BLOW UP FOR THE GENERALIZED SURFACE QUASI-GEOSTROPHIC EQUATION WITH SUPERCRITICAL DISSIPATION

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Abstract. We prove the existence of singularities for the generalized surface quasi-geostrophic (GSQG) equation with supercritical dissipation. Analogous results are obtained for the family of equations interpolating between GSQG and 2D Euler.

1. Introduction and main results

In this article we consider a family of generalized active scalar equations arising from fluid mechanics. Of particular interest is the generalized surface quasi-geostrophic (GSQG) equation

$$\begin{cases} \theta_t + u \cdot \nabla\theta + \kappa\Lambda^\gamma\theta = 0 & (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ \theta(x,0) = \theta_0(x) & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\kappa \geq 0$, $\gamma \in (0,2]$ are fixed parameters. The dissipative term is given by a fractional Laplacian, Λ^γ, that is defined by means of the Fourier transform; we have

$$\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \hat{f}(\xi).$$

To complete the system, the velocity u is related to the scalar θ by:

$$u = -Q_\beta(-\Delta)^{-1/2}\nabla\theta, \tag{1.2}$$

with

$$Q_\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

Here $\beta \in [0,2\pi)$ is also a fixed parameter. When $\beta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, we recover the usual 2D surface quasi-geostrophic equation (SQG). In this sense we regard the system (1.1)-(1.2) as a generalization of the usual SQG. We remark that the usual SQG equation, $\beta = \frac{\pi}{2}, \frac{3\pi}{2}$, is an incompressible model, while for all other β it is easy to check that the divergence of u is not 0.

The other systems we will consider arise as a generalization of the family of interpolating models between SQG and 2D Euler. More precisely, the evolution of the active scalar θ is still given by equation (1.1) but now the velocity is given by

$$u = -Q_\beta(-\Delta)^{-(1-\frac{\alpha}{2})}\nabla\theta, \tag{1.3}$$

for $0<\alpha<1$. Notice that the endpoint case $\alpha=1$ corresponds to the previously described generalized surface quasi-geostrophic equation while the case $\alpha=0$ and $\beta=\frac{\pi}{2},\frac{3\pi}{2}$ produces the classical 2D Euler.

We will not review here in detail the known results for the standar surface quasi-geostrophic equation. We refer the reader to [2], [5], [6], [8], [11], [14], [15], [17], [18] and [19] for more details both from the theoretical and numerical point of view. We briefly recall some recent results for the generalized problems. For GSQG ($\alpha=1$) without dissipation ($\kappa=0$), Dong and Li (see [12]) obtained the blow up of smooth radial solutions, while Balodis and Córdoba (see [1]) proved the existence of singularities for the equation

$$\partial_t\theta+(-R\theta)\nabla\theta=0 \tag{1.4}$$

which corresponds to $\alpha=1$, $\beta=0$ or π. Here R stands for the Riesz transform. In [1], they obtained some new bilinear estimates for the Riesz transform and used them to show the existence of singularities for general (smooth) initial data (not necessarily radial, and without any restrictions on the sign of θ).

Both results ([1] and [12]) are inspired by the work of A. Córdoba, D. Córdoba and M. Fontelos (see [9]) for the following one dimensional model for the surface quasi-geostrophic equation

$$\partial_t f-(H\,f)f_x=-\kappa\Lambda^\gamma f \tag{1.5}$$

where H is the Hilbert transform. In an elegant way they obtained some new bilinear estimates for the Hilbert transform and as a result proved the ill-posedness of the equation without the dissipative term (i.e. $\kappa=0$) We refer the reader to [9] and [10] for more details. In [16] the authors were able to extend their result to include the dissipative term for $\gamma<\frac{1}{2}$.

The results we present here are the first ones to answer the question of the global well posedness for the supercritical case for any of the two dimensional models mentioned before. We will restrict our attention to radial solutions of constant sign (determined by β). This will allow us to present an elementary argument for the blow up, providing a simpler proof of the blow up result of Balodis and Córdoba.

We will concentrate on the GSQG model, presenting the required modifications for the other equations at the end of the paper.

For GSQG we will prove the following

Theorem 1.1 (GSQG)**.** *Let $\theta_0\in C^\infty(\mathbb{R})$ be a smooth, bounded, even function. Assume $\|\theta_0\|_{L^\infty}=M$. Let $0\le\gamma<1/2$ and $0<\delta<1-2\gamma$ be arbitrary but fixed. Let β in* (1.2) *be different from $\frac{\pi}{2},\frac{3\pi}{2}$, but otherwise arbitrary. Then there exists a constant $C=C(\gamma,\delta,\beta,M)>0$ such that if θ_0 satisfies*

$$\int_0^\infty\frac{\theta_0(y)-\theta_0(0)}{y^{1+\delta}}dy>C,$$

then the solution to (1.1)-(1.2) *, with initial data $\theta_0(|x|)$, blows up in finite time.*

Remark 1.2. In the course of the proof of the above Theorem we will actually prove that the blow up happens at the level of $\|\nabla\theta\|_{L^\infty}$.

