Refined blowup criteria and nonsymmetric blowup of an aggregation equation

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Abstract

We consider an aggregation equation in $\mathbb{R}^d$, $d \geq 2$ with fractional dissipation: $u_t + \nabla \cdot (u \nabla K * u) = -\nu \Lambda^\gamma u$, where $\nu \geq 0$, $0 < \gamma < 1$ and $K(x) = e^{-|x|}$. We prove a refined blowup criteria by which the global existence of solutions is controlled by its L^q_x norm, for any $\frac{d}{d-1} \leq q \leq \infty$. We prove for a general class of nonsymmetric initial data the finite time blowup of the corresponding solutions. The argument presented works for both the inviscid case $\nu = 0$ and the supercritical case $\nu > 0$ and $0 < \gamma < 1$. Additionally, we present new proofs of blowup which does not use free energy arguments.

Key words:
aggregation equations, fractional dissipation, blowup, blowup criteria
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1 Introduction and main results

In this paper we are concerned with the following aggregation equation in $\mathbb{R}^d$ with fractional dissipation:

$$\begin{cases} u_t + \nabla \cdot (u \nabla K * u) = -\nu \Lambda^\gamma u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $K(x) = e^{-|x|}$. The unknown function $u = u(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ typically represents the population density in biology or the density of particles in material science. The parameters $\nu \geq 0$ and $0 < \gamma < 1$ control the strength of the dissipation term. For any function f on $\mathbb{R}^d$, the fractional Laplacian Λ^γ is defined via the Fourier transform:

$$\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \hat{f}(\xi).$$

Throughout this paper we will consider the specific choice of the kernel $K(x) = e^{-|x|}$ for convenience of presentation, although much of our analysis can be easily extended to similar kernels K that are nonnegative, decreasing, radial and have a Lipschitz point at the origin. In addition, the kernel K has to satisfy the definition of acceptable potential introduced by Laurent (21).

Equations similar to (1) with fractional diffusion have been studied in the literature (see (7), (10), (9) and (24)). Concerning the problem we consider here, the natural range for the viscosity power is $0 < \gamma \leq 2$. The case $\gamma = 2$ corresponds to the usual diffusion, while the regime $0 < \gamma < 2$ corresponds to the so-called anomalous diffusion which in probabilistic terms has a connection with stochastic equations driven by Lévy α-stable flights [3] . As was mentioned in (7), an important technical difficulty lies in the fact that non-Gaussian Lévy α-stable ($0 < \alpha < 2$) semigroups have densities which decay only at an algebraic rate $|x|^{-d-\alpha}$ as $|x| \to \infty$ while the Gaussian kernel $\alpha = 2$ decays exponentially fast.

In equation (1), the strength of the dissipation term is controlled by two parameters ν and γ. For any fixed $\nu > 0$, given the natural scales of the equation (1) we have 3 different ranges to the parameter γ. Namely $0<\gamma < 1$, $\gamma = 1$ and $1 < \gamma \leq 2$, known as the supercritical, critical and subcritical regimes. The choice of the three regimes is connected with the a priori L^1_x conservation of solutions to (1), namely for positive initial data, one has $\|u(t)\|_{L^1_x} = \|u_0\|_{L^1_x}$ for any $t \geq 0$. One can then understand the choice of the three regimes by replacing ∇K by its homogeneous part $-x/|x|$ in (1), from which one obtains

[3] We choose the letter α to be consistent with the standard notation. One should regard $\gamma = \alpha$ here

that the L^1_x space is a critical space for the case $\gamma = 1$, hence the three regimes(cf. (25)). In the subcritical case the a priori L^1_x conservation allows us to prove the global wellposednes of (1) and in the critical case one can prove the global wellposedness for solutions with a small L^1_x norm (cf. (25), (26)). In this work we shall focus on studying (1) in the inviscid case $\nu = 0$ and the supercritical case $\nu > 0$ and $0 < \gamma < 1$.

Aggregation equations of the form (1), with more general kernels (and other modifications) arise in many problems in biology, chemistry and population dynamics (see (11), (31), (35), (12), (23), (30), (39), (13) and (34)). We will not discuss aggregation equations from the modelling point of view. We refer the reader to (30),(31), (34), (14), (12), (40), (36), (15), (16), (17), (18), (20), (32) (33), (38), (37) and (20)

In the mathematics literature, aggregation equations have been studied extensively (see e.g. (2), (3), (4), (5), (6), (21), (25) and (38)). In connection with the problem we study here, Laurent (21) has studied in detail the case of (1) *without the diffusion term* (i.e. $\nu = 0$) and proved several local and global existence results for a class of kernels K with different regularity. More recently Bertozzi and Laurent (2) have obtained finite-time blowup of solutions for the case of (1) without diffusion (i.e. $\nu = 0$) in $\mathbb{R}^d (d \geq 2)$ assuming compactly supported radial initial data with highly localized support. Li and Rodrigo (25) (26) studied the case of (1) with $\nu > 0$ and proved finite time blowup for a class of radial initial data in the case $0 < \gamma < 1$ and global wellposedness for L^1_x initial data in the case $\gamma > 1$. Also, Bertozzi and Brandman (1) have recently constructed $L^1_x \cap L^\infty_x$ weak solutions to (1) in $\mathbb{R}^d$ $(d \geq 2)$ with no dissipation ($\nu = 0$) by following Yudovich's work on incompressible Euler equations (41). We refer the interested reader to (36), (15), (16), (17), (18), (20), (32) and (33) and the references therein for some further rigorous studies.

The purpose of this work is to give a detailed study of (1) in the inviscid and supercritical case. By using L^1_x conservation combined with a logarithmic Sobolev inequality, we obtain a refined blowup criteria of the solution in terms of its L^q_x norm where $\frac{d}{d-1} \leq q \leq \infty$. Previous results require $q > 2$ for $d = 2$ and $q \geq 2$ for $d \geq 3$. We also solve an open problem posed in (1). Namely the existence of nonsymmetric blowing up solutions to (1). Previous results in the literature all relies on the radial assumption. We emphasize that our construction works for both the inviscid case and the supercritical case. We mention that in the inviscid case, we also obtain several new results for initial data which are even but not necessarily compactly supported. As a particular corollary, we also show that all compactly supported even initial data will lead to blowup in finite time. This is in contrast with all previous results that rely on assuming the initial data is radial and has sufficiently localized support.

Before we state the main results, we recall the following theorem previously obtained by the authors (see (25) and (26)) which we state here as a proposition.

