FINITE-TIME SINGULARITIES OF AN AGGREGATION EQUATION IN $\mathbb{R}^n$ WITH FRACTIONAL DISSIPATION

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ABSTRACT. We consider an aggregation equation in $\mathbb{R}^n$, $n \geq 2$, with fractional dissipation, namely, $u_t + \nabla \cdot (u \nabla K * u) = -\nu(-\Delta)^{\gamma/2} u$, where $0 \leq \gamma \leq 2$ and K is a nonnegative decreasing radial kernel with a Lipschitz point at the origin, e.g. $K(x) = e^{-|x|}$. We prove that for $0 \leq \gamma < 1$ the solutions develop blow-up in finite for a general class of initial data. In contrast we prove that for $1 < \gamma \leq 2$ the equation is globally well-posed.

1. INTRODUCTION AND MAIN RESULTS

We consider the following aggregation equation in $\mathbb{R}^n$ with fractional dissipation:

$$u_t + \nabla \cdot (u \nabla K * u) = -\nu(-\Delta)^{\gamma/2} u, \tag{1.1}$$

where K is a nonnegative radial decreasing kernel with a Lipschitz point at the origin, e.g. $K(x) = e^{-|x|}$. As usual, $*$ denotes spatial convolution. Here $\nu \geq 0$ and $0 \leq \gamma < 1$ are parameters controlling the strength of the dissipative term. For any (reasonable) function f on $\mathbb{R}^n$, the fractional Laplacian $(-\Delta)^{\gamma/2}$ is defined via the Fourier transform:

$$\widehat{(-\Delta)^{\gamma/2} u}(\xi) = |\xi|^\gamma \hat{u}(\xi).$$

Aggregation equations of the form (1.1), with more general kernels (and other modifications) arise in many problems in biology, chemistry and population dynamics. In particular, these type of equations have applications in modeling the swarming phenomenon in biology. We use the term swarm here to describe the collective behavior of an aggregation of similar biological individuals cruising in the same direction. An overview of the modeling aspects of swarming can be found in [15], [32] and [36]. Some Lagrangian type models in which each individual is regarded as a discrete point are studied in [1], [11], [13], [14], [26], [30], [41], [44] and [45]. In the Eulerian setting, in which the individuals are approximated by a continuum population density field, several earlier models are constructed in [16], [26], [31], [44], [17], [32] and [35]. As it has already been pointed out by several authors (see [43] and [39]) the challenge with these continuum models has been obtaining biologically realistic swarm solutions with sharp boundaries (often referred to as clumping, see [40] and [39]), relatively constant internal population densities and long life times.

In one space dimension, some analytic studies have been conducted by Mogilner and Edelstein-Keshet [31], where they considered an integro-differential population model of the form (based on traditional population models, see [32], [35] and [18]):

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(D(f)\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(V(f)f\right) + B(f), \tag{1.2}$$

where $D(f)$ is the density-dependent diffusion coefficient, $B(f)$ is the growth-rate of the population and $V(f)$ is the advection velocity which takes the form

$$V(f) = a_e f + A_a(K_a * f) - A_r f(K_r * f),$$

with constants a_e, A_a and A_r representing density-dependent motion, attraction and repulsion respectively. Here the kernels K_a and K_r are called attraction and repulsion kernels (they belong to the so called social interaction kernels). Based on perturbation analysis and numerical studies, they identified the conditions when aggregation occurs and also the stability of travelling swarm profiles. As noted in [31], the clumping behavior does not seem to be supported in the one-dimensional model (1.2) under realistic assumptions on the social interaction kernels. We refer the reader to [31], [16], [46], [38], [19], [20], [21], [22], [23], [33] and [34] and the references therein for more extensive background and reviews on these one-dimensional models.

As a multi-dimensional generalization of the model (1.2), Topaz and Bertozzi [43] constructed a kinematic two-dimensional swarming model which takes the form

$$u_t + \nabla \cdot (u\,(G * u)) = 0, \tag{1.3}$$

where the (vector-valued) kernel G is called the social interactional kernel which is spatially decaying. By applying the Hodge decomposition theorem [29], one can write

$$G = G^{(I)} + G^{(P)} := \nabla^{\perp} N + \nabla P,$$

where N and P are scalar functions. In the language of [43], the kernel $G^{(I)}$ introduces incompressible motion which leads to pattern formation (e.g. vortex patterns), while the potential kernel $G^{(P)}$ models repulsion or attraction between biological organisms which in turn leads to either dispersion or aggregation. In a related paper, Topaz, Bertozzi and Lewis [42] modified the classical model of Kawasaki [23] and derived a model similar to [31], which takes the form

$$u_t + \nabla \cdot (uK * \nabla u - ru^2 \nabla u) = 0, \tag{1.4}$$

where the kernel K has fast decay in space. We remark that the clumping can be observed in these two-dimensional models (1.3) and (1.4) which were also found numerically in Levine, Rappel and Cohen [26]. We refer the reader to [24] and [3] and references therein for more details about aggregation models in this context. Aggregation equations have also been applied to image processing (see for example [2] and [37] for more details).

From the mathematical point of view the aggregation equations have been studied extensively (see e.g. [3], [5], [6], [7], [8], [24] and [43]). In one space dimension with C^1 initial data, Bodnar and Velázquez [6] proved global well-posedness for some classes of interaction potentials and finite-time blow-up for others. Burger and Di Francesco [7] and also Burger, Capasso and Morale [8] studied the well-posedness of the model with an additional smoothing term. In connection with the problem we study here, Laurent [24] has developed the existence theory for a general class of equations containing the nondissipative version of (1.1) (i.e. $\nu = 0$) and studied the connections between the regularity of the potential K and the global existence of the solution. More recently, Bertozzi and Laurent [3] have obtained finite-time blow-up of solutions for (1.1) without dissipation ($\nu = 0$). The goal of this paper is to extend this result to the dissipative equation for the range $0 \le \gamma < 1$. Additionally, we show that if the dissipation is sufficiently strong, i. e., $1 < \gamma \le 2$, the solutions don't develop any singularities.

