\|_{H^1} + \|f\|_{H^1}\|g\|_{H^a}$$

The following lemma provides a family of interpolation inequalities that we use throughout the following sections.

Lemma 2. *For functions defined in a compact domain in $\mathbf{R}^d$, and for all $l \leq m \leq n$*

$$\|\psi\|_{C^m} \lesssim \|\psi\|_{C^n}^{\frac{m-l}{n-l}} \|\psi\|_{C^l}^{\frac{n-m}{n-l}}$$

$$\|\psi\|_{H^m} \lesssim \|\psi\|_{H^n}^{\frac{m-l}{n-l}} \|\psi\|_{H^l}^{\frac{n-m}{n-l}}$$

The proofs of the above two lemmas can be found in any standard text about interpolation, see for example [Be-Lo].

We make the observation that the above lemma holds in $\mathbb{R}/_{\mathbb{Z}} \times [0,1]$ as well.

Corollary 1. *Let $\psi_i, i=1,...,a$ be a set of smooth functions on $(\mathbb{R}/_{\mathbb{Z}})^d \times [0,1]$ with $\|\psi_i\|_{H^l} \leq C$ uniformly in i. Then given $m_i, i=1,...,a$ with $l \leq m_i \leq n$ satisfying $\sum_{i=1}^a (m_i - l) = n - l$ we have*

$$\prod_{i=1}^a \|\psi_i\|_{H^{m_i}} \lesssim \sum_{i=1}^a \|\psi_i\|_{H^n}$$

Lemma 3. *Let Q be an operator given by multiplication by a function that behaves like $log|k|$ for large k in Fourier space. Then, for f, g functions on $\mathbb{R}/_{\mathbb{Z}}$ we have*

$$\|Q^n(f \cdot g)\|_{L^2} \lesssim \|g\|_{L^\infty} \|Q^n(f)\|_{L^2} + \|\partial g\|_{L^\infty} \|f\|_{L^2}$$

Lemma 4. *Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be smooth functions satisfying the following tame estimates,*

$$\|\partial_x^a \partial_t^b f\|_{L_x^\infty L_t^\infty} \lesssim 1 + \|h\|_{a+b+s_1}$$

$$\|\partial_x^a \partial_t^b g\|_{L_x^\infty L_t^\infty} \lesssim 1 + \|h\|_{a+b+s_2}$$

If also $\|h\|_{L_{x,t}^\infty} \leq C_0$ then there exists some r depending only on s_1 and s_2 so that we have

$$\|\partial_x^a \partial_t^b [f(x,t)g(x,t)]\|_{L_x^\infty L_t^\infty} \lesssim C_0(1 + \|h\|_{a+b+r})$$

Lemma 5. *Suppose $f : \mathbf{R} \to \mathbf{R}$ is a smooth function satisfying*

$$\|\partial_x^k f(x)\|_{L^\infty} \leq C_k$$

where C_k is independent of x. Now, if $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfies the tame estimate

$$\|\partial_x^a \partial_t^b g(x,t)\|_{L_x^\infty L_t^\infty} \lesssim 1 + \|h\|_{a+b+r}$$

for some positive r then the composition $f(g(x,t))$ satisfies

$$\|\partial_x^a \partial_t^b f(g(x,t))\|_{L_x^\infty L_t^\infty} \lesssim \tilde{C}_0(1 + \|h\|_{a+b+s})$$

provided $\|h\|_r \leq \tilde{C}_0$.

As a Corollary of the previous lemma we can obtain the following result.

Lemma 6. *Assume that $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfies the tame estimate*

$$\|\partial_x^a \partial_t^b g(x,t)\|_{L_x^\infty L_t^\infty} \lesssim 1 + \|h\|_{a+b+r}$$

Assume also that g is bounded below, precisely

$$0 < C \leq \|g(x,t)\|_{L_x^\infty L_t^\infty}$$

Then the functions $\frac{1}{g}$ and $\frac{1}{g^{\frac{1}{2}}}$ are tame with respect to h provided $\|h\|_r \leq \tilde{C}_0$.

Lemma 7. *Suppose $f(u,v)$ satisfies*

$$\|\partial_u^a \partial_v^b f\|_{L_x^\infty L_t^\infty} \leq C_{a,b}$$

with $C_{a,b}$ independent of u and v. Then if $g_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $g_2 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfy the estimates

$$\|\partial_x^a \partial_t^b g_1(x,t)\|_{L_x^\infty L_t^\infty} \lesssim 1 + \|h\|_{a+b+s_1}$$

$$\|\partial_x^a \partial_t^b g_2(x,t)\|_{L_x^\infty L_t^\infty} \lesssim 1 + \|h\|_{a+b+s_2}$$

we have the estimate

$$\|\partial_x^a \partial_t^b f(g_1(x,t), g_2(x,t))\|_{L_{x,t}^\infty} \lesssim (1 + \|h\|_{a+b+s_1+s_2})$$

provided $\|h\|_{s_1} \leq \tilde{C}_0$ *and* $\|h\|_{s_2} \leq \tilde{C}_0$.