Proposition 1 (LWP, smoothing and blowup criteria - (25), (26)) *Let $\nu \geq 0$ and $0 < \gamma \leq 1$. Assume the initial data $u_0 \geq 0$ and $u_0 \in H^s_x \cap L^1_x$ with $s \geq 1$, $s \in \mathbb{R}$. Then there exists a unique maximal-lifespan solution to* (1) *$u \in C([0,T); H^s_x \cap L^1_x) \cap C^1([0,T); H^{s-1}_x \cap L^1_x)$. Here $[0,T)$ is the lifespan of u. The solution u satisfies $u(t) \geq 0$ and $\|u(t)\|_{L^1_x} = \|u_0\|_{L^1_x}$ for any $0 \leq t < T$. If $\nu > 0$, then due to smoothing effect we have $u \in C((0,T); H^{s'}_x)$ for any $s' \geq s$.*

Additionally, we have the following blowup criteria: either $T = +\infty$ in which case we have a global solution or $T < \infty$ and then we have

$$\lim_{t \to T} \int_0^t \|u(s)\|_{L^q_x(\mathbb{R}^d)} ds = +\infty,$$

where q can be any number satisfying:

$$\begin{cases} 2 \leq q \leq \frac{2d}{d-2s}, & \text{if } d \geq 3 \text{ and } s < \frac{d}{2} \\ 2 \leq q < \infty, & \text{if } d \geq 3 \text{ and } s = \frac{d}{2} \\ 2 \leq q \leq \infty, & \text{if } d \geq 3 \text{ and } s > \frac{d}{2} \\ 2 < q < \infty, & \text{if } d = 2 \text{ and } s = 1 \\ 2 < q \leq \infty, & \text{if } d = 2 \text{ and } s > 1. \end{cases} \tag{2}$$

Remark 2 *We stress that the proof of local wellposedness actually only requires $u_0 \in H^s_x$ (see (26)). By standard methods one can weaken the assumptions on the inital data although we shall not do it here. The positivity and L^1_x assumption is physically meaningful since u typically represents the population density in biology.*

Remark 3 *We point out that the range for q in* (2) *is only descriptive in the inviscid case, since due to the smoothing effect of the viscosity one instantly obtains additional regularity for the solution and this yields the biggest ranges, namely $2 \leq q \leq \infty$ in dimension 3 and $2 < q \leq \infty$ in dimension 2.*

We now state the main results. The first theorem gives an improved blowup criteria than that given by Proposition 1.

Theorem 4 (Refined blowup criteria) *Let $\nu \geq 0$ and $0 < \gamma \leq 1$ in* (1)*. Let $u_0 \in H^s_x(\mathbb{R}^d) \cap L^1_x(\mathbb{R}^d)$ with $u_0 \geq 0$, $s \geq 1$ ($s > \frac{d}{2}$ in the inviscid case) and $d \geq 2$. Assume u is the corresponding maximal-lifespan solution with lifespan $[0,T)$ obtained by Proposition 1. Then either $T = +\infty$ in which case we have*

a global solution or $T < \infty$, and we have

$$\lim_{t\to T}\int_0^t \|u(s)\|_{L^q_x} ds = +\infty,$$

where q can be any number satisfying

$$\frac{d}{d-1} \le q \le \infty.$$

Remark 5 *This result improves significantly the blowup criteria given in the inviscid case (2) and the general case (26), where one requires $q > 2$ if $d = 2$ and $q \ge 2$ if $d \ge 3$. There the main requirement $q > 2$ is due to the fact that $D^2K \notin L^2_x$ in $\mathbb{R}^2$. The crucial point which allows us to go below the threshold $q = 2$ is the use of L^1_x conservation combined with a logarithmetic Sobolev inequality. Also the requirement $s > \frac{d}{2}$ on the initial data is not very restrictive. In particular by Proposition 1, in the diffusive case $\nu > 0$, $0 < \gamma \le 1$, one can start with H^1_x initial data and obtain a smooth solution in H^s_x for any $s \ge 1$.*

Remark 6 *A close examination of the proof of Theorem 4 (see Section 2) will reveal that the assumption $s > \frac{d}{2}$ on the initial data is actually not used in establishing the logarithmetic Sobolev inequality. This assumption is only used to show by Sobolev embedding that the constructed solution $u \in L^q_x(\mathbb{R}^d)$ for any $2 \le q \le \infty$. Similar to Proposition 1, in the general inviscid case with $s \ge 1$, the range of q can be as follows:*

$$\begin{cases} \frac{d}{d-1} \le q \le \frac{2d}{d-2s}, & \text{if } d \ge 2 \text{ and } s < \frac{d}{2} \\ \frac{d}{d-1} \le q < \infty, & \text{if } d \ge 2 \text{ and } s = \frac{d}{2} \\ \frac{d}{d-1} \le q \le \infty, & \text{if } d \ge 2 \text{ and } s > \frac{d}{2}. \end{cases} \tag{3}$$

However to simplify the presentation we only choose to state the last case in Theorem 4. The proof of the other two cases are similar.

By Theorem 4 we have the following definition.

Definition 7 (Blowup) *For any nonnegative initial data $u_0 \in H^s_x(\mathbb{R}^d) \cap L^1_x(\mathbb{R}^d)$, $s \ge 1$, we say that the corresponding solution u to (1) blows up in finite time if there exists $T < \infty$, such that*

$$\lim_{t\to T}\int_0^t \|u(s)\|_{L^q_x} ds = +\infty,$$

where q can be any number satisfying (3).

The next theorem is concerned with the inviscid case of (1). We show that solutions corresponding to general initial data with non-compact, non-localized support will develop blowup in finite time. In the following theorem we re-

quire the definition of the class of functions Σ given by $\Sigma := \{f : \ f : \mathbb{R}^d \to \mathbb{R}$ is even, $f \geq 0$, and $\int_{\mathbb{R}^d} f(x)e^{2|x|}dx < \infty\}$.

Theorem 8 (Inviscid case blowup in Σ space) *Let $\nu = 0$ in (1). Let $u_0 \in \Sigma \cap H^s_x$ for some $s \geq 1$ and u_0 is not identically zero. Let u be the corresponding maximal-lifespan solution. Then u blows up in finite time in the sense of Definition 7.*

Remark 9 *We stress that Theorem 8 is already an improvement of the corresponding blowup result in (2). In (2) the existence of finite time blowing up solution was proved for a class of radially symmetric smooth initial data with highly localized support. Their argument was based on studying the free energy associated with the solution u to (1), namely one considers the quantity*

$$E(t) = \int_{\mathbb{R}^d} u(t,x)(K * u)(t,x)dx.$$

and show by contradiction that $E(t)$ grows fast enough to attain, in finite time, a value which exceeds the a priori L^1_x bound. The argument presented here is not based on free energy and we remove the radial or localized support assumption.