Aggregation equations with a dissipation term have been considered by several authors (see [24] and references therein for more details). For example, Topaz, Bertozzi and Lewis [42] have considered the equation

$$u_t = -\nabla \cdot [u(u * \nabla G)] + \nabla \cdot (u^2 \nabla u) \tag{1.5}$$

in cell-based models for the case in which we have a long range social attraction and short range dispersal. We remark that (1.5) contains the same type of aggregation term considered here and a local, nonlinear, diffusion term.

We have chosen a diffusion term that contains different features, namely it is linear (which will translate into a milder diffusion process) and nonlocal. We believe the nonlocality should be an interesting feature for many applications. It is the interest in this features, linearity and nonlocality that leads directly into the use of the Laplacian for the dissipative term. We introduce fractional powers of the Laplacian to have a scale of strength for the dissipative terms against which we can study well-posedness. Given the natural scales of the equation (1.1) we have 3 different ranges to the parameter γ. Namely $0 \leq \gamma < 1$, $\gamma = 1$ and $1 < \gamma \leq 2$, known as the supercritical, critical and subcritical regimes. We motivate the choice of the three regimes as follows. Since the kernel ∇K scales as $\frac{x}{|x|}$ near the origin, heuristically our equation (1.1) which is not scale invariant can be approximated by the homogeneous version

$$u_t + \nabla \cdot \left(u \frac{x}{|x|} * u \right) = -\nu(-\Delta)^{\frac{\gamma}{2}} u. \tag{1.6}$$

Equation 1.6 has a scaling symmetry in the sense that if u is a solution, then for any $\lambda > 0$,

$$u_\lambda(t,x) = \lambda^{n+\gamma-1} u(\lambda^\gamma t, \lambda x)$$

is also a solution with initial data $u_\lambda(0,x) = \lambda^{n+\gamma-1} u_0(\lambda x)$. Here n is the space dimension where we are considering the problem. For positive initial data, it follows from Lemma 2.1 that the L^1_x norm of the solutions of equation (1.1) is preserved for all time. The critical threshold of γ is then determined by the relation

$$\|u_\lambda\|_{L^\infty_t L^1_x} = \|u\|_{L^\infty_t L^1_x}.$$

Solving this equations yields, $\gamma = 1$ which is then referred to as the critical case. For $\gamma > 1$, the a priori control of the L^1_x norm then allows us to prove the global wellposedness of the solution (with L^1_x initial data, see Theorem 1.3 below) and hence the name subcritical. In the supercritical case $\gamma < 1$, we prove the blow up of solutions in finite time (see Theorem 1.2 below). We refer the reader to [10],[9] and [27] where this type of dissipation has been used in the context of the surface quasi-geostrophic equation and other one dimensional models, for a more detailed explanation of the 3 regimes. A detailed study of the well-posedness issues, regularity of solutions will be contained in a forthcoming paper [28].

We state our results starting with an extension of the local existence theorem and continuation result proved by Bertozzi and Laurent [3] in the case $\nu = 0$. It is an analogy of the Beale-Kato-Majda result for 3D Euler [4]. In this case we have the following

Theorem 1.1 (Local existence and continuation [3])**.** *Let $\nu \geq 0$ and $0 \leq \gamma \leq 2$. Given initial data $u_0 \in H^s(\mathbb{R}^n)$, $n \geq 2$, for positive integer $s \geq 2$, there exists a unique solution u of* (1.1) *with life span $[0, T^*)$ such that either $T^* = +\infty$ or $\lim_{t \to T^*} \sup_{0 \leq \tau \leq t} \|u(\tau, \cdot)\|_{L^q_x} = +\infty$. The result holds for all $q \geq 2$ for $n > 2$ and $q > 2$ for $n = 2$.*

Proof. We refer the reader to [3] for the prove of the inviscid case $\nu = 0$. We sketch here the main modification needed to prove the general result. Notice that the changes needed are very similar to the ones used to prove local existence and continuation for Euler and Navier-Stokes. We refer the reader to [28] for a detailed explanation of the necessary modifications introduced by the presence of viscosity. As in the case of Euler and Navier-Stokes, the main difference appears at the level of energy estimates. The presence of the viscosity term produces a regularizing effect and consequently a gain of derivatives. More precisely we have the following energy estimates for the approximate solutions u^ϵ

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|_{H^2}^2 + \nu\|u^\epsilon\|^2_{H^{s+\frac{\gamma}{2}}} \le c_s\|u^\epsilon\|_{H^{s-1}}\|u^\epsilon\|^2_{H^s} \tag{1.7}$$

which provides control of a higher norm, $\|u\|_{H^{s+\frac{\gamma}{2}}}$, than in the inviscid case (see Proposition 1 in [3] for the inviscid energy estimate). From these estimates, Theorem 1.1 follows easily.

□

In the inviscid case $\nu = 0$, Bertozzi and Laurent [3] proved the existence of finite-time blow up for a class of compactly supported smooth initial data. It is conceivable that when there is some amount of weak diffusion term, the blow up phenomenon should still persist. Indeed we show that, in the case of supercritical dissipation $0 \le \gamma < 1$, there exist finite-time singularities of equation (1.1) for a suitable class of initial conditions (subset of $H^s, s \ge 2$). Postponing the definition of this class of initial data (denoted below by $A_{\delta,C,w}$, see (2.19) (2.20)) and the technical definition of admissible weight (see 2.6) we state our result in the supercritical case

Theorem 1.2 (Blow up for the supercritical case)**.** *Let w be an admissible weight function and let $\nu \ge 0$ and $0 \le \gamma < 1$. There exists constants $\delta = \delta(n) > 0$, $C = C(n, w, \nu, \gamma) > 0$ such that if $u_0 \in H^s \cap A_{\delta,C,w}$, $s \ge 2$, then there exists a finite time T^* and a unique local solution $u \in C([0, T^*); H^s) \cap C^1([0, T^*); H^{s-1})$ for* (1.1) *that blows at time T. Furthermore, we have, for every $q \ge 2$ ($q > 2$ for $n = 2$), $\sup_{0\le\tau\le t} \|u(\cdot, \tau)\|_{L^q} \to \infty$, as $t \uparrow T^*$.*

In contrast with the above Theorem, when the dissipation power is bigger, that is, in the subcritical regime, the solutions don't develop a singularity. More precisely, we have the following result.