Remark 1.3. For the GSQG equation, $\gamma = 1$ is the critical exponent. Our argument only provides blow up for $\gamma < \frac{1}{2}$, but we believe that this restriction is just a limitation of our approach, and conjecture that singularities exist for all the supercritical range ($0 < \gamma < 1$).

For the interpolating models we have a similar result for $\alpha \geq \frac{1}{2}$. We have

Theorem 1.4 (Interpolating Models)**.** *Let $\theta_0 \in C^\infty(\mathbb{R})$ be a smooth, bounded, even function. Assume $\|\theta_0\|_{L^\infty} = M$. Let $\alpha \geq \frac{1}{2}$, $0 \leq \gamma < \frac{\alpha}{2}$ and $0 < \delta < \alpha - 2\gamma$ be arbitrary but fixed. Let β in* (1.3) *be different from $\frac{\pi}{2}, \frac{3\pi}{2}$, but otherwise arbitrary. Then there exists a constant $C = C(\gamma, \delta, \beta, M) > 0$ such that if θ_0 satisfies*

$$\int_0^\infty \frac{\theta_0(y) - \theta_0(0)}{y^{1+\delta}} dy > C,$$

then the solution to (1.1),(1.3) *, with initial data $\theta_0(|x|)$, blows up in finite time.*

2. Reduction to a one dimensional model

Following [9] and [16] the main strategy in the proof of Theorem 1.1 is to establish the blow up of

$$\int_0^\infty \int_0^\infty \frac{\theta(x_1, x_2, t) - \theta(0,0,t)}{(x_1^2 + x_2^2)^{\frac{2+\delta}{2}}} dx_1 dx_2 \tag{2.1}$$

for some positive δ.

By restricting our attention to radial solutions we will be able to reduce the study of (2.1) to a 1 dimendional problem.

To obtain an evolution equation for (2.1) we observe that

$$\begin{aligned} \partial_t \theta(x_1, x_2, t) + u \cdot \nabla \theta(x_1, x_2, t) &= -\kappa \Lambda^\gamma \theta(x_1, x_2, t) \\ \partial_t \theta(0,0,t) + \qquad\qquad &= -\kappa \Lambda^\gamma \theta(0,0,t) \end{aligned} \tag{2.2}$$

since the velocity at the origin is 0 for radial solutions

$$u(0,0,t) = -Q_\beta \int\int \frac{\nabla \theta(y_1, y_2, t)}{|(y_1, y_2)|} dy_1 dy_2 =$$

$$= -Q_\beta \int_0^\infty \int_0^{2\pi} \partial_r \bar{\theta}(r,t)(\cos\mu, \sin\mu) d\mu dr = (0,0)$$

And so using (2.2) we obtain an equation for (2.1), namely

$$\frac{d}{dt} \int_0^\infty \int_0^\infty \frac{\theta(x_1, x_2, t) - \theta(0,0,t)}{(x_1^2 + x_2^2)^{\frac{2+\delta}{2}}} dx_1 dx_2 = \tag{2.3}$$

$$= -\int_0^\infty \int_0^\infty \frac{u \cdot \nabla \theta}{(x_1^2 + x_2^2)^{\frac{2+\delta}{2}}} dx_1 dx_2 - \int_0^\infty \int_0^\infty \frac{\Lambda^\gamma \theta(x_1, x_2, t) - \Lambda^\gamma \theta(0,0,t)}{(x_1^2 + x_2^2)^{\frac{2+\delta}{2}}} dx_1 dx_2$$

Since we are only considering radial functions we have $\theta(x_1, x_2, t) = \bar{\theta}(|x|, t)$ for some function $\bar{\theta}$ and so

$$\frac{d}{dt}\int_0^\infty\int_0^\infty \frac{\theta(x_1,x_2,t)-\theta(0,0,t)}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}}dx_1dx_2 = 2\pi\frac{d}{dt}\int_0^\infty \frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr$$

In order to handle the dissipative term we will prove the following

Lemma 2.1. Let δ and γ satisfy the conditions of Theorem 1.1. Given $f(x_1,x_2) = \bar f(|x|)$ a positive, smooth (bounded) radial function. We have

$$\int_0^\infty\int_0^\infty \frac{\Lambda^\gamma f(x_1,x_2)-\Lambda^\gamma f(0,0)}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}}dx_1dx_2 = 2\pi\int_0^\infty \frac{\bar f(r)-\bar f(0)}{r^{1+\delta+\gamma}}dr$$

$$\leq \epsilon\int_0^\infty \frac{(\bar f(r)-\bar f(0))^2}{r^{2+\delta}}dr + \frac{c}{\epsilon} + c\,\|f\|_{L^\infty} \tag{2.4}$$

for any positive ϵ.