As a particular corollary of Theorem 8, we have

Corollary 10 (Blow up - radial compactly supported positive data) *Let $\nu = 0$ in (1) and let u_0 be non-negative, smooth, radially symmetric and compactly supported and with nonzero L^1_x norm. Then the corresponding solution u must blow up in finite time in the sense of Definition 7.*

Remark 11 *Although Corollary 10 is a particular case of Theorem 8, we also give a rather straightforward proof in Section 3 which is only based on the method of characteristics. For comparison, the corresponding result in (2) needs to assume further that the initial data has highly localized support and a free energy argument was used there.*

The next theorem removes any symmetry assumption and works for both the inviscid case and the case with supercritical dissipation. The only drawback of our approach is that we have to assume the initial data is sufficiently localized. On the other hand, this assumption is quite natural in view of the fact that even for compactly supported initial data the dissipation term makes the solution immediately non-compactly supported, due to the infinite speed of propagation of the heat semigroup $\exp(-\nu\Lambda^\gamma t)$. We shall show the existence of blowing up solutions for a class of nonsymmetric initial data. Postponing the definition of this class of initial data (denoted below by $A_{\delta,a,b}$, see Definition 19), we have

Theorem 12 (Blow up for non-symmetric initial data) *Let $\nu \geq 0$ and $0<\gamma<1$ in (1). There exist parameters $0<\delta<\frac{1}{100}$, $a>0$ and $b>0$, such that for any $u_0 \in H^s_x \cap A_{\delta,a,b}$, $s \geq 1$, the corresponding solution blows up in finite time in the sense of Definition 7.*

Outline of the paper. This paper is organized as follows. In Section 2 we prove Theorem 4. Section 3 is devoted to the proof of Theorem 8 and a new proof of Corollary 10. Finally we give the construction of blowing up solutions for nonsymmetric initial data (Theorem 12) in Section 4.

Notation. Throughout the paper we denote by $L^p_x = L^p_x(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ the usual Lebesgue space on $\mathbb{R}^d$. For $s>0$, s being an integer and $1 \leq p \leq \infty$, $W^{s,p}_x = W^{s,p}_x(\mathbb{R}^d)$ denotes the usual Sobolev space

$$W^{s,p}_x = \Big\{ f \in S'(\mathbb{R}^d) : \|f\|_{W^{s,p}} = \sum_{0 \leq j \leq s} \|\partial^j_x f\|_{L^p_x(\mathbb{R}^d)} < \infty \Big\}.$$

When $p=2$, we denote $H^m_x = H^m_x(\mathbb{R}^d) = W^{2,p}_x(\mathbb{R}^d)$ and $\|\cdot\|_{H^m_x}$ as its norm. We will also use the Sobolev space of fractional power $H^s_x(\mathbb{R}^d)$ for fraction s, which is defined via the Fourier transform:

$$\|f\|_{H^s} = \|(1+|\xi|)^s \hat{f}(\xi)\|_{L^2_\xi}.$$

Finally, for any two quantities X and Y, we use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C>0$. A constant C with subscripts implies the dependence on these parameters. We use $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ holds.

2 Proof of Theorem 4

To prove Theorem 4 we will need some basic harmonic analysis. Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N>0$, we define the Fourier multipliers

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi) \\
\widehat{P_{>N} f}(\xi) &:= (1-\varphi(\xi/N))\hat{f}(\xi) \\
\widehat{P_N f}(\xi) &:= \psi(\xi/N)\hat{f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi)
\end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. We will need the following standard estimate which we include here for the sake of completeness.

Lemma 13 (Bernstein estimates) *For* $1 \leq p \leq q \leq \infty$,

$$\begin{aligned} \left\| |\nabla|^{\pm s} P_N f \right\|_{L^p_x(\mathbb{R}^d)} &\sim N^{\pm s} \|P_N f\|_{L^p_x(\mathbb{R}^d)}, \\ \|P_{\leq N} f\|_{L^q_x(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L^p_x(\mathbb{R}^d)}, \\ \|P_N f\|_{L^q_x(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p_x(\mathbb{R}^d)}. \end{aligned}$$

Using Lemma 13, we now give

PROOF. [Proof of Theorem 4] We first treat the case $d \geq 3$. By Proposition 1 (see also (26)), we only need to obtain a priori control of the L^2_x norm of u. To this end, by using (1), we compute

$$\frac{d}{dt} \|u(t)\|^2_{L^2_x} \lesssim \|\Delta K * u(t)\|_{L^\infty_x} \|u(t)\|^2_{L^2_x}. \tag{4}$$

It is easy to see that up to a multiple constant we can write $\Delta K * u = \Delta \langle \nabla \rangle^{-(d+1)} u$, with $\langle \cdot \rangle$ denoting the Japanese bracket, defined by $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$, and where we take the Fourier transform as $\int f(x) e^{-ix\cdot\xi} dx$ to avoid irrelevant constants . Let N_0 be a number to be chosen later. By using the LP decomposition and Bernstein's inequality (Lemma 13), we have

$$\begin{aligned} &\|\Delta \langle \nabla \rangle^{-(d+1)} u\|_{L^\infty_x} \lesssim \\ \lesssim &\sum_{N \leq 0} 2^N \|P_{2^N} u\|_{L^1_x} + \sum_{0 < N \leq N_0} \|P_{2^N} u\|_{L_x^{\frac{d}{d-1}}} + \sum_{N > N_0} 2^{N^{1-\frac{d}{2}}} \|P_{2^N} u\|_{L^2_x} \lesssim \\ \lesssim &\|u\|_{L^1_x} + N_0 \cdot \|u\|_{L_x^{\frac{d}{d-1}}} + 2^{N_0^{1-\frac{d}{2}}} \|u\|_{L^2_x} \lesssim \\ \lesssim &\|u_0\|_{L^1_x} + N_0 \cdot \|u\|_{L_x^{\frac{d}{d-1}}} + 2^{N_0^{1-\frac{d}{2}}} \|u\|_{L^2_x}, \end{aligned}$$

where in the last inequality we used the L^1_x conservation of positive solutions (see Proposition 1). Now if $\|u\|_{L^2_x} \leq 16$, then we choose $2^{N_0} = 8$. Otherwise we choose N_0 such that

$$2^{N_0} \leq \|u\|_{L^2_x}^{\frac{2}{d-2}} < 2^{N_0+1}.$$

This immediately gives us

$$\begin{aligned}&\|\Delta\langle\nabla\rangle^{-(d+1)}u\|_{L^\infty_x}\\ \lesssim&(1+\|u_0\|_{L^1_x})\log(5+\|u\|^2_{L^2_x})\cdot(1+\|u\|_{L^{\frac{d}{d-1}}_x}).\end{aligned}$$

Plugging this estimate into (4), we obtain

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{L^2_x}\lesssim&(1+\|u_0\|_{L^1_x})\cdot(1+\|u(t)\|_{L^{\frac{d}{d-1}}_x})\cdot(5+\|u(t)\|^2_{L^2_x})\\ &\cdot\log(5+\|u(t)\|^2_{L^2_x}).\end{aligned}$$

A simple Gronwall inequality applied to the quantity $X(t)=\log\log(5+\|u(t)\|^2_{L^2_x})$ immediately gives us

$$\begin{aligned}\|u(t)\|^2_{L^2_x}\leq&(5+\|u_0\|^2_{L^2_x})\exp\Big(\exp\Big(const\cdot\int_0^t(1+\|u_0\|_{L^1_x})\\ &\cdot(1+\|u(s)\|_{L^{\frac{d}{d-1}}_x})ds\Big)\Big).\end{aligned}$$

This concludes the proof of the case $d\geq 3$. It remains for us to treat the case $d=2$.