Theorem 1.3 (Global wellposedness for positive initial data in the subcritical case)**.** *Let $\nu > 0$ and $1 < \gamma \le 2$. Assume the initial data $u_0 \in L^1_x(\mathbb{R}^n)$ and $u_0 \ge 0$ for a.e. x. Then there exists a unique global solution $u \in C([0, \infty), L^1_x) \cap C((0, \infty), W^{1,1}_x)$ of equation* (1.1).

2. Proof of Theorem 1.2

We will argue by contradiction. Under the the assumption that there is global existence for all initial data in $H^s, s \ge 2$ we will prove contradicting estimates for the energy of the system. As in the context of gradient flows and following Bertozzi and Laurent [3] (see also Topaz, Bertozzi and Lewis [42]), it is convenient to define the (free) energy as

$$E(t) = \int u(x, t)(K * u)(x, t)dx. \tag{2.1}$$

We will restrict our attention to positive initial data, and since the kernel K is positive, E is also positive. We recall the following lemma

Lemma 2.1 (Persistence of positivity and L^1 norm [24]). *Let $\nu \geq 0$ and $0 \leq \gamma \leq 2$. Assume $u_0 \geq 0$ for a.e. x. Let u be the solutions as described in Theorem 1.1. Then for each $t \in [0, T^*)$, the solution u is nonnegative and $\|u(t)\|_{L^1_x} = \|u_0\|_{L^1_x}$.*

By using Hölder's inequality, together with Young's inequality and Lemma 2.1, it is easy to see that the energy has an a priori bound $E(t) \leq \|u\|^2_{L^1}$. The main estimate that we will obtain is a growth estimate for the energy, more precisely we will prove

$$E'(t) > c(\|u_0\|_{L^1}) > 0, \quad \text{for } t \text{ up to some time } T. \tag{2.2}$$

We will arrive at a contradiction by showing that at time T (from (2.2)) the energy $E(T)$ exceeds the a priori bound.

In order to obtain (2.2) we notice that an elementary calculation yields (using the fact that K is radial)

$$E'(t) = 2\int_{\mathbb{R}^n} u|\nabla K * u|^2 dx - 2\nu \int_{\mathbb{R}^n} (-\Delta)^{\gamma/2} u(K * u) dx. \tag{2.3}$$

We will explicitely describe a set of initial conditions for which the first term dominates the second, that is the nonlinear term controls the difussion.

The bulk of estimate (2.2) is obtaining a lower bound for the first integral coming from the nonlinear term. Dealing with the second integral, involving the diffusion term is elementary. We have

$$\left|2\nu \int_{\mathbb{R}^n} (-\Delta)^{\gamma/2} u(K * u) dx\right| \leq 2\nu \left|\int u \|(-\Delta)^{\gamma/2} K\|_{L^\infty} \|u\|_{L^1} dx\right| \leq$$

$$\leq 2\nu \|(-\Delta)^{\gamma/2} K\|_{L^\infty} \|u_o\|_{L^1} \leq C_K \|u_o\|_{L^1} \tag{2.4}$$

where

$$2\nu \|(-\Delta)^{\gamma/2} K\|_{L^\infty} \leq 2\nu \,\|\, |\xi|^\gamma K^(\xi)\|_{L^1} =: C_K \tag{2.5}$$

Remark 2.2. We notice that C_K, given by $\|\, |\xi|^\gamma K^(\xi)\|_{L^1}$, is only finite for $0 \leq \gamma < 1$. This is precisely where the arguement for the existence of singularities breaks down for $\gamma = 1$. Notice that if we take K to be exactly $e^{-|x|}$, its Fourier transform is given by the Poisson Kernel, which up to a constant multiple equals

$$((2\pi)^{-2} + (\xi)^2)^{-\frac{n+1}{2}}$$

making the function ($\gamma = 1$)

$$|\xi|^1 K(\xi)((2\pi)^{-2} + (\xi)^2)^{-\frac{n+1}{2}}$$

not integrable in $\mathbb{R}^n$.

We return now to the estimate for the first term in (2.3). Since we are only considering potentials K that are nonnegative, decreasing, radial and with a Lipschitz point at the origin, we can rewrite the gradient of K as

$$\nabla K(x) = a\frac{x}{|x|} + S(x), \tag{2.6}$$

where $a \neq 0$ is a constant, $S \in L^\infty(\mathbb{R}^n)$ is continuous at $x = 0$ with $S(0) = 0$.

In order for the nonlinearity to generate a singularity it is clear we need $\nabla K * u$ sufficiently large. Since for positive functions the L^1 norm is preserved, the main problem is the cancellation arising in $\frac{x}{|x|} * u$ if u is essentially constant over a large ball centered at the origin. It is clear from this observation, and the work of Bertozzi and Laurent [3] on the inviscid equation that we need to consider solutions that are highly concentrated near the origin.

We will now estimate several integrals arising in the evolution of E involving $\frac{x}{|x|} * u$ and $\nabla K * u$, for functions highly concentrated around the origin. The right definition of highly concentrate is made precise in Lemma 2.5

Define $N(x) = \frac{x}{|x|}$. We have the following lemma which gives a lower bound of the contribution due to the homogeneous kernel N (a multiple of the homogeneous part of ∇K (see (2.6)).

Lemma 2.3 (Lower bound for the homogenous kernel)**.** *There exists a constant $C_1 = C_1(n) > 0$ such that for any nonnegative radial function $g \in L^1_{rad}(\mathbb{R}^n)$ we have*

$$\int g(x)|(N * g)(x)|dx \geq C_1 \|g\|^2_{L^1}.$$

Proof. It is clear that we can assume that $\|g\|_{L^1} = 1$. By the Cauchy-Schwartz inequality we have

$$\begin{aligned}
&\int g(x)|(N * g)(x)|dx \\
\geq &\int g(x)\langle (N * g)(x), \frac{x}{|x|}\rangle dx \\
= &\int\int g(x)g(y)\frac{(x-y)\cdot x}{|x-y|\cdot|x|}dxdy, \qquad (2.7)
\end{aligned}$$