Proof. First, using the representation of a fractional derivative as an integral, and the fact that $\Lambda^\gamma(\Lambda^\beta f) = \Lambda^{\gamma+\beta} f$ we have

$$\int_0^\infty\int_0^\infty \frac{\Lambda^\gamma f(x_1,x_2)-\Lambda^\gamma f(0,0)}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}}dx_1dx_2 = \int_0^\infty\int_0^\infty \frac{f(x_1,x_2)-f(0,0)}{(x_1^2+x_2^2)^{\frac{2+\delta+\gamma}{2}}}dx_1dx_2 =$$

$$= 2\pi\int_0^\infty \frac{\bar f(r)-\bar f(0)}{r^{1+\delta+\gamma}}dr \leq 2\pi\int_0^1 \frac{\bar f(r)-\bar f(0)}{r^{1+\delta+\gamma}}dr + c\|f\|_{L^\infty} \leq \tag{2.5}$$

$$\leq 2\pi\int_0^1 \frac{\bar f(r)-\bar f(0)}{r^{1+\frac{\delta}{2}}}\frac{1}{r^{\frac{\delta}{2}+\gamma}}dr + c\|f\|_{L^\infty} \leq$$

$$\leq \epsilon\int_0^\infty \frac{(\bar f(r)-\bar f(0))^2}{r^{2+\delta}}dr + \frac{c}{\epsilon} + c\,\|f\|_{L^\infty}$$

where we have used the fact that $\delta + 2\gamma < 1$ for the range of γ and δ we are considering. □

Remark 2.1. The need for the inequality $\delta + 2\gamma < 1$ in the above calculation (to make $r^{-\frac{\delta}{2}-\gamma}$ be in $L^2[(0,1)]$ is the only reason why we need to restrict to $\gamma < \frac{1}{2}$. This restriction on γ seems rather unnatural since the critial exponent is $\gamma = 1$ and it just seems to be a limitation of the techniques involved in the proof.

In order to analyze the nonlinear term we will stablish the following

Lemma 2.2. Given θ and u as above we have

$$\int_0^\infty\int_0^\infty \frac{u\cdot\nabla\theta}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}}dx_1dx_2 = -2\pi\int_0^\infty \frac{T(\bar\theta)(r,t)\ \partial_1\bar\theta(r,t)}{r^{1+\delta}}dr$$

where

$$Tf(x) = \cos\beta\int_0^\infty f'(r)g(\frac{|x|}{r})dr$$

and

$$g(x) = \int_0^{2\pi} \frac{\cos\mu}{\sqrt{x^2+1-2x\,\cos\mu}}\,d\mu$$

Proof. Since we are only considering the radial case it is sufficient to compute $u\cdot\nabla\theta$ at $(|x|,0,t)$. We have

$$u\cdot\nabla\theta(|x|,0,t) = -Q_\beta(-\triangle)^{-\frac{1}{2}}\nabla\theta\ (|x|,0,t)\cdot\nabla\theta(|x|,0,t) =$$

$$= -Q_\beta\int\int\frac{(\partial_1\theta(y_1,y_2,t),\partial_2\theta(y_1,y_2,t))}{\left|\begin{pmatrix}|x|\\0\end{pmatrix}-\begin{pmatrix}y_1\\y_2\end{pmatrix}\right|}dy_1dy_2\cdot\partial_1\bar\theta(|x|,t)(1,0) =$$

$$= -Q_\beta\int_0^\infty\int_0^{2\pi}\frac{\partial_1\bar\theta(r,t)(\cos\,\mu,\sin\,\mu)}{\sqrt{|x|^2+r^2-2|x|r\cos\,\mu}}r\,dr\,d\mu\cdot\partial_1\bar\theta(|x|,t)(1,0) =$$

$$= -\int_0^\infty\partial_1\bar\theta(r,t)\int_0^{2\pi}\frac{\cos\beta\,\cos\,\mu-\sin\,\beta\sin\,\mu}{\sqrt{(\frac{|x|}{r})^2+1-2\frac{|x|}{r}\cos\,\mu}}d\mu dr\ \ \partial_1\bar\theta(|x|,t) =$$