In the case $d=2$, by Proposition 1 (see also (26)), we need to obtain a priori control of the L^p_x norm of u for some $p>2$. For simplicity we shall consider the L^4_x norm. Then by (1), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}u^4dx+\nu\int_{\mathbb{R}^2}(\Lambda^\gamma u)u^3dx\\ \lesssim&\|\Delta K*u\|_{L^\infty_x(\mathbb{R}^2)}\cdot\|u\|^4_{L^4_x(\mathbb{R}^2)}.\end{aligned}\tag{5}$$

At this point we recall the following positivity lemma by Ju (19), which improves on work of Córdoba and Córdoba (8).

Lemma 14 *Let $0\leq\alpha\leq 2$ and $p\geq 2$, then*

$$\int_{\mathbb{R}^2}|u|^{p-2}u\Lambda^\alpha udx\geq\frac{2}{p}\int_{\mathbb{R}^2}(\Lambda^{\frac{\alpha}{2}}|u|^{\frac{p}{2}})^2dx.$$

Specializing to our case, this means that we can drop the second term on the LHS of (5). We then have

$$\frac{d}{dt}\|u(t)\|_{L^4_x}\lesssim\|\Delta K*u\|_{L^\infty_x}\|u(t)\|_{L^4_x}.\tag{6}$$

Now similar to the $d \geq 3$ case, we use LP decomposition and Bernstein's inequality to estimate

$$\begin{aligned}\|\Delta K * u\|_{L^\infty_x} &\lesssim \|\Delta\langle\nabla\rangle^{-3}u\|_{L^\infty_x(\mathbb{R}^2)} \lesssim \\ &\lesssim (1+\|u_0\|_{L^1_x})\log(5+\|u(t)\|_{L^4_x})\cdot(1+\|u(t)\|_{L^2_x}).\end{aligned}$$

Plugging this estimate into (6), we get

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|_{L^4_x} \lesssim \\ \lesssim&(1+\|u_0\|_{L^1_x})\cdot(1+\|u(t)\|_{L^2_x})\log(5+\|u(t)\|_{L^4_x})\cdot(5+\|u(t)\|_{L^4_x}).\end{aligned}$$

A simple Gronwall argument applied to the quantity $X(t) = \log\log(5+\|u(t)\|_{L^4_x})$ immediately yields

$$\begin{aligned}&\|u(t)\|_{L^4_x} \\ \leq&(5+\|u_0\|_{L^4_x})\exp\Big(\exp\Big(const\cdot\int_0^t(1+\|u_0\|_{L^1_x})\cdot(1+\|u(s)\|_{L^2_x})ds\Big)\Big).\end{aligned}$$

This ends the case $d = 2$ and the proof of Theorem 4.

As a direct application of Theorem 4, we consider the following variant of (1):

$$\begin{cases} v_t + \nabla\cdot(v\nabla\tilde{K} * v) = -\nu\Lambda^\gamma v, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \\ v(0,x) = v_0(x), & x\in\mathbb{R}^d, \end{cases} \tag{7}$$

where $\tilde{K}(x) = -e^{-|x|}$. The solutions to (7) and solutions to (1) can be related to each other through a simple sign change: $u \to v = -u$. This means for example that positive solutions to (7) corresponds to negative solutions of (1) and vice versa. It is not diffcult to check that Poposition 1 and Theorem 4 hold also for our modified equation (7). In fact in accordance with the usual terminology, we can call $\tilde{K}(x)$ a repulsive potential in which case on expects global wellposedness even without diffusion and we call $K(x)$ an attractive potential in which case we expect finite time blowup occurs (cf. (27) for a related model). Now we state precisely results concerning equation (7)

Corollary 15 (Global wellposedness for equation (7) for general initial data) *Let $\nu \geq 0$ and $0<\gamma\leq 2$ in (7). Let $v_0 \in L^1_x \cap H^s_x$ with $v_0 \geq 0$, $s \geq 1$. The the corresponding solution v is global.*

PROOF. [Proof of Corollary 15] By Theorem 4, we only need to give a priori control of the L^2_x norm of v. By (7) and a direct calculation, we have

$$\begin{aligned}\partial_t \|v(t)\|^2_{L^2_x} &\lesssim -\int_{\mathbb{R}^d} \Delta(\tilde{K} * v) v(t,x)^2 dx = \\ &= \int_{\mathbb{R}^d} \Delta(K * v) v(t,x)^2 dx. \end{aligned} \tag{8}$$

Now a simple calculation gives

$$\Delta K(x) = e^{-|x|} - \frac{d-1}{|x|} e^{-|x|}.$$

Plugging this equality into (8) and using positivity of the solution v, we have

$$\begin{aligned}\partial_t \|v(t)\|^2_{L^2_x} &\lesssim \int_{\mathbb{R}^d} (e^{-|x|} * v) v(t,x)^2 dx \\ &\lesssim \|v_0\|_{L^1_x} \|v(t)\|^2_{L^2_x},\end{aligned}$$

where the last step follows from L^1_x conservation. A simple Gronwall immediately yields

$$\|v(t)\|^2_{L^2_x} \leq \|v_0\|^2_{L^2_x} \exp\Big(const \cdot \|v_0\|_{L^1_x} t\Big)$$

which shows that L^2_x norm of v is a priori controlled. Therefore by Theorem 4 v must be global. The corollary is proved.

Remark 16 *We remark that it is possible to prove Corollary 15 by appealing directly to the blowup criteria in Proposition 1. However, in that case, one has to discuss two cases: $d = 2$ and $d \geq 3$. In the case of $d = 2$, one has to use for example the L^4_x norm and Lemma 14. Theorem 4 allows us to use L^2_x for all cases $d \geq 2$ and therefore the argument is much simpler.*

3 Proof of Theorem 8 and Corollary 10

We begin with the proof of Corollary 10. As mentioned above, the proof we present below relies only on the method of characteristics and no free energy is used.