By symmetrizing in the integral in x and y and using the fact that g is nonnegative, we obtain

$$\begin{aligned}
\text{RHS of (2.7)} =& 2\int\int g(x)g(y)\frac{x-y}{|x-y|}\cdot\left(\frac{x}{|x|}-\frac{y}{|y|}\right)dxdy \\
=& \int\int_{|y|\leq|x|} g(x)g(y)\frac{x-y}{|x-y|}\cdot\left(\frac{x}{|x|}-\frac{y}{|y|}\right)dxdy \\
=& \int\int_{|y|\leq|x|} g(x)g(y)\frac{(|x|+|y|)\cdot(1-\frac{x\cdot y}{|x||y|})}{|x-y|}dxdy \\
\geq& C_2\int\int_{\substack{|y|\leq|x| \\ x\cdot y\leq 0}} g(x)g(y)dxdy \\
\geq& \frac{C_2}{2}\int\int_{x\cdot y\leq 0} g(x)g(y)dxdy \qquad (2.8)
\end{aligned}$$

where C_2 is a constant depending only on n. In the last inequality we symmetrized again in the variables x, y. To bound this last integral, we now use the fact that g is a radial function. Denoting by $d\sigma$ as the surface measure on S^{n-1}, with a simple scaling argument we obtain

$$\begin{aligned}
\text{RHS of (2.8)} =& \frac{C_2}{2}\int_0^\infty\int_0^\infty g(\rho_1)g(\rho_2)\int_{\substack{|x|=\rho_1,\,|y|=\rho_2 \\ x\cdot y\leq 0}} d\sigma(x)d\sigma(y)d\rho_1 d\rho_2 \\
\geq& \frac{C_2}{2}\left(\int_0^\infty g(\rho)\rho^{n-1}d\rho\right)^2\int_{\substack{|x|=1,\,|y|=1 \\ x\cdot y\leq 0}} d\sigma(x)d\sigma(y) \\
\geq& C_1\|g\|^2_{L^1_x}, \qquad (2.9)
\end{aligned}$$

where C_1 is a positive constant depending only on n. $\square$

Remark 2.4. The proof of Lemma 2.3 is the only place in our blow-up argument where we need the radial assumption of the solution u. It is possible to remove the radial assumption although we shall not do it here.

In the next lemma we establish a similar conclusion for the whole kernel ∇K. Because of the presence of the inhomogeneous part, we need to consider functions having mass localized near the origin so that the contribution due to $S(x)$ (see (2.6)) is small and the whole integral is still bounded below by a large constant.

Lemma 2.5 (Lower bound for the kernel ∇K for mass localized functions)**.** *There exists a constant $\delta = \delta(n, K) > 0$ such that the following holds true:*
For any nonnegative radial function f on $\mathbb{R}^n$ with the property

$$\int_{|x|\geq\delta} f(x)dx \leq \delta\|f\|_{L^1}, \tag{2.10}$$

we have

$$\int_{\mathbb{R}^n} f|\nabla K * f|^2 dx \geq \frac{(aC_1)^2}{2}\|f\|_{L^1}^3,$$

where C_1 is the same constant as in Lemma 2.3 and a is defined in the decomposition (2.6).

Proof. Without loss of generality we will assume that $\|f\|_{L^1} = 1$. Recall the decomposition (2.6), since $S(x)$ is continuous at $x = 0$ with $S(0) = 0$, we know that for any $\epsilon_1 > 0$, there exists $\delta_1 = \delta_1(K, \epsilon_1)$, such that

$$|S(x)| \leq \epsilon_1, \quad \forall\, |x| \leq \delta_1.$$

On the other hand since S is assumed to be bounded, we have

$$|S(x)| \leq D_1, \quad \forall\, |x| \geq 0, \tag{2.11}$$

where D_1 is another constant depending only on K. Take $\epsilon_1 = \frac{aC_1}{100}$ and let $\delta > 0$ be sufficiently small such that

$$\delta < \min\left\{\frac{aC_1}{100D_1}, \frac{\delta_1(\epsilon_1, K)}{4}\right\}. \tag{2.12}$$

Fix this δ and assume that f satisfies the localization property (2.10). For $|x| \leq \delta$, by splitting the integral and using the fact that $\|f\|_{L^1} = 1$, we have

$$\begin{aligned}
|(S * f)(x)| &\leq \int_{|x|\leq 2\delta} |f(x-y)||S(y)|dy + \int_{|y|>2\delta} |f(x-y)||S(y)|dy \\
&\leq \epsilon_1 + D_1 \int_{|y|>\delta} |f(y)|dy \\
&\leq \epsilon_1 + \delta D_1,
\end{aligned} \tag{2.13}$$

where the last inequality follows from the localization assumption (2.10). For any $|x| \geq 0$, we have by Young's inequality and (2.11),

$$|(S * f)(x)| \leq D_1. \tag{2.14}$$

In view of our choice of ϵ_1, δ (see (2.12)) and the pointwise bounds on $(S * f)(x)$ (2.13) (2.14), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} f|(S * f)(x)|dx &\leq \int_{|x|\leq\delta} |f(x)|dx(\epsilon_1 + \delta D_1) + \int_{|x|\geq\delta} |f(x)|dx D_1 \\
&\leq \epsilon_1 + 2\delta D_1 \\
&\leq \frac{aC_1}{10}.
\end{aligned} \tag{2.15}$$

Now by the Cauchy-Schwartz inequality and Lemma 2.3, we have

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} f|\nabla K * f|^2 dx\Big)^{\frac{1}{2}} \\
=&\Big(\int_{\mathbb{R}^n} f|\nabla K * f|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} f dx\Big)^{\frac{1}{2}} \\
\geq& \int_{\mathbb{R}^n} f|\nabla K * f| dx \\
\geq& aC_1 - \int_{\mathbb{R}^n} f|S * f| dx \\
\geq& \frac{aC_1}{\sqrt{2}},
\end{aligned}$$

where the last inequality follows from the bound (2.15). The lemma is proved. □

We remark that both Lemma 2.3 and Lemma 2.5 deal with time independent estimates but require high concentration of mass near the origin. It is crucial for our proof that we show that if u_0 is concentrated near the origin, then the solution $u(\cdot, t)$ remains concentrated near the origin for at least some short time t. In the inviscid case $\nu = 0$, Bertozzi and Laurent [3] showed that if one starts with compactly supported data then it remains being compactly supported during the time of existence. The situation changes dramatically in the dissipative case $\nu > 0$. In the case we considered here, even if the initial data is compactly supported, the solution at any $t > 0$ will have nonzero support on the whole space due to the infinite speed of propagation of the fractional heat semigroup $e^{-t(-\Delta)^{\gamma/2}}$. It is for this reason that we need to prove the non-evacuation of mass for a short time. As we shall see later, the mass localization will follow from a weighted estimate for u. To this end, we need the following definition

Definition 2.6 (Admissible weight functions)**.** A function $w \in C^\infty(\mathbb{R}^n)$ is said to be an admissible weight function if w is a nonnegative radial function such that $w(0) = 0$ and $w(x) = 1$ for all $|x| \geq 1$.