$$= -\cos\,\beta\int_0^\infty\partial_1\bar\theta(r,t)g(\frac{|x|}{r})dr\ \ \partial_1\bar\theta(|x|,t) = -T(\bar\theta)(|x|,t)\ \ \partial_1\bar\theta(|x|,t)$$

where we have used that

$$\int_0^{2\pi}\frac{\sin\,\mu}{(x^2+1-2x\cos\,\mu)^{\frac{1}{2}}}d\mu = 0$$

And so we have

$$-\int_0^\infty\int_0^\infty\frac{u\cdot\nabla\theta}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}}dx_1dx_2 = \int_0^\infty\int_0^\infty\frac{T(\bar\theta)(|x|,t)\ \ \partial_1\bar\theta(|x|,t)}{|x|^{2+\delta}}dx_1dx_2 =$$

$$= 2\pi\int_0^\infty\frac{T(\bar\theta)(r,t)\ \ \partial_1\bar\theta(r,t)}{r^{1+\delta}}dr$$

□

Remark 2.2. The presence of the factor $\cos\,\beta$ in the expression for T in the above operator means that the nonliner term is not present for the cases $\beta=\frac{\pi}{2}$ or $\frac{3\pi}{2}$, making our approach inapplicable for the standard surface quasi geostrophic equation.

Using Lemma 2.1 and 2.2 we have reduced the problem to the one-dimensional equation

(2.6)
$$\frac{d}{dt}\int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr = \int_0^\infty\frac{T(\bar\theta)(r,t)\ \ \partial_1\bar\theta(r,t)}{r^{1+\delta}}dr - \int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta+\gamma}}dr$$

To simplify the presentation we have taken $\kappa=1$. It is clear that this has no effect in the result. We remark that $T(\bar{(\theta)})$ contains the factor $\cos\beta$, whose sign depends on β. We will assume without loss of generality that $\cos\beta$ is positive, since otherwise one can consider the equation for $-\theta$. When $\cos\beta>0$ we consider positive solutions.

We want to prove that $\displaystyle\int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr$ blows up finite time. We will use the following Theorem to estimate the nonlinear term.

Theorem 2.3. *Let $f \geq 0$ be a smooth, bounded function in $[0, \infty)$. Let $\beta \neq \frac{\pi}{2}, \frac{3\pi}{2}$, but otherwise arbitrary. Then for every δ in $[0, 1)$, there exist a constant $c_{\delta,\beta}$, independent of f, such that*

$$\int_0^\infty \frac{Tf(x)\, f'(x)}{x^{1+\delta}}\, dx \geq c_{\delta,\beta} \int_0^\infty \frac{(f(x) - f(0))^2}{x^{2+\delta}}\, dx \tag{2.7}$$

where

$$Tf(x) = \cos\,\beta \int_0^\infty f'(r) g(\frac{|x|}{r}) dr$$

and

$$g(x) = \int_0^{2\pi} \frac{\cos\,\mu}{\sqrt{x^2 + 1 - 2x \cos\,\mu}}\, d\mu$$

Proof. Without loss of generality we will assume that $f(0) = 0$. Using the Parseval identity for the Mellin Transform we have

$$\int_0^\infty \frac{Tf(x)\, f'(x)}{x^{1+\delta}}\, dx = \frac{1}{2\pi} \int_{-\infty}^\infty M_1(\lambda) \overline{M_2(\lambda)} d\lambda$$

where

$$M_1(\lambda) = \int_0^\infty x^{i\lambda - \frac{1}{2} - \frac{\delta}{2}} f'(x) dx$$

$$M_2(\lambda) = \int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} Tf(x) dx$$

Integrating by parts we obtain

$$M_1(\lambda) = -(i\lambda - \frac{1}{2} - \frac{\delta}{2}) \int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f(x) dx$$

As for M_2

$$M_2(\lambda) = \int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} \cos\,\beta \int_0^\infty f'(r) g(\frac{x}{r}) dr dx = \cos\,\beta \int_0^\infty \int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f'(r) g(\frac{x}{r}) dx dr$$

$$= \cos\,\beta \int_0^\infty \int_0^\infty r^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} y^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f'(r) g(y) r dy dr =$$

$$= \cos\,\beta \int_0^\infty y^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} g(y) dy \Big[- (i\lambda - \frac{1}{2} - \frac{\delta}{2}) \int_0^\infty r^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f(r) dr \Big]$$

And so

$$\frac{1}{2\pi} \int_{-\infty}^\infty M_1(\lambda) \overline{M_2(\lambda)} d\lambda = \frac{1}{2\pi} (i\lambda - \frac{1}{2} - \frac{\delta}{2}) [\overline{(i\lambda - \frac{1}{2} - \frac{\delta}{2})}] \cos\,\beta \times$$

$$\times \int_0^\infty y^{-i\lambda - \frac{3}{2} - \frac{\delta}{2}} g(y) dy \int_{-\infty}^\infty \int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f(x) dx \overline{\int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f(x) dx}\; d\lambda =$$

$$= \frac{\cos\beta}{2\pi}(\lambda^2+a^2)\int_0^\infty y^{-i\lambda-\frac{3}{2}-\frac{\delta}{2}}g(y)dy\int_{-\infty}^\infty\int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}f(x)dx\overline{\int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}f(x)dx}\ d\lambda$$

where we have defined $a = \frac{1}{2}+\frac{\delta}{2}$.