PROOF. [Proof of Corollary 10] We argue by contradiction . Assume the corresponding solution u is global. By Proposition 1, u is a smooth solution. Then consider the characteristic curves defined by

$$\begin{cases} \frac{d}{dt} X(t,\alpha) = (\nabla K * u)(X(t,\alpha), t), & t > 0, \\ X(0,\alpha) = \alpha \in \mathbb{R}^d. \end{cases}$$

By standard ODE theory and the fact that u is smooth, we have $X(t,\alpha)$ is smooth and globally well defined. Assume the initial data $supp(u_0) \subset B(0,R_0)$ for some $R_0 > 0$. It is not difficult to check (as we will see shortly) that $supp(u(t)) \subset B(0,R(t))$, where $R(t) = |X(t,R_0)|$ and $R(t) \leq R_0$ for any $t \geq 0$. Here we have slightly abused the notation and denote $X(t,\alpha)$ as $X(t,R_0)$ for any $|\alpha| = R_0$. This is reasonable in view of the radial assumption. Next we compute the radial velocity

$$\begin{aligned}
v(t,R_0) &= \frac{X(t,R_0)}{|X(t,R_0)|} \cdot (\nabla K * u)(X(t,R_0),t) = \\
&= -\int_{|y|\leq|X(t,R_0)|} \frac{X(t,R_0)}{|X(t,R_0)|} \cdot \frac{X(t,R_0)-y}{|X(t,R_0)-y|} e^{-|X(t,R_0)-y|} u(t,y)dy \leq \\
&\leq -e^{-2R_0} \int_{y\cdot X(t,R_0)\leq 0} \frac{X(t,R_0)}{|X(t,R_0)|} \cdot \frac{X(t,R_0)-y}{|X(t,R_0)-y|} u(t,y)dy \leq \\
&\leq -C_1 \cdot \|u_0\|_{L^1_x},
\end{aligned}$$

where the last inequality follows from the radial assumption and C_1 is a positive constant depending only on R_0 and d. The minus sign here means that the radial velocity is pointing towards the origin. This estimate shows that the boundary of the support of the solution at any moment $t \geq 0$ moves inward toward the origin with a constant velocity independent of time. This immediately implies that the solution must collapse into a point in finite time. We have obtained a contradiction and the corollary is proved.

As mentioned in the introduction, Theorem 8 deals with initial data more general than that of Corollary 10.

Since the kernel $K(x)$ is a radial function, it is not difficult to check that the property of being even is preserved by (1). Recall, that we are concerned with the following class of functions

$$\Sigma := \Big\{ f : \ f : \mathbb{R}^d \to \mathbb{R} \text{ is even, } f \geq 0, \text{ and } \int_{\mathbb{R}^d} f(x) e^{2|x|} dx < \infty \Big\}. \tag{9}$$

As a first step, we aim to show that if the initial data u_0 is in the set Σ, the for any $t \geq 0$ we also have $u(t) \in \Sigma$. This is

Lemma 17 (Wellposedness in Σ space) *Let $\nu = 0$ in (1). Let $u_0 \in H^s_x \cap \Sigma$ for some $s \geq 1$. Let u be the corresponding maximal-lifespan solution with lifespan $[0,T)$. Then we have $u(t) \in H^s_x \cap \Sigma$ for any $0 \leq t < T$.*

PROOF. [Proof of Lemma 17] Let $u_0 \in H^s \cap \Sigma$ and assume u is the associated

maximal-lifespan solution. Since the property of being even is preserved by (1), we only need to show that

$$\int_{\mathbb{R}^d} e^{2|x|} u(t,x)dx < \infty,$$

for any $0 \le t < T$. We first show that

$$\sup_{0\le\tau\le t} \int_{\mathbb{R}^d} |x| u(\tau,x)dx < \infty,$$

for any $0 \le t < T$. Let $\phi(x) = e^{-|x|}$ and $R > 0$. Later we will let R tend to infinity. We now compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)|x|\phi(\frac{x}{R})dx = \\
&= \int_{\mathbb{R}^d} |x|\phi(\frac{x}{R}) \cdot (-\nabla \cdot (u\nabla K * u))dx \lesssim \\
&\lesssim \int_{\mathbb{R}^d} |\nabla K * u|\phi(\frac{x}{R})u(t,x)dx + \\
&\qquad + \int_{\mathbb{R}^d} |\nabla K * u| \cdot \frac{|x|}{R} \cdot |(\nabla\phi)(\frac{x}{R})|u(t,x)dx \lesssim \\
&\lesssim \|u_0\|^2_{L^1_x} + \frac{1}{R}\|u_0\|_{L^1_x} \int_{\mathbb{R}^d} \phi(\frac{x}{R})|x|u(t,x)dx,
\end{aligned}$$

where we have used the L^1_x conservation in the last inequality and the fact that $|\nabla\phi| \lesssim |\phi|$ for our choice $\phi(x) = e^{-|x|}$. Now let $0 < t_0 < T$ be arbitrary. A simple application of Gronwall's inequality then gives us

$$\int_{\mathbb{R}^d} u(t_0,x)|x|\phi(\frac{x}{R})dx \lesssim e^{Ct_0}\int_{\mathbb{R}^d} u_0(x)|x|dx + e^{Ct_0}\|u_0\|_{L^1_x},$$

where C is a constant. Taking $R \to \infty$ and using Lebesgue 's Monotone Convergence Theorem we immediately obtain

$$\sup_{0\le t\le t_0} \int_{\mathbb{R}^d} u(t,x)|x|dx < \infty, \qquad \forall\, 0 < t_0 < T. \tag{10}$$

This is the first estimate we need. Next we let $\psi(x) = e^{-|x|^2}$ and $R > 0$. Later

we will let R tend to infinity. We compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)e^{2|x|}\psi(\frac{x}{R})dx = \\
&= \int_{\mathbb{R}^d} e^{2|x|}\psi(\frac{x}{R})\cdot(-\nabla\cdot(u\nabla K * u))dx \lesssim \\
&\lesssim \int_{\mathbb{R}^d} |\nabla K * u| e^{2|x|}\psi(\frac{x}{R})u(t,x)dx + \\
&\qquad + \frac{1}{R}\int_{\mathbb{R}^d} e^{2|x|}|(\nabla\psi)(\frac{x}{R})|\cdot|\nabla K * u|u(t,x)dx \lesssim \\
&\lesssim \|u_0\|_{L^1_x}\int_{\mathbb{R}^d} u(t,x)e^{2|x|}\psi(\frac{x}{R})dx + \\
&\qquad + \|u_0\|_{L^1_x}\int_{|x|\le R^2} e^{2|x|}\cdot\frac{1}{R}\frac{|x|}{R}\cdot e^{-\frac{|x|^2}{R^2}}u(t,x)dx + \\
&\qquad + \|u_0\|_{L^1_x}\int_{|x|\ge R^2}\frac{1}{R^2}\cdot|x|u(t,x)dx \lesssim \\
&\lesssim \|u_0\|_{L^1_x}\int_{\mathbb{R}^d} u(t,x)e^{2|x|}\psi(\frac{x}{R})dx + \\
&\qquad + \|u_0\|_{L^1_x}\cdot\frac{1}{R^2}\cdot\int_{\mathbb{R}^d}|x|u(t,x)dx.
\end{aligned}$$

Let $0 < t_0 < T$ be arbitrary. A simple Gronwall argument yields

$$\begin{aligned}
\int_{\mathbb{R}^d} u(t_0,x)e^{2|x|}\psi(\frac{x}{R})dx \lesssim e^{Ct_0}\int_{\mathbb{R}^d} u_0(x)e^{2|x|}dx + \\
+ \frac{1}{R^2}\|u_0\|_{L^1_x}\cdot e^{Ct_0}\int_0^{t_0}\int_{\mathbb{R}^d}|x|u(t,x)dxdt.
\end{aligned}$$

Now use (10). Taking $R \to \infty$ and applying again Lebesgue 's Monotone Convergence Theorem immediately yields the result. The lemma is proved.