Remark 1. *The above lemmas can be easily extended to the case of functions of more that two variables.*

Lemma 8. *If* $f(x,y,t)$ *is tame and satisfies* $f(x,y,t) = 0$ *when* $x = y$ *then* $\dfrac{f(x,y,t)}{x-y}$ *is also tame.*

Lemma 9. *Given a smooth function* $P(u,v,t)$ *such that* $P(u,v,t) = 0$ *for* v *in a neighborhood of* $u + \frac{1}{2}$ *we have*

$$\partial_x^a \int_x^{x+\frac{1}{2}} P(x,y,t)dy = -\sum_{j=0}^{a-1} \frac{d^{a-1-j}}{dx^{a-1-j}} \Big(\frac{\partial^j P}{\partial u^j}(x,x,t)\Big) + \int_x^{x+\frac{1}{2}} \frac{\partial^a P}{\partial x^a}(x,y,t)dy$$

References

[Al-Ge] Alinhac, S. and Gerard, P. "Opérateurs pseudo-différentials et théorème de Nash-Moser" Editions du CNRS (1991).

[Be-Lo] Bergh, J. and Löfström, J. "Interpolation spaces. An introduction". Springer-Verlag, **223** (1976).

[Be-Co] Bertozzi, A. L. and Constantin, P. "Global regularity for vortex patches". Commun. Math. Phys. **152** (1993), no.1, 19-28.

[Ch] Chemin, J. Y. "Persistance de structures géométriques dans les fluides incompressibles bidimensionnels". Ann. Ec. Norm. Supér. **26** (1993), no.4, 1-16.

[Co1] Constantin, P. "Regularity results for incompressible fluids". Proceedings to the Workshop on the Earth Climate as a Dynamical System, Argone National Laboratory, 25-26 September, 1992. (ANL/MCSTM-170)

[Co2] Constantin, P. "Geometric statistics in turbulence". SIAM Rev. **36** (1994), no.1, 73-98.

[Co-Ma-Ta1] Constantin P., Majda A. and Tabak E. "Singular front formation in a model for quasigesotrophic flow". Phys. Fluids **6** (1994), no. 1, 9-11.

[Co-Ma-Ta2] Constantin P., Majda, A. and Tabak E. "Formation of strong fronts in the $2-D$ quasigeostrophic thermal active scalar". Nonlinearity **7** (1994), n0. 6, 1495-1533.

[Co-Ni-So1] Constantin, P., Nie, Q. and Schörghofer, N. "Front formation in an active scalar equation". Phys. Rev. E (3) **60** (1999), no. 3, 2858–2863.

[Co-Ni-So2] Constantin,P. and Nie, Q and Schörghofer, N. "Nonsingular surface quasi-geostrophic flow". Phys. Lett. A **241** (1998), no. 3, 168–172.

[Cor] Córdoba, D. "Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation". Ann. of Math. (2) **148** (1998), no. 3, 1135-1152.

[Fe-St] Fefferman, C. and Stein, E. "H^p spaces of several variables" Acta Math. **129** (1972), no. 3-4, 137–193.

[Fo] Folland, G. B. "Introduction to Partial Differential Equations" Second Edition. Princeton University Press, Princeton, NJ (1995)

[G-H-P-S] Garner, S.T., Held, I.M., Pierrehumbert, R.T. and Swanson, K. "Surface quasi-geostrophic dynamics, J. Fluid Mech. **282** (1995), 1–20.

[Ha] Hamilton, R. "The inverse function theorem of Nash and Moser". Bull. Amer. Math. Soc. (N.S.) bf 7 (1982), no. 1, 65-222.

[H-P-S] Held, I.M., Pierrehumbert, R.T. and Swanson. K. "Spectra of local and nonlocal two dimensional turbulence". Chaos, Solitons and Fractals **4** (1994), 1111-1116.

[Ho] Hörmander, L. "Implicit function theorem. Lectures at Standford University". Summer quarter (1977).

[Kl] Klainerman, S. "Global existence for nonlinear wave equation". Comm. Pure Appl. Math. **33** (1980), no. 1, 43–101.

[Ma-Be] Majda A. and Bertozzi, A. "Vorticity and incompressible flow". Cambridge Texts in applied Mathematics, 27 (2002).

[Ma-Ta] Madja, A. and Tabak, E. "A two-dimensional model for quasi-geostrophic flow: comparison with the two-dimensional Euler flow". Physisa D **98** (1996), n0. 2-4, 515-522.

[Mo] Moser, J.K. and Siegel, C.L. "Lectures on celestial mechanics". Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[OhYa] Ohkitani, K. and Yamada, M. "Inviscid and inviscid-limit behavior of a surface quasigeostrophic flow". Phys. Fluids **9** (1997), no. 4, 876–882.

[Pe] Pedlosky, J. "Geophysical Fluid Dynamics". Springer-Verlag, New York (1987), 345-368 and 653-670.

[Ra] Rauch, J. "Partial Differential Equations". Graduate Text in Mathematics, v. 128, Springer-Verlag, (1991).

[St1] Stein, E. "Singular integrals and differentiability properties of functions". Princeton Mathematical Series, no. 30 Princeton University Press, Princeton, N.J. 1970

[St2] Stein, E. "Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals." Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[Ze] Zehnder, E. "Generalized implicit function theorems with applications to some small divisor problems. I." Comm. Pure Appl. Math. **28** (1975), 91–140.