QUANTIZATION OF KÄHLER MANIFOLDS. IV

Michel Cahen, Simone Gutt and John Rawnsley

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Abstract. We use Berezin's dequantization procedure to define a formal $*$-product on the algebra of smooth functions on the bounded symmetric domains. We prove that this formal $*$-product is convergent on a dense subalgebra of the algebra of smooth functions.

Introduction.

The notion of $*$-product as an operator-free approach to quantization was introduced in [1, 2]. The method of quantization of Kähler manifolds due to Berezin, [3], as the inverse of taking symbols of operators, has been used systematically to construct such $*$-products from geometric data in [5, 6]. The method involves introducing a suitable parameter into the Berezin composition of symbols, taking the asymptotic expansion in this parameter on a large algebra of functions and then showing that the coefficients of this expansion satisfy the cocycle conditions to define a $*$-product on the smooth functions. We have only been able to establish these cocycle conditions by showing that the asymptotic expansion actually converges on this subalgebra. This restricts the applicability of our results to the situation where we can find a large enough subalgebra on which the expansion converges.

In [5] we showed the existence of the asymptotic series for coadjoint orbits of compact Lie groups and its convergence for compact Hermitian symmetric spaces. In [6] we showed how the symbols of polynomial differential operators gave a dense subalgebra on which the asymptotic expansion converges in the case of the unit disk in $\mathbb{C}$.

In this paper we turn our attention to general Hermitian symmetric spaces of non-compact type, and use their realization as bounded domains to define an analogous algebra of symbols of polynomial differential operators. The main result of the paper is thus

Theorem. *Let $\mathcal{D}$ be a bounded symmetric domain and $\mathcal{E}$ the algebra of symbols of polynomial differential operators on a homogeneous holomorphic line bundle L over $\mathcal{D}$ which gives a realization of a holomorphic discrete series representation of G_0, then for f and g in $\mathcal{E}$ the Berezin product $f *_k g$ has an asymptotic expansion in powers of k^{-1} which converges to a rational function of k. The coefficients of the asymptotic expansion are bidifferential operators which define an invariant and covariant $*$-product on $C^\infty(\mathcal{D})$.*

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Existence and convergence of the $*$-product for bounded domains.

Let $\mathcal{D}$ denote a bounded symmetric domain. We shall use the Harish-Chandra embedding to realize $\mathcal{D}$ as a bounded subset of its Lie algebra of automorphisms. More precisely, if G_0 is the connected component of the group of holomorphic isometries then $\mathcal{D}$ is the homogeneous space G_0/K_0 where G_0 is a non-compact semi-simple Lie group and K_0 is a maximal compact subgroup. Let $\mathfrak{g}_0$ be the Lie algebra of G_0, $\mathfrak{k}_0$ the subalgebra corresponding with K_0, $\mathfrak{g}$ and $\mathfrak{k}$ the complexifications and G, K the corresponding complex Lie groups containing G_0 and K_0. The complex structure on $\mathcal{D}$ is determined by K_0-invariant abelian subalgebras $\mathfrak{m}_+$ and $\mathfrak{m}_-$ with

$$\mathfrak{g} = \mathfrak{m}_+ + \mathfrak{k} + \mathfrak{m}_-, \qquad [\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{k}, \qquad \overline{\mathfrak{m}_+} = \mathfrak{m}_-$$

where $^-$ denotes conjugation over the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. The exponential map sends $\mathfrak{m}_\pm$ diffeomorphically onto subgroups $M_\pm$ of G such that M_+KM_- is an open set in G containing G_0 and the multiplication map

$$M_+ \times K \times M_- \to M_+KM_-$$

is a diffeomorphism. KM_- is a parabolic subgroup of G and the quotient G/KM_- a generalized flag manifold. The G_0-orbit of the identity coset can be identified with G_0/K_0 and lies inside $M_+KM_-/KM_- \cong M_+$. Composing this identification with the inverse of the exponential map gives the desired Harish-Chandra embedding of $\mathcal{D}$ as a bounded open subset of $\mathfrak{m}_+$. We shall assume from now on that $\mathcal{D} \subset \mathfrak{m}_+$ via this embedding. In this realization it is clear that the action of K_0 on $\mathcal{D}$ coincides with the adjoint action of K_0 on $\mathfrak{m}_+$.

In [6] we considered the special case of the unit disk in $\mathbb{C}$. In [9, p395] it is shown that every bounded symmetric domain contains a product of 1-dimensional disks (which we call a polydisk), $\mathcal{D}_0$, which is totally geodesically embedded in $\mathcal{D}$. $\mathcal{D}_0$ is homogeneous under a subgroup of G_0 which is locally isomorphic to a product of copies of $SL(2,\mathbb{R})$. The K_0-orbit of $\mathcal{D}_0$ is the whole of $\mathcal{D}$. It is obvious that, for a disk, and hence the polydisk $\mathcal{D}_0$, if ξ is in the disk then so is any multiple $r\xi$ for $0 < r < 1$. This immediately implies:

Lemma 1. *If $0 < a < b$ then $a\mathcal{D} \subset b\mathcal{D}$. $\overline{\mathcal{D}} \subset r\mathcal{D}$, $\forall r > 1$ (here $\overline{\mathcal{D}}$ means the topological closure of $\mathcal{D}$).*

Proof. If $\xi \in \mathcal{D}$ then $\mathrm{Ad}\, k\xi \in \mathcal{D}_0$ for some $k \in K_0$. Hence $(a/b)\,\mathrm{Ad}\, k\xi \in \mathcal{D}_0$ so $a\xi \in b\mathcal{D}$. If ξ_m is a sequence in $\mathcal{D}$ converging to $\xi \in \overline{\mathcal{D}}$, then there is a sequence $k_m \in K_0$ such that $\mathrm{Ad}\, k_m\xi_m \in \mathcal{D}_0$ for all m. Since K_0 is compact, by passing to a subsequence if necessary, we can assume that k_m converges to some $k \in K_0$. Thus $\mathrm{Ad}\, k_m\xi_m$ converges to $\mathrm{Ad}\, k\xi \in \overline{\mathcal{D}_0}$. In the case of the unit disk it is obvious that its closure is contained in any disk of larger radius. Hence the lemma.