We will prove the following result in the next section

Lemma 2.3 (Main Lemma). Let $g(x) = \displaystyle\int_0^{2\pi}\frac{\cos\mu}{(x^2+1-2x\cos\mu)^{\frac{1}{2}}}\ d\mu$ and a and δ as above. Then for every $\lambda\in\mathbb{R}$ we have

$$(\lambda^2+a^2)Re\int_0^\infty y^{-i\lambda-\frac{3}{2}-\frac{\delta}{2}}\,g(y)\,dy > c_\delta(1+|\lambda|)$$

where $c_\delta > 0$ depends only on δ. We postpone the proof until the next section.

Using this result we have

$$\int_0^\infty\frac{Tf(x)\,f'(x)}{x^{1+\delta}}dx \geq c_{\delta,\beta}\int_{-\infty}^\infty\int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}f(x)dx\overline{\int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}f(x)dx}\ d\lambda =$$

$$= c_{\delta,\beta}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{i\lambda x-\frac{1}{2}\ -\frac{\delta}{2}x}f(e^x)dx\overline{\int_{-\infty}^\infty e^{i\lambda x-\frac{1}{2}\ -\frac{\delta}{2}x}f(e^x)dx d\lambda} =$$

$$= c_{\delta,\beta}\int_{-\infty}^\infty e^{-\frac{1}{2}x-\frac{\delta}{2}x}f(e^x)e^{-\frac{1}{2}x-\frac{\delta}{2}x}f(e^x)dx = c_{\delta,\beta}\int_0^\infty\frac{f^2(y)}{y^{2+\delta}}dy$$

□

Using Lemma 2.1 and Theorem 2.3 we have the following estimates for the one dimensional equation (2.6)

$$\frac{d}{dt}\int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr \geq c_\delta\int_0^\infty\frac{(\bar\theta(r,t)-\bar\theta(0,t))^2}{r^{2+\delta}}dr - \epsilon\int_0^\infty\frac{(\bar\theta(r,t)-\bar\theta(0,t))^2}{r^{2+\delta}}dr - c\|\theta\|_{L^\infty} - \frac{c}{\epsilon}$$

and so choosing $\epsilon = \frac{c_\delta}{2}$ we have

$$\frac{d}{dt}\int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr \geq \frac{c_\delta}{2}\int_0^\infty\frac{(\bar\theta(r,t)-\bar\theta(0,t))^2}{r^{2+\delta}}dr - c\|\theta\|_{L^\infty} - \frac{c}{\epsilon}$$

$$\geq \frac{c_\delta}{2}\left[\int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr\right]^2 - c\,\|\theta_0\|_{L^\infty} - \frac{c}{\epsilon}$$

where we have used the fact that since GSQG is an advection-difusion equation we have $\|\theta\|_{L^\infty}\leq\|\theta_0\|_{L^\infty}$.

If we denote by

$$J(t) := \int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr$$

we have established

$$\frac{dJ(t)}{dt} \geq c_1 J(t)^2 - c_2(1+\|\theta_0\|_{L^\infty}), \tag{2.8}$$

It is obvious that if we choose $\theta_0 \in C^\infty(\mathbb{R}^+)$ with $\|\theta_0\|_{L^\infty} = 1$, and

$$J(0) = \int_0^\infty \frac{\theta_0(x) - \theta_0(0)}{x^{1+\delta}} dx$$

to be sufficiently large, then $J(t)$ in (2.8) will blow up in some finite $T < \infty$, i.e., $J(t) \to \infty$ as $t \uparrow T$.

To conclude the proof of the main Theorem 2.3 we notice that

$$\begin{aligned} J(t) &\leq \sup_{0<r<1} \frac{|\theta(r,t) - \theta(0,t)|}{r} \cdot \frac{1}{1-\delta} + \int_1^\infty \frac{2}{r^{1+\delta}} dr \\ &\leq \frac{\|\theta_r(\cdot,t)\|_{L^\infty}}{1-\delta} + \frac{2}{\delta} \end{aligned}$$

and so we conclude that $\|\theta_x\|_{L^\infty}$ also blows up in finite time.