An admissible weight function can be regarded as a smoothed out version of the spatial cut-off function $\chi_{\{|x|\geq 1\}}$. Let w be an admissible weight function and let $\delta > 0$ be the same constant as in Lemma 2.5. We define

$$I(t) = \int_{\mathbb{R}^n} u(t,x) w(\frac{x}{\delta}) dx.$$

Intuitively speaking, the integral $I(t)$ quantifies the mass of u outside of a small ball of size δ near the origin. The growth of $I(t)$ provides an upper bound of the mass of u away from the origin. Let $w_1(x) = w(x) - 1$. Clearly by definition $w_1 \in C_c^\infty(\mathbb{R}^n)$. By integration by parts, Young's inequality and Lemma 2.1, We compute

$$\begin{aligned}
\frac{d}{dt}I(t) &= -\int_{\mathbb{R}^n} \nabla\cdot(u\nabla K * u)w(\frac{x}{\delta})dx - \nu\int_{\mathbb{R}^n}(-\Delta)^{\gamma/2}u(x)w(\frac{x}{\delta})dx \\
&= \int_{\mathbb{R}^n} u\nabla K * u\cdot\frac{1}{\delta}(\nabla w_1)(\frac{x}{\delta})dx - \nu\int_{\mathbb{R}^n}u(x)\frac{1}{\delta^\gamma}\left((-\Delta)^{\frac{\gamma}{2}}w_1\right)(\frac{x}{\delta})dx \\
&\le \frac{1}{\delta}\|\nabla w_1\|_{L^\infty_x}\int_{\mathbb{R}^n}|u\nabla K*u|dx - \nu\|u\|_{L^1_x}\frac{1}{\delta^\gamma}\|(-\Delta)^{\frac{\gamma}{2}}w_1\|_{L^\infty_x} \\
&\le \frac{1}{\delta}\|\nabla w_1\|_{L^\infty_x}\|\nabla K\|_{L^\infty_x}\|u_0\|^2_{L^1_x} - \|u_0\|_{L^1_x}\cdot\frac{\nu}{\delta^\gamma}\||\xi|^\gamma\hat{w}_1(\xi)\|_{L^1_\xi} \\
&\le C_3\cdot(\|u_0\|^2_{L^1}+1), \qquad (2.16)
\end{aligned}$$

where $C_3 = C_3(n,\nu,\gamma,w,\delta)$ is a constant.

Now if we choose

$$T = \frac{\delta\|u_0\|_{L^1}}{2C_3\cdot(\|u_0\|^2_{L^1}+1)},$$

then we have

$$\sup_{0\le t\le T} I(t) \le I(0) + \frac{\delta}{2}\|u_0\|_{L^1}, \qquad (2.17)$$

where

$$I(0) = \int_{\mathbb{R}^n} u_0(x)w(\frac{x}{\delta})dx.$$

Since $w(x/\delta) = 1$ for $|x|\ge\delta$, (2.17) implies the bound,

$$\sup_{0\le t\le T}\int_{|x|\ge\delta} u(t,x)dx \le \int_{\mathbb{R}^n} u_0(x)w(\frac{x}{\delta})dx + \frac{\delta}{2}\|u_0\|_{L^1}.$$

Now if we choose u_0 such that

$$\int_{\mathbb{R}^n} u_0(x)w(\frac{x}{\delta})dx \le \frac{\delta}{2}\|u_0\|_{L^1},$$

Then clearly

$$\sup_{0\le t\le T}\int_{|x|\ge\delta} u(t,x)dx \le \delta\|u_0\|_{L^1}. \qquad (2.18)$$

This is the mass localization property we need.

Based on the results above we will specify the set of initial conditions for which one can easily obtain blow-up. Let $\delta>0$, $C>0$ be two constants. We define $A = A_{\delta,C,w}\subset L^1_{rad}(\mathbb{R}^n)$ to be the class of nonnegative radial functions u satisfying the following properties:

(1) The mass of u is comparable to its energy:

$$|K(0)|\|u\|^2_{L^1} < \int_{\mathbb{R}^n} u(K*u)dx + 1. \qquad (2.19)$$

(2) u is localized near the origin:

$$\int_{\mathbb{R}^n} u(x)w(\frac{x}{\delta})dx < \frac{\delta}{2}\|u\|_{L^1}. \qquad (2.20)$$

(3) The mass of u is sufficiently large: $\|u\|_{L^1} > C$.

For any $\delta>0$, $C>0$ and any admissible weight w, it is not too difficult to see that the class $A_{\delta,C,w}$ is nonempty. Indeed one can take any $f\in L^1_{rad}(\mathbb{R}^n)$ such that $\|f\|_{L^1}>C$, then define $f_\lambda(\cdot) = \lambda^{-n}f(\lambda^{-1}\cdot)$. For all sufficiently small $\lambda>0$, one can check directly that $u = f_\lambda$ satisfies (2.19) and (2.20) due to the assumption that $K(0) = \|K\|_{L^\infty}$ and $w(0)=0$.

We are now ready to complete the proof of the main theorem.

Proof of Theorem 1.2. Take δ to be the same constant as in Lemma 2.5 and choose a constant C sufficiently large such that

$$C > \max\{\frac{4C_3 + C_K}{(a\,C_1)^2}, 1\}, \tag{2.21}$$

where C_3 was defined in (2.16) and C_K is given in (2.5) in the estimate for the difussion term.