Next we need an elementary lemma.

Lemma 18 *For any x, $y \in \mathbb{R}^d$, we have*

$$I := (x-y)\cdot(\frac{x}{|x|}e^{2|x|} - \frac{y}{|y|}e^{2|y|}) \ge 0$$

Furthermore, if $x\cdot y \le 0$ we have

$$I \ge \frac{1}{2}(|x|+|y|)(e^{2|x|}+e^{2|y|})$$

PROOF. First we notice that

$$2|x|e^{2|x|} + 2|y|e^{2|y|} \geq (|x| + |y|)(e^{2|x|} + e^{2|y|}) \tag{11}$$

since it is equivalent to

$$(|x| - |y|)(e^{2|x|} - e^{2|y|}) \geq 0$$

which is obviously true as both $|z|$ and $e^{2|z|}$ are increasing functions.

Expanding the product, I becomes

$$I = |x|e^{2|x|} + |y|e^{2|y|} - \frac{x \cdot y}{|x|\,|y|}[|y|e^{2|x|} + |x|e^{2|y|}]$$

and using the fact that $x \cdot y \leq 0$ and (11) we obtain

$$\geq |x|e^{2|x|} + |y|e^{2|y|} \geq \frac{1}{2}(|x| + |y|)[e^{2|x|} + e^{2|y|}]$$

which proves the second statement. In order to prove the positivity of I we have

$$I = |x|e^{2|x|} + |y|e^{2|y|} - \frac{x \cdot y}{|x|\,|y|}[|y|e^{2|x|} + |x|e^{2|y|}] \geq$$

$$\geq \frac{1}{2}(|x| + |y|)[e^{2|x|} + e^{2|y|}] - \frac{x \cdot y}{|x|\,|y|}[|y|e^{2|x|} + |x|e^{2|y|}] \geq$$

$$\geq \frac{1}{2}(|x| + |y|)[e^{2|x|} + e^{2|y|}] - [|y|e^{2|x|} + |x|e^{2|y|}] \geq$$

$$\geq \frac{1}{2}|x|e^{2|x|} + \frac{1}{2}|y|e^{2|y|} - \frac{1}{2}|y|e^{2|x|} - \frac{1}{2}|x|e^{2|y|} =$$

$$= \frac{1}{2}\Big[2|x|e^{2|x|} + 2|y|e^{2|y|} - (|x| + |y|)(e^{2|x|} + e^{2|y|})\Big] > 0$$

as proved in (11).

We now complete the

PROOF. [Proof of Theorem 8] Assume $u_0 \in \Sigma \cap H^s_x$ for some $s \geq 1$ and let u be the corresponding maximal-lifespan solution with lifespan $[0, T)$. By Lemma 17, we have $u(t) \in \Sigma \cap H^s_x$ for any $0 \leq t < T$. We obtain the result by a contradiction argument. We assume that $T = +\infty$ and we will derive a contradiction. To this end, we compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} e^{2|x|}u(t,x)dx = \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla(e^{2|x|}) \cdot \nabla K(x-y)u(t,x)u(t,y)dxdy = \\
&= -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{x}{|x|} \cdot \frac{x-y}{|x-y|} e^{-|x-y|}e^{2|x|}u(t,x)u(t,y)dxdy = \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|} e^{-|x-y|}(x-y) \cdot (\frac{x}{|x|}e^{2|x|} - \frac{y}{|y|}e^{2|y|}) \cdot \\
&\qquad\qquad \cdot u(t,x)u(t,y)dxdy, \qquad (12)
\end{aligned}$$

where the last equality follows from symmetrizing the integral in x and y. Now by Lemma 18, we have

$$\begin{aligned}
\text{RHS of (12)} &\leq -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 - \frac{x \cdot y}{|x| \cdot |y|}) \cdot \frac{|x| + |y|}{|x-y|} \cdot e^{-|x-y|} \cdot \\
&\qquad \cdot (e^{2|x|} + e^{2|y|}) \cdot u(t,x)u(t,y)dxdy \leq \\
&\leq -\frac{1}{2}\int_{x \cdot y \leq 0} u(t,x)u(t,y)dxdy = \\
&= -\frac{1}{4}\|u_0\|^2_{L^1_x},
\end{aligned}$$

where the last equality follows from the fact that u is an even function of x. This estimate shows that

$$\int_{\mathbb{R}^d} e^{2|x|}u(t,x)dx \leq \int_{\mathbb{R}^d} e^{2|x|}u_0(x)dx - \frac{1}{4}\|u_0\|^2_{L^1_x}t,$$

for all $t \geq 0$. This implies that $\int_{\mathbb{R}^d} e^{2|x|}u(t,x)dx$ becomes negative in finite time. This is clearly a contradiction to the fact that u is non-negative. The theorem is proved.

4 Proof of Theorem 12

In this section we give the proof of Theorem 12. We begin with the following

Definition 19 (Admissable initial conditions) *Let $0 < \delta < \frac{1}{100}$, $a > 0$, $b > 0$ be constants. The set $A_{\delta,a,b}$ consists of functions $f \in L^1_x(\mathbb{R}^d)$, $f \geq 0$ satisfying the following conditions:*

(1) The L^1_x norm of f is not small, i.e.

$$\|f\|_{L^1_x(\mathbb{R}^d)} \geq a. \tag{13}$$

(2) f is localized in a neighborhood near the origin:

$$\int_{|x|\geq \frac{\delta}{100}} f(x)dx \leq \frac{\delta}{100}\|f\|_{L^1_x}. \tag{14}$$

(3) f satisfies the following inequality.

$$\|f\|^2_{L^1_x} < b\|f\|^2_{L^1_x} + \int_{\mathbb{R}^d} f(x)(K*f)(x)dx, \tag{15}$$

where the kernel $K(x) = e^{-|x|}$ is the same as in (1).