3. The main lemma and its proof

In the previous section we have used the following estimate which we state here as a

Lemma 3.1. Let

$$g(x) = \int_0^{2\pi} \frac{\cos\,\mu}{\sqrt{x^2 + 1 - 2x\cos\,\mu}} d\mu$$

then for every λ

$$(\lambda^2 + (\frac{1+\delta}{2})^2) Re \int_0^\infty y^{-i\lambda - \frac{3}{2} - \frac{\delta}{2}} g(y) dy > c_\delta(1 + |\lambda|) \tag{3.1}$$

Proof. Notice that since the expression is even with respect to λ, it is enough to deal with $\lambda > 0$. Denote by $M(\lambda)$ the left hand side of (3.1) and let $a = \frac{1+\delta}{2}$. We will prove (3.1) in two stages. We will first prove that $Re \int_0^\infty y^{-i\lambda - \frac{3}{2} - \frac{\delta}{2}} g(y) dy$ is always positive, and then that it is of order $\frac{1}{\lambda}$ for large λ.

The main tool in the proof is to transform the expression for $M(\lambda)$ into the cosine transform of some function, and use a general result about positive cosine transforms to prove the lemma. We have

$$\begin{aligned} M(\lambda) &= (\lambda^2 + a^2) Re \int_{-\infty}^\infty e^{i\lambda x} e^{-ax} g(e^x) dx \\ &= (\lambda^2 + a^2) 2 \int_0^\infty \cos(\lambda y) \Big[e^{-ay} g(e^y) + e^{ay} g(e^{-y}) \Big] dy \end{aligned}$$

In order to prove the lemma we will use the following sufficient condition for a cosine transform to be positive

Theorem 3.1 (Pólya)**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a convex function ($f > 0$). Then its cosine transform is always positive. More precisely, for every λ*

$$\int_0^\infty f(x) \cos(\lambda\, x) dx > 0 \tag{3.2}$$

Using Pólya's theorem it is sufficient to prove that

$$W(y) = e^{-a\,y} g(e^y) + e^{a\,y} g(e^{-y}) \tag{3.3}$$

is convex for $y > 0$, since W is trivially strictly positive.

A simple calculation yields

$$\begin{aligned} W''(y) &= a^2 e^{-ay} g(e^{ay}) + (1-2a)e^{(1-a)y} g'(e^y) + e^{(2-a)y} g''(e^y) + \\ &+ a^2 e^{ay} g(e^{-ay}) + (1-2a)e^{(-1+a)y} g'(e^{-y}) + e^{(2-a)y} g''(e^{-y}) \end{aligned} \tag{3.4}$$

Since e^y is increasing, it is sufficient to prove that

$$\begin{aligned} &a^2 x^{-a} g(x) + (1-2a)x^{1-a} g'(x) + x^{2-a} g''(x) + \\ &+ a^2 x^a g(\frac{1}{x}) + (1-2a)x^{-1+a} g'(\frac{1}{x}) + x^{-2+a} g''(\frac{1}{x}) > 0 \end{aligned} \tag{3.5}$$

for all $x > 1$

In order to prove the above inequality we will need a deeper analysis of the function g, in particular a representation of in terms of hypergeometric functions.

Definition 3.2. We will denote by (Gauss) Hypergeometric Function ($F(a, b; c; z)$) the solution of the equation

$$z(1-z)F'' + (c - (a+b+1)z)F' - ab\,F = 0 \tag{3.6}$$

with initial condition $F(a, b; c; 0) = 1$.

The funtion F has a representation as a power series given by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{3.7}$$

where the symbol $(a)_n$ is the rising factorial (or Pochhammer symbol) given by $a(a+1)(a+2)\cdots(a+n-1)$.

Using simple identities it is elementary to obtain the following expression for g

$$g(x) = \frac{\pi x}{(1+x^2)^{\frac{3}{2}}} F(\frac{3}{4}, \frac{5}{4}; 2; \frac{4x^2}{(1+x^2)^2}) \tag{3.8}$$

The hypergeometric function involved in the expression of g, that we will denote by $F(z)$, satisfies the equation

$$z(1-z)F''(z) + (2-3z)F'(z) - \frac{15}{16} F(z) = 0 \tag{3.9}$$

where $z(x) = \dfrac{4x^2}{(1+x^2)^2}$.

We will prove (3.5) by showing that

$$a^2 x^{-a} g(x) + (1-2a)x^{1-a} g'(x) + x^{2-a} g''(x) > 0 \tag{3.10}$$

and

$$a^2x^ag(\frac{1}{x})+(1-2a)x^{-1+a}g'(\frac{1}{x})+x^{-2+a}g''(\frac{1}{x})>0 \tag{3.11}$$

for $x>1$. Due to the structure of the expressions involved it is sufficient to prove that (3.10) is positive for $x>0, x\neq 1$ (notice that g is not defined at 1). We only sketch the result, leaving the details to the interested reader.