Take $u_0 \in H^s \cap A_{\delta,C,w}$ and recall that

$$E(t) = \int_{\mathbb{R}^n} u(t,x)(K * u)(t,x)dx.$$

Then obviously

$$E(t) \leq \|u_0\|_{L^1}^2 \|K\|_{L^\infty} = \|u_0\|_{L^1}^2 K(0).$$

On the other hand we have

$$\frac{d}{dt}E(t) = 2\int_{\mathbb{R}^n} u|\nabla K * u|^2 dx - 2\nu \int_{\mathbb{R}^n} (-\Delta)^{\gamma/2} u(K * u)dx.$$

Let

$$T = \frac{\delta \|u_0\|_{L^1}}{2C_3 \cdot (\|u_0\|_{L^1}^2 + 1)},$$

then by the mass localization property (2.18) and Lemma 2.5, together with the estimate (2.4) for the diffusion term we have

$$\frac{d}{dt}E(t) \geq (a\,C_1)^2 \|u_0\|_{L^1}^3 - C_K \|u_0\|_{L^1}^2.$$

By our choice of u_0 and the choice of the constant C (see (2.21)), it is not difficult to check that

$$(a\,C_1)^2 \|u_0\|_{L^1}^3 - C_K \|u_0\|_{L^1}^2 > \frac{1}{T} = \frac{2C_3 \cdot (\|u_0\|_{L^1}^2 + 1)}{\delta \|u_0\|_{L^1}}.$$

This gives us

$$E(T) \geq E(u_0) + 1.$$

But this is impossible since we have

$$E(T) \leq \|u_0\|_{L^1}^2 \|K\|_{L^\infty} = \|u_0\|_{L^1}^2 K(0) < E(u_0) + 1,$$

where the last inequality is due to the fact that $u_0 \in A_{\delta,C,w}$. The theorem is proved. □

3. GLOBAL WELL-POSEDNESS AND SMOOTHING FOR THE SUBCRITICAL CASE $1 < \gamma \leq 2$

In this section we consider the aggregation equation in the subcritical regime $1 < \gamma \leq 2$. We first prove local well-posedness in $L_x^1(\mathbb{R}^n)$. We shall do this by constructing mild solutions. This is

Theorem 3.1 (Local well-posedness in L_x^1 for the subcritical case)**.** *Let $\nu > 0$ and $1 < \gamma \leq 2$. Assume the initial data $u_0 \in L_x^1(\mathbb{R}^n)$. Then there exists a time $T = T(\|u_0\|_{L_x^1}, \nu, \gamma, \|\nabla K\|_{L_x^\infty}) > 0$ and a unique mild solution of* (1.1) *in the space $C([0,T), L_x^1(\mathbb{R}^n))$. In fact the uniqueness of mild solutions holds in a slightly stronger sense: for any $T' > 0$, there exists at most one solution in the space $C([0,T'), L_x^1(\mathbb{R}^n))$ with initial data $u_0 \in L_x^1$.*

Remark 3.2. As we shall see in the proof of Theorem 3.1, the time of existence of the constructed mild solution has an upper bound of the form

$$T < \left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} \cdot \nu^{\frac{1}{\gamma-1}} \cdot \left(\|\nabla K\|_{L_x^\infty}\|u_0\|_{L_x^1}\right)^{-\frac{\gamma}{\gamma-1}},$$

provided of course $\|u_0\|_{L_x^1} \neq 0$.

We shall prove Theorem 3.1 by the classical fixed point theorem for general Banach spaces. We state it as the following lemma.

Lemma 3.3 ([25])**.** *Let X be a Banach space endowed with norm $\|\cdot\|_X$ and let $B : X \times X \to X$ be a bilinear map such that for any x_1, $x_2 \in X$, we have*

$$\|B(x_1, x_2)\|_X \leq C\|x_1\|_X\|x_2\|_X.$$

Then for any $y \in X$ such that

$$4C\|y\|_X < 1,$$

the equation

$$x = y + B(x, x)$$

has a solution in X with $\|x\|_X \leq 2\|y\|_X$. Moreover the solution is unique in the Ball $\bar{B}(0, \frac{2}{C})$.

Proof. The proof can be found in [25]. We reproduce here for the sake of completeness and also for the comparison with the two-normed version Lemma 3.5 (see below). Define $x_0 = y$ and $x_n = y + B(x_{n-1}, x_{n-1})$. By induction it is easy to show that $\|x_n\|_X \leq 2\|y\|_X$; Moreover,

$$\begin{aligned}\|x_{n+1} - x_n\|_X &\leq \|B(x_n, x_n - x_{n-1})\|_X + \|B(x_n - x_{n-1}, x_{n-1})\|_X \\ &\leq 4C\|y\|_X\|x_n - x_{n-1}\|_X.\end{aligned}$$

Since $4C\|y\|_X < 1$, this shows that (x_n) is a Cauchy sequence and hence has a limit x. The uniqueness of x in the Ball $\bar{B}(0, \frac{2}{C})$ is obvious. □

As we shall see below, we only need the existence part of Lemma 3.3. The uniqueness of the constructed mild solution will be proved independently. We now write $S(t) = e^{-\nu(-\Delta)^{\frac{\gamma}{2}}t}$. Our equation (1.1) in the mild formulation can be written as

$$\begin{aligned}u(t) &= S(t) * u_0 - \int_0^t \nabla S(\tau) * (u\nabla K * u)(t-\tau)d\tau \\ &= S(t) * u_0 + B(u, u)(t),\end{aligned} \tag{3.1}$$

where for any two functions f, g, we define the Bilinear form $B(f, g)(t)$ as

$$B(f, g)(t) = -\int_0^t \nabla S(\tau) * (f\nabla K * g)(t-\tau)d\tau. \tag{3.2}$$

We shall consider our equation (3.1) in the Banach space $X_T = C([0, T), L_x^1)$. The following simple lemma gives the boundedness of the bilinear operator (3.2) on $X_T \times X_T$.