Remark 20 *It is not difficult to show that the set $A_{\delta,a,b}$ is nonempty for any $0 < \delta < \frac{1}{100}$, $a > 0$, $b > 0$. In the extreme case one can think of f as an approximation of the delta function centered at the origin. In that case it can be easily checked that the last integral on the RHS of (15) is approximately equal to $\|f\|^2_{L^1_x}$. Therefore it is natural that (15) holds for such a class of functions localized near the origin. Note also that we do not impose any symmetry assumption on the candidate functions (cf. (25) for earlier constructions where radial symmetry is used). The conditions (13), (14) force the solution to be sharply peaked near the origin. Condition 15 is a technical condition needed for the blowup argument (cf. (26)). In fact all the conditions listed above are not very restrictive and one can easily come up with several other alternatives and weaker conditions. However for the simplicity of presentation we shall not do it here.*

Now let $\phi \in C^\infty_c(\mathbb{R}^d)$ be a radially symmetric function such that $0 \leq \phi(x) \leq 1$ for any $x \in \mathbb{R}^d$ and

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

Let $0 < \delta < \frac{1}{100}$. Define $w_\delta(x) = \phi(\frac{2x}{\delta})$ and denote

$$M = \Big(\|\nabla w_\delta\|_{L^\infty_x} + 1\Big) \cdot \|\nabla K\|_{L^\infty_x} + \nu\|\Lambda^\gamma w_\delta\|_{L^\infty_x} + 1, \tag{16}$$

and also define

$$T = \frac{\delta}{40d}\frac{1}{\tilde{M}\|u_0\|_{L^1_x} + \tilde{\tilde{M}}}. \tag{17}$$

where

$$\tilde{M} = \Big(\|\nabla w_\delta\|_{L^\infty_x} + 1\Big) \cdot \|\nabla K\|_{L^\infty_x} \qquad \text{and} \qquad \tilde{\tilde{M}} = \nu\|\Lambda^\gamma \omega\|_{L^\infty_x} + 1$$

We have the following

Lemma 21 (Localization of weighted averages for short time) *Let* $\omega \in C_c^\infty(\mathbb{R}^d)$*. Then for any* T_0 *satisfying*

$$T_0 \leq \frac{\delta}{40 \cdot d \cdot (\|\nabla \omega\|_{L^\infty_x} \cdot \|\nabla K\|_{L^\infty_x} \cdot \|u_0\|_{L^1_x} + \nu\|\Lambda^\gamma \omega\|_{L^\infty_x})}, \tag{18}$$

we have

$$\sup_{0 \leq t \leq T_0} \left| \int_{\mathbb{R}^d} u(t,x)\omega(x)dx - \int_{\mathbb{R}^d} u_0(x)\omega(x)dx \right| \leq \frac{\delta}{40d}\|u_0\|_{L^1_x}.$$

PROOF. [Proof of Lemma 21] By (1), we compute

$$\begin{aligned}
&\left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t,x)\omega(x)dx \right| \leq \\
\leq &\left| \int_{\mathbb{R}^d} \nabla\omega \cdot (\nabla K * u)u(t,x)dx \right| + \nu \left| \int_{\mathbb{R}^d} u(t,x)(\Lambda^\gamma \omega)(x)dx \right| \leq \\
\leq &\|\nabla\omega\|_{L^\infty_x} \cdot \|\nabla K\|_{L^\infty_x} \|u_0\|^2_{L^1_x} + \nu\|\Lambda^\gamma\omega\|_{L^\infty_x} \cdot \|u_0\|_{L^1_x},
\end{aligned}$$

where in the last inequality we have used the L^1_x conservation. Taking T_0 as in (18) and integrating in time immediately yields the result.

We have the following

Corollary 22 (Mass localization for short time) *Let* $u_0 \geq 0$ *and satisfying* (13) *and* (14)*. Let* T *be as in* (17)*. Then we have*

$$\int_{|x| \leq \delta} u(t,x)dx \geq (1 - \frac{\delta}{10})\|u_0\|_{L^1_x}, \quad \forall\, 0 \leq t \leq T.$$

PROOF. [Proof of Corollary 22] Take $\omega = w_\delta$. It is easy to check that $T \leq T_0$ where T_0 is given as in (18). Therefore by Lemma 21, we have

$$\sup_{0 \leq t \leq T} \left| \int_{\mathbb{R}^d} u(t,x)w_\delta(x)dx - \int_{\mathbb{R}^d} u_0(x)w_\delta(x)dx \right| \leq \frac{\delta}{40d}\|u_0\|_{L^1_x}.$$

By the definition of w_δ, we have

$$\begin{aligned}\int_{|x|\le\delta} u(t,x)dx &\ge \int_{\mathbb{R}^d} u(t,x)w_\delta(x)dx \ge \\ &\ge \int_{\mathbb{R}^d} u_0(x)w_\delta(x)dx - \frac{\delta}{40d}\|u_0\|_{L^1_x} \ge \\ &\ge (1-\frac{\delta}{10})\|u_0\|_{L^1_x}.\end{aligned}$$

The corollary is proved.

Next we need the following elementary inequality.

Lemma 23 (Trigonometric inequality) *Let α, β, γ be angles of a triangle on the plane. Then we have*

$$\cos\alpha + \cos\beta + \cos\gamma > 1. \tag{19}$$

PROOF. [Proof of Lemma 23] This is a standard exercise in plane geometry. However we provide a proof here for the sake of completeness. By a few elementary manipulations involving only trigonometric identities, we arrive at

$$\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

Now (19) follows immediately by observing that $0 < \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} < \frac{\pi}{2}$.

As a result we have the following

Corollary 24 (Three-point inequality) *Let x, y, $z \in \mathbb{R}^d$ with $d \ge 2$. Assume that they are not collinear. Then we have*

$$\frac{x-y}{|x-y|}\cdot\frac{x-z}{|x-z|} + \frac{y-x}{|y-x|}\cdot\frac{y-z}{|y-z|} + \frac{z-y}{|z-y|}\cdot\frac{z-x}{|z-x|} > 1. \tag{20}$$

PROOF. [Proof of Corollary 24] Since x, y, z are not collinear, there is a (hyper)plane passing through all three points such that x, y, z are vertices of a triangle on that plane. The sum of cosines of the internal angles of the triangle are precisely given by LHS of (20). Now clearly (20) holds true by Lemma 23.

We will need estimates for the nonlinear terms involving the kernel ∇K. Denote $N(x) = -\frac{x}{|x|}$. $N(x)$ is clearly the homogeneous part of the kernel ∇K. We are now ready to prove the following crucial lemma.