A simple calculation yields

$$g(x)=\frac{x}{(1+x^2)^{\frac{3}{2}}}F$$

$$g'(x)=\frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}F+\frac{x}{(1+x^2)^{\frac{3}{2}}}F'\,z' \tag{3.12}$$

$$g''(x)=\frac{-9x+6x^3}{(1+x^2)^{\frac{7}{2}}}F+\Big[z'\frac{2-4x^2}{(1+x^2)^{\frac{5}{2}}}+z''\frac{x}{(1+x^2)^{\frac{3}{2}}}\Big]F'+(z')^2\frac{x}{(1+x^2)^{\frac{3}{2}}}F''$$

and so (3.10) becomes (after multiplying both sides by $(1+x^2)^{\frac{3}{2}}x^{a-1}$

$$\begin{aligned} 0\;<\;F\Big[a^2+(1-2a)\frac{1-2x^2}{1+x^2}+x\frac{-9x+6x^3}{(1+x^2)^2}\Big]+ \\ +F'\Big[(1-2a)xz'+z'x\frac{2-4x^2}{1+x^2}+z''x^2\Big]+F''(z')^2x^2 \end{aligned} \tag{3.13}$$

Using (3.9) we have

$$F=\frac{16}{15}\Big[z(1-z)F''+(2-3z)F'\Big]$$

and so (3.14) becomes

$$0\;<\;F'I+F''II \tag{3.14}$$

where

$$I:=(1-2a)xz'+z'x\frac{2-4x^2}{1+x^2}+z''x^2+\frac{16}{15}(2-3z)\Big[a^2+(1-2a)\frac{1-2x^2}{1+x^2}+x\frac{-9x+6x^3}{(1+x^2)^2}\Big]$$

$$II:=(z')^2x^2+\frac{16}{15}z(1-z)\Big[a^2+(1-2a)\frac{1-2x^2}{1+x^2}+x\frac{-9x+6x^3}{(1+x^2)^2}\Big]$$

In order to complete the proof, we make two simple observations. First $II\geq 0$, and second $(1-z)F''>F'>0$. The first one is just a simple calculation, using the fact that $\frac{1}{2}\leq a<1$, while the second can be easily verified using the power series expansion for F.

Then (3.14) becomes

$$0\;<\;F'[I(1-z)+II] \tag{3.15}$$

and a simple calculation yields $I(1-z)+II>0$, for $x>0, x\neq 0$, completing the argument.

In order to complete the argument we note that

$$Re \int_0^\infty y^{-i\lambda-\frac{3}{2}-\frac{\delta}{2}} g(y)dy = O(\frac{1}{\lambda})$$

The reason for this is that g is not smooth, and actually has a (mild) singularity at 1. The contribution outside a neighborhood of 1 is $O(\lambda^{-l})$ for any positive l (this can be proved by using a smooth cut-off function and a localization argument). We analyze the contribution of an interval around 1. It is easy to see that near 1

$$g(x) \approx -2Log(|1-x|)$$

Since we also have

$$\Big|\int_{\frac{1}{2}}^{\frac{3}{2}} \cos(\lambda\, x) 2Log(|1-x|)dx\Big| = O(\frac{1}{\lambda})$$

for large λ. We leave the details of the calculation to the reader.

□

4. Generalized interpolating models

We sketch the modification to the arguments in the previous sections in order to handle the generalized interpolation models. Recall that the main difference is the fact that the velocity is now given by

$$u = -Q_\beta(-\Delta)^{-(1-\frac{\alpha}{2})}\nabla\theta,$$

Proceeding as above before we obtain equation (2.3)

Since there have been no changes in the dissipation term, we will obtain an estimate by Lemma 2.1. For the nonlinear term we have a new version of Lemma 2.2 We have

Lemma 4.1.

$$-\int_0^\infty\int_0^\infty \frac{u\cdot\nabla\theta}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}}dx_1dx_2 = -2\pi\int_0^\infty \frac{T_\alpha(\bar\theta)(r,t)\ \partial_1\bar\theta(r,t)}{r^{1+\delta}}dr$$

where

$$T_\alpha f(x) = \cos\,\beta\int_0^\infty f'(r)g(\frac{|x|}{r})r^{1-\alpha}\,dr$$

and

$$g_\alpha(\xi) = \int_0^{2\pi}\frac{\cos\mu}{(\xi^2+1-2\xi\cos\mu)^{\frac{\alpha}{2}}}\ d\mu$$