Lemma 3.4 (Boundedness of the bilinear operator)**.** *The bilinear operator* (3.2) *is continuous on $X_T \times X_T$, more precisely, we have*

$$\|B(f, g)\|_{X_T} \leq \frac{\gamma}{\gamma-1}\nu^{-\frac{1}{\gamma}}T^{1-\frac{1}{\gamma}}\|\nabla K\|_{L_x^\infty}\|f\|_{X_T}\|g\|_{X_T}.$$

Proof. By Minkowski's inequality, Young's equality we have

$$\begin{aligned}\|B(f,g)\|_{X_T} &\le \left\| \int_0^t (\nu\tau)^{-\frac{1}{\gamma}} \|f\nabla K * g\|_{X_T} d\tau \right\|_{L_t^\infty} \\ &\le (\nu)^{-\frac{1}{\gamma}} \int_0^T (\tau)^{-\frac{1}{\gamma}} d\tau \|f\|_{X_T} \|g\|_{X_T} \|\nabla K\|_{L_x^\infty} \\ &\le (\nu)^{-\frac{1}{\gamma}} \frac{\gamma}{\gamma-1} T^{1-\frac{1}{\gamma}} \|\nabla K\|_{L_x^\infty} \|f\|_{X_T} \|g\|_{X_T}.\end{aligned}$$

The lemma is proved. □

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We choose $T > 0$ such that

$$4 \cdot \frac{\gamma}{\gamma-1} \cdot \nu^{-\frac{1}{\gamma}} T^{1-\frac{1}{\gamma}} \|\nabla K\|_{L_x^\infty} \|u_0\|_{L_x^1} < 1.$$

Then by the inequality $\|S(t) * u_0\|_{X_T} \le \|u_0\|_{L_x^1}$, the strong continuity of the semi-group $S(t)$ in L_x^1, the boundedness of the bilinear operator Lemma 3.4 and the fixed point Lemma 3.3, we conclude that there exists a solution of the equation (3.1) in the space X_T. It only remains for us to prove the uniqueness part of Theorem 3.1. Let $T' > 0$ be arbitrary and u_1, u_2 be two solutions of (3.1) with the same initial data u_0. Denote

$$M = \max \left\{ \|u_1\|_{X_{T'}},\, \|u_2\|_{X_{T'}} \right\}.$$

Let T'' be sufficiently small such that

$$\frac{\gamma}{\gamma-1} \cdot \nu^{-\frac{1}{\gamma}} (T'')^{1-\frac{1}{\gamma}} \|\nabla K\|_{L_x^\infty} M < \frac{1}{10}.$$

Then since u_1 and u_2 has the same initial data u_0, we have by Lemma 3.4

$$\begin{aligned}\|u_1 - u_2\|_{X_{T''}} &\le \|B(u_1, u_1 - u_2)\|_{X_{T''}} + \|B(u_1 - u_2, u_2)\|_{X_{T''}} \\ &\le \frac{1}{2} \|u_1 - u_2\|_{X_{T''}}.\end{aligned}$$

This implies that $u_1 \equiv u_2$ on $[0, T'')$. A finite iteration of the argument then gives $u_1 \equiv u_2$ on the whole time interval $[0, T')$. The Theorem is proved. □

We now show that our constructed mild solution has additional regularity. This is achieved by another contraction argument in the subspace of X_T. We first formulate a two-normed version of the fixed point Lemma 3.3.

Lemma 3.5 (Two-normed fixed point lemma)**.** *Assume that Z is a Banach space endowed with the norms $\|\cdot\|_Z$, $\|\cdot\|_X$ and seminorm $\|\cdot\|_Y$ such that*

$$\|\cdot\|_Z = \max\{\|\cdot\|_X, \|\cdot\|_Y\}.$$

Let $B : Z \times Z \to Z$ be a bilinear map such that for any x_1, $x_2 \in Z$, we have

$$\|B(x_1,x_2)\|_Z \le C(\|x_1\|_Z \|x_2\|_X + \|x_1\|_X \|x_2\|_Z),$$

and

$$\|B(x_1,x_2)\|_X \le C\|x_1\|_X \|x_2\|_X.$$

Then for any $y \in Z$ such that

$$8C\|y\|_X < 1,$$

the equation $x = y + B(x,x)$ has a solution in Z with $\|x\|_Z \le 2\|y\|_Z$. Moreover by Lemma 3.3 the solution is unique in the ball $\{z : \|z\|_X \le \frac{2}{C}\}$.

Proof. Again we construct the solution x by iteration. Define $x_0 = y$ and $x_n = y + B(x_{n-1}, x_{n-1})$ for $n \geq 1$. Then since

$$\begin{aligned}\|x_n\|_Z &\leq \|y\|_Z + 2\|x_{n-1}\|_Z\|x_{n-1}\|_X \\ &\leq \|y\|_Z + 4C\|y\|_X\|x_{n-1}\|_Z,\end{aligned}$$

it is easy to prove by induction that $\|x_n\|_Z \leq 2\|y\|_Z$. To show (x_n) is Cauchy in Z we calculate

$$\begin{aligned}\|x_{n+1} - x_n\|_Z &\leq \|B(x_n, x_n - x_{n-1})\|_Z + \|B(x_n - x_{n-1}, x_{n-1})\|_Z \\ &\leq 4C\|y\|_Z\|x_n - x_{n-1}\|_X + 4C\|y\|_X\|x_n - x_{n-1}\|_Z.\end{aligned}$$

From the proof of Lemma 3.3 we know that $\|x_n - x_{n-1}\|_X \leq \theta^n$ for some constant $0 < \theta < 1$. This together with the fact that $4C\|y\|_X < 1$ and a few elementary manipulations implies that $\|x_{n+1} - x_n\|_Z \leq (\theta')^n$ for another constant $0 < \theta' < 1$. This immediately shows that x_n is Cauchy in Z and hence converges to a fixed point x. □

In what follows, it is useful to consider the $\|\cdot\|_{Y_T}$ norm of u defined by

$$\|u\|_{Y_T} := \|t^{\frac{1}{\gamma}}\nabla u\|_{L_t^\infty L_x^1([0,T)\times\mathbb{R}^n)}.$$

We first prove that the $\|\cdot\|_{Y_T}$ norm of the bilinear operator (3.2) is bounded.