Lemma 25 (Lower bound for the homogeneous kernel) *There exists a constant $C_1 > 0$ such that for any nonnegative function $g \in L^1_x(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} g(x)|(N * g)(x)|^2 \, dx \geq C_1 \|g\|^3_{L^1_x}. \tag{21}$$

PROOF. [Proof of Lemma 25] By direct computation, we have

$$\begin{aligned}\text{LHS of (21)} &= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{x-y}{|x-y|} \cdot \frac{x-z}{|x-z|} \right) g(x)g(y)g(z) = dxdydz \\ &= \frac{1}{3} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \tilde{k}(x,y,z) g(x)g(y)g(z) dxdydz,\end{aligned}$$

where the last equality follows from symmetrizing the integral in x, y, z, and

$$\tilde{k}(x,y,z) = \frac{x-y}{|x-y|} \cdot \frac{x-z}{|x-z|} + \frac{y-x}{|y-x|} \cdot \frac{y-z}{|y-z|} + \frac{z-x}{|z-x|} \cdot \frac{z-y}{|z-y|}.$$

By Corollary 24, we have

$$\tilde{k}(x,y,z) > 1, \quad \text{for all } x, y, z$$

except on a set of measure 0 (in the Lebesgue measure $dxdydz$) on which x, y, z are possibly collinear. Now inequality (21) follows immediately by using this lower bound and direct integration in dx, dy, dz. The lemma is proved.

Finally we complete the

PROOF. [Proof of Theorem 12] Let $0 < \delta < \frac{1}{100}$. Let $u_0 \geq 0$, $u_0 \in H^s_x \cap A_{\delta,a,b}$ (trivially non-empty) for some $s \geq 1$, $a > 0$, $b > 0$, where the set $A_{\delta,a,b}$ is defined in Definition 19. We shall specify the choice of the constants a, b later in the proof. We will argue by contradiction and assume that the corresponding solution u is global. Now we compute

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^d} u(t,x)(K * u)(t,x) dx &= 2 \int_{\mathbb{R}^d} (\partial_t u)(t,x)(K * u)(t,x) dx = \\ &= -2 \int_{\mathbb{R}^d} \nabla \cdot (u \nabla K * u)(K * u)(t,x) dx + \\ &\quad - 2\nu \int_{\mathbb{R}^d} (\Lambda^\gamma u)(t,x)(K * u)(t,x) dx = \\ &= 2 \int_{\mathbb{R}^d} u(t,x) |(\nabla K * u)(t,x)|^2 dx + \\ &\quad - 2\nu \int_{\mathbb{R}^d} (\Lambda^\gamma K * u)(t,x) u(t,x) dx = \\ &=: 2A - 2\nu B. \end{aligned} \tag{22}$$

We now estimate the terms A and B separately.

`Estimate of` A. Recall that $N(x) = -\frac{x}{|x|}$. Denoting

$$G(x) = \frac{x}{|x|}(1 - e^{-|x|}),$$

we have

$$\begin{aligned} A &\geq \frac{1}{2}\int_{\mathbb{R}^d} u(t,x)|(N * u)(t,x)|^2 dx - \int_{\mathbb{R}^d} u(t,x)|(G * u)(t,x)|^2 dx \geq \\ &\geq \frac{1}{2} C_1 \|u_0\|_{L^1_x}^3 - \int_{\mathbb{R}^d} u(t,x)|(G * u)(t,x)|^2 dx, \end{aligned} \tag{23}$$

where the last inequality follows from Lemma 25 and the L^1_x conservation of the solution. To estimate the last integral on the RHS of (23), we shall use the mass localization of the solution u on the time interval $[0, T]$, where T is chosen the same as in (17). By Corollary 22, for any $0 \leq t \leq T$, we have

$$\int_{|x|\geq\delta} u(t,x)dx \leq \frac{\delta}{10}\|u_0\|_{L^1_x}.$$

Therefore if $|x| \leq \delta$ and δ is sufficiently small, we then have

$$\begin{aligned} |(G * u)(t,x)| &\leq \int_{|y-x|\leq 2\delta} |G(x-y)|u(t,y)dy + \int_{|y-x|>2\delta} |G(x-y)|u(t,y)dy \\ &\leq 3\delta\|u_0\|_{L^1_x} + \int_{|y|\geq\delta} u(t,y)dy \leq \\ &\leq 4\delta\|u_0\|_{L^1_x}. \end{aligned} \tag{24}$$

For general $|x| \geq 0$, we have the trivial estimate

$$|(G * u)(t,x)| \leq \|u_0\|_{L^1_x}. \tag{25}$$

Plugging (24), (25) into (23), we obtain

$$\begin{aligned} A &\geq \frac{1}{2} C_1 \|u_0\|_{L^1_x}^3 - \int_{|x|\geq\delta} u(t,x)|(G * u)(t,x)|^2 dx \\ &\qquad - \int_{|x|\leq\delta} u(t,x)|(G * u)(t,x)|^2 dx \\ &\geq \frac{1}{2} C_1 \|u_0\|_{L^1_x}^3 - \delta\|u_0\|_{L^1_x}^3 - 16\delta^2\|u_0\|_{L^1_x}^3 \\ &\geq \frac{1}{4} C_1 \|u_0\|_{L^1_x}^3, \end{aligned}$$

where the last inequality follows if we take δ sufficiently small. This finishes the estimate of the term A.

`Estimate of` B. We have

$$|B| \leq \|\Lambda^\gamma K\|_{L^\infty_x} \|u_0\|^2_{L^1_x}.$$

By Fourier transform, it is not difficult to show that

$$\|\Lambda^\gamma K\|_{L^\infty_x} \leq \frac{1}{2} C_2 < \infty,$$

where C_2 is a constant. This finishes the estimate of the term B.

Finally collecting all the estimates, we arrive at

$$\frac{d}{dt} \int_{\mathbb{R}^d} u(t,x)(K * u)(t,x) dx \geq \frac{1}{4} C_1 \|u_0\|^3_{L^1_x} - C_2 \nu \|u_0\|^2_{L^1_x},$$

By our choice of T (see (17)) and the fact that $\|K\|_{L^\infty_x} = 1$, we obtain

$$\begin{aligned} \|u_0\|^2_{L^1_x} &\geq \int_{\mathbb{R}^d} u(T,x)(K * u)(T,x) dx \geq \\ &\geq \left(\frac{1}{4} C_1 \|u_0\|^3_{L^1_x} - C_2 \nu \|u_0\|^2_{L^1_x} \right) \cdot T + \\ &\quad + \int_{\mathbb{R}^d} u_0(x)(K * u_0)(x) dx \geq \\ &\geq \delta \cdot C_3 \cdot \|u_0\|^2_{L^1_x} + \int_{\mathbb{R}^d} u_0(x)(K * u_0)(x) dx, \end{aligned} \tag{26}$$

where C_3 is a positive constant and we require that

$$\|u_0\|_{L^1_x} \geq \frac{8 C_2 \nu}{C_1}.$$

Now if we choose $a = \frac{8C_2\nu}{C_1}$, $b = \delta \cdot C_3$ and $u_0 \in A_{\delta,a,b}$, then we have obtained a contradiction since the inequalities (26) and (15) contradict each other. Finally it is not difficult to see that the set of parameters $0 < \delta < \frac{1}{100}$, $a > 0$, $b > 0$ forms an open set for which our construction of blowing up solutions works. The theorem is proved.

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