We have reduced the problem to the one dimensional equation

(4.1)
$$\frac{d}{dt}\int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta}}dr = \int_0^\infty\frac{T_\alpha(\bar\theta)(r,t)\ \partial_1\bar\theta(r,t)}{r^{1+\delta}}dr - \int_0^\infty\frac{\bar\theta(r,t)-\bar\theta(0,t)}{r^{1+\delta+\gamma}}dr$$

For the dissipation we use the following Lemma

Lemma 4.2. Let $0 \leq \gamma < \frac{\alpha}{2}$ and $0 < \delta < \alpha - 2\gamma$. Given $f(x_1, x_2) = \bar{f}(|x|)$ a positive, smooth (bounded) radial function. We have

$$\int_0^\infty \int_0^\infty \frac{\Lambda^\gamma f(x_1,x_2) - \Lambda^\gamma f(0,0)}{(x_1^2+x_2^2)^{\frac{2+\delta}{2}}} dx_1 dx_2 = 2\pi \int_0^\infty \frac{\bar{f}(r) - \bar{f}(0)}{r^{1+\delta+\gamma}} dr$$
$$\leq \epsilon \int_0^\infty \frac{(\bar{f}(r) - \bar{f}(0))^2}{r^{1+\alpha+\delta}} dr + \frac{c}{\epsilon} + c\,\|f\|_{L^\infty} \tag{4.2}$$

for any positive ϵ.

For the nonlinear term we will use a modified version of Theorem 2.3

Theorem 4.1. *Let $f \geq 0$ be a smooth, bounded function in $[0, \infty)$. Let $\beta \neq \frac{\pi}{2}, \frac{3\pi}{2}$, but otherwise arbitrary. Then for every δ in $[0, 1)$, there exist a constant $c_{\delta,\beta}$, independent of f such that*

$$\int_0^\infty \frac{T_\alpha f(x)\, f'(x)}{x^{1+\delta}} dx \geq c_{\delta,\beta} \int_0^\infty \frac{(f(x) - f(0))^2}{x^{1+\alpha+\delta}} dx \tag{4.3}$$

where T_α and g_α are as above.

Proof. We proceed as before using Parseval's identity for Mellin Transforms. This time we define

$$M_1(\lambda) = \int_0^\infty x^{i\lambda - \frac{\alpha}{2} - \frac{\delta}{2}} f'(x) dx$$
$$M_2(\lambda) = \int_0^\infty x^{i\lambda - 2 + \frac{\alpha}{2} - \frac{\delta}{2}} T_\alpha f(x) dx \tag{4.4}$$

With a similar analysis we obtain the desired result, provided we can prove a modified version of the Main Lemma

□

Lemma 4.3 (Generalized Main Lemma). Let α, δ and g_α as above. Then

$$\Big[\lambda^2 + \Big(\frac{\alpha+\delta}{2}\Big)^2\Big] Re \int_0^\infty y^{-i\lambda - 2 + \frac{\alpha}{2} - \frac{\delta}{2}} g_\alpha(y) dy\Big] > c_{\delta,\alpha}(1 + |\lambda|)$$

for all $\lambda \in \mathbb{R}$.

Proof. The proof follows the same argument as for GSQG. We have to prove

$$a^2 x^{-a} g_\alpha(x) + (1-2a) x^{1-a} g'_\alpha(x) + x^{2-a} g''_\alpha(x) +$$
$$+ a^2 x^a g_\alpha(\frac{1}{x}) + (1-2a) x^{-1+a} g'_\alpha(\frac{1}{x}) + x^{-2+a} g''_\alpha(\frac{1}{x}) > 0 \tag{4.5}$$

where a is now $\frac{2-\alpha+\delta}{2}$ and we have the following expression for g_α in terms of hypergeometric functions

$$g_\alpha(x) = \frac{2\alpha\pi x}{(1+x^2)^{1+\frac{\alpha}{2}}} [F(\frac{1}{2} + \frac{\alpha}{4}, 1 + \frac{\alpha}{4}; 2; \frac{4x^2}{(1+x^2)^2}) \tag{4.6}$$

For the corresponding ranges of a, α and δ we still have $II > 0$ and $(1-z)F'' > \frac{1+\alpha}{2}F' > 0$. and so a similar argument to the one presented for GSQG using Polya's Thm concludes the proof.

□

Remark 4.2. We remark that the range for α can be improved to $\alpha \geq \frac{3}{10}$ by improving the estimate $(1-z)F'' > \frac{1+\alpha}{2}F' > 0$, but for α small enough, for example $\frac{1}{5}$, the analogue of the function W ceases to be convex, making the application of Polya's Theorem imposible. All other elementary criterias to verify that the cosine transform of a function is always positive also fail.

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