Lemma 3.6 ($\|\cdot\|_{Y_T}$ norm boundedness of the bilinear operator)**.** *The bilinear operator* (3.2) *is bounded in the following sense:*

$$\|B(f,g)\|_{Y_T} \leq (\|f\|_{Y_T}\|g\|_{X_T} + \|f\|_{X_T}\|g\|_{Y_T}) \cdot \|\nabla K\|_{L_x^\infty} \cdot C_1\nu^{-\frac{1}{\gamma}} \cdot T^{\frac{\gamma-1}{\gamma}},$$

where $C_1 = C_1(\gamma)$ is a positive constant depending only on γ.

Proof. We have

$$\begin{aligned}\|B(f,g)\|_{Y_T} &= \|t^{\frac{1}{\gamma}}\nabla B(f,g)\|_{L_t^\infty L_x^1([0,T)\times\mathbb{R}^n)} \\ &\leq \nu^{-\frac{1}{\gamma}}\left\|t^{\frac{1}{\gamma}}\int_0^t (t-\tau)^{-\frac{1}{\gamma}}\|(\nabla f\cdot\nabla K * g)(\tau)\|_{L_x^1}d\tau\right\|_{L_t^\infty([0,T))} \\ &\quad + \nu^{-\frac{1}{\gamma}}\left\|t^{\frac{1}{\gamma}}\int_0^t (t-\tau)^{-\frac{1}{\gamma}}\|(f\nabla K * \nabla g)(\tau)\|_{L_x^1}d\tau\right\|_{L_t^\infty([0,T))} \\ &\leq (\|f\|_{Y_T}\|g\|_{X_T} + \|f\|_{X_T}\|g\|_{Y_T}) \cdot \|\nabla K\|_{L_x^\infty} \cdot \nu^{-\frac{1}{\gamma}}\left\|t^{\frac{1}{\gamma}}\int_0^t (t-\tau)^{-\frac{1}{\gamma}}\tau^{-\frac{1}{\gamma}}d\tau\right\|_{L_t^\infty([0,T))} \\ &\leq (\|f\|_{Y_T}\|g\|_{X_T} + \|f\|_{X_T}\|g\|_{Y_T}) \cdot \|\nabla K\|_{L_x^\infty} \cdot C_1\nu^{-\frac{1}{\gamma}}T^{\frac{\gamma-1}{\gamma}},\end{aligned}$$

where C_1 is an constant depending only on γ. The lemma is proved. □

We can now upgrade the regularity of our constructed mild solution. We define $Z_T \subset C([0,T), L_x^1)$ as a Banach space with the norm

$$\begin{aligned}\|u\|_{Z_T} &= \max\{\|u\|_{X_T}, \|u\|_{Y_T}\} \\ &= \max\{\|u\|_{L_t^\infty L_x^1([0,T)\times\mathbb{R}^n)}, \|t^{\frac{1}{\gamma}}\nabla u\|_{L_t^\infty L_x^1([0,T)\times\mathbb{R}^n)}\}.\end{aligned}$$

Theorem 3.7 (Local well-posedness in Z_T for the subcritical case)**.** *Let $\nu > 0$ and $1 < \gamma \leq 2$. Assume the initial data $u_0 \in L_x^1(\mathbb{R}^n)$. Then there exists a time $T = T(\|u_0\|_{L_x^1}, \nu, \gamma, \|\nabla K\|_{L_x^\infty}) > 0$ and a unique mild solution of* (1.1) *in the space Z_T. By Theorem 3.1 the uniqueness of the mild solutions holds in a larger space: for any $T' > 0$, there exists at most one solution in the space $C([0,T'), L_x^1(\mathbb{R}^n))$ with initial data $u_0 \in L_x^1$.*

Remark 3.8. As we will see in the proof below, the time of existence of the constructed mild solution has an upper bound of the form

$$T < C_2 \cdot \nu^{\frac{1}{\gamma-1}} \cdot \left(\|\nabla K\|_{L_x^\infty} \|u_0\|_{L_x^1}\right)^{-\frac{\gamma}{\gamma-1}},$$

where $C_2 = C_2(\gamma)$ is a positive constant depending only on γ.

Proof of Theorem 3.7. We only need to prove the existence. The uniqueness part is already in Theorem 3.1. Choose $T > 0$ such that

$$8C_1 \cdot \nu^{-\frac{1}{\gamma}} T^{1-\frac{1}{\gamma}} \|\nabla K\|_{L_x^\infty} \|u_0\|_{L_x^1} < 1,$$

where C_1 is the same constant as in Lemma 3.6. By the inequality $\|\nabla S(t) * u_0\|_{L_x^1} \leq t^{-\frac{1}{\gamma}} \|u_0\|_{L_x^1}$, the boundedness of the bilinear operator Lemma 3.6 and the two-normed fixed point Lemma 3.5, we conclude that there exists a solution of the equation (3.1) in the space Z_T. □

By a standard bootstrap argument, we can obtain the following corollary.

Corollary 3.9 (Maximal time of existence of solutions)**.** *Let $\nu > 0$ and $1 < \gamma \leq 2$. Assume the initial data $u_0 \in L_x^1(\mathbb{R}^n)$. Then there exists a maximal time of existence $T^* \in (0, \infty]$ and a unique solution $u \in C([0, T^*), L_x^1) \cap C((0, T^*), W_x^{1,1})$. Moreover if $T^* < \infty$, then necessarily $\lim_{t \to T^*} \|u(\cdot, t)\|_{L_x^1} = \infty$.*

Proof. This is a standard argument which follows from Theorem 3.7. □

By Corollary 3.9, to obtain a global solution, it suffices for us to control the $L_x^1(\mathbb{R}^n)$. Concerning positive initial data, the following result was originally proved by Laurent [24] for the inviscid case $\nu = 0$ and with different assumptions on the initial data. By using time splitting approximation, it is straightforward to obtain the same result for the dissipative case $\nu > 0$. By another approximation argument, we obtain the following

Lemma 3.10 (Persistence of positivity and L^1 norm [24])**.** *Let $\nu \geq 0$ and $1 < \gamma \leq 2$. Assume $u_0 \in L_x^1$ and $u_0 \geq 0$ for a.e. x. Then for each $t \in [0, T^*)$, the solution u is nonnegative and $\|u(t)\|_{L_x^1} = \|u_0\|_{L_x^1}$.*

We are now ready to complete

Proof of Theorem 1.3. It follows directly from Corollary 3.9 and Lemma 3.10. □

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