Lemma 2. $\mathcal{D} = \bigcup_{0<r<1} r\mathcal{D}$

Proof. By lemma 1, the union is contained in $\mathcal{D}$. Conversely, suppose $\xi \in \mathcal{D}$. The map $\lambda \mapsto \lambda\xi$ is a continuous map of $\mathbb{R}$ into $\mathfrak{m}_+$ and its value is in the open subset $\mathcal{D}$ when

$\lambda = 1$. Hence there is a neighbourhood of 1 where the values are in $\mathcal{D}$. Thus there is $\lambda > 1$ with $\lambda\xi \in \mathcal{D}$ and hence $\xi \in (1/\lambda)\mathcal{D}$.

Corollary. *If X is a compact subset of $\mathcal{D}$ there is an r with $0 < r < 1$ such that $X \subset r\mathcal{D}$.*

Proof. The $\{r\mathcal{D} \mid 0 < r < 1\}$ form an open covering of X. Hence a finite number cover X and, since they are nested, X is contained in one of them.

Following Satake [11] we define maps

$$k\colon \mathcal{D} \times \mathcal{D} \to K, \qquad m_\pm\colon \mathcal{D} \times \mathcal{D} \to \mathfrak{m}_\pm$$

by

$$\exp -\overline{Z'} \exp Z = \exp m_+(Z, Z')\; k(Z, Z')^{-1} \exp m_-(Z, Z'), \qquad Z, Z' \in \mathcal{D}.$$

They satisfy

$$\overline{k(Z, Z')} = k(Z', Z)^{-1}, \qquad \overline{m_+(Z, Z')} = -m_-(Z', Z).$$

In what follows the derivatives of k will be important, so we pause to prove the following lemma.

Lemma 3. *For $\xi \in \mathfrak{m}_+$ we have*

(i) $$\left.\frac{d}{dt}\right|_{t=0} k(Z + t\xi, Z')\, k(Z, Z')^{-1} = -[m_-(Z, Z'), \xi];$$

(ii) $$\left.\frac{d}{dt}\right|_{t=0} m_-(Z + t\xi, Z') = \frac{1}{2}[m_-(Z, Z'), [m_-(Z, Z'), \xi]];$$

(iii) $$\left.\frac{d}{dt}\right|_{t=0} m_-(Z, Z' + t\xi) = -\operatorname{Ad} k(Z, Z')\,\overline{\xi}.$$

Proof.

$$\begin{aligned} k(Z + t\xi, Z') &= \exp m_-(Z + t\xi, Z') \exp -t\xi \,(\exp -\overline{Z'} \exp Z)^{-1} \exp m_+(Z + t\xi, Z') \\ &= \exp m_-(Z + t\xi, Z') \exp -t\xi \exp -m_-(Z, Z') \\ &\qquad \times k(Z, Z') \exp -m_+(Z, Z') \exp m_+(Z + t\xi, Z') \end{aligned}$$

so

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} k(Z + t\xi, Z')\, k(Z, Z')^{-1} &= -\left(\operatorname{Ad} \exp m_-(Z, Z')\, \xi\right)^{\mathfrak{k}} \\ &= -[m_-(Z, Z'), \xi] \end{aligned}$$

using the fact that $\mathfrak{m}_-$ is abelian. Similarly

$$\begin{aligned} \exp m_-(Z + t\xi, Z') &= k(Z + t\xi, Z') \exp -m_+(Z + t\xi, Z') \exp -\overline{Z'} \exp Z \exp t\xi \\ &= k(Z + t\xi, Z') \exp -m_+(Z + t\xi, Z') \exp m_+(Z, Z') \\ &\qquad \times k(Z, Z')^{-1} \exp m_-(Z, Z') \exp t\xi \end{aligned}$$

so

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} m_-(Z+t\xi,Z') &= \left.\frac{d}{dt}\right|_{t=0} \exp m_-(Z+t\xi,Z')\exp -m_-(Z,Z')\\ &= (\mathrm{Ad}\exp m_-(Z,Z')\,\xi)^{\mathfrak{m}_-}\\ &= \frac{1}{2}[m_-(Z,Z'),[m_-(Z,Z'),\xi]].\end{aligned}$$

Finally

$$\begin{aligned}\exp m_-(Z,Z'+t\xi) &= k(Z,Z'+t\xi)\exp -m_+(Z,Z'+t\xi)\exp -t\overline{\xi}\exp -\overline{Z'}\exp Z\\ &= k(Z,Z'+t\xi)\exp -m_+(Z,Z'+t\xi)\exp -t\overline{\xi}\exp m_+(Z,Z')\\ &\qquad \times k(Z,Z')^{-1}\exp m_-(Z,Z')\end{aligned}$$

so

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} m_-(Z,Z'+t\xi) &= \left.\frac{d}{dt}\right|_{t=0} \exp m_-(Z,Z'+t\xi)\exp -m_-(Z,Z')\\ &= -\left(\mathrm{Ad}\,k(Z,Z')\,\mathrm{Ad}\exp -m_+(Z,Z')\,\overline{\xi}\right)^{\mathfrak{m}_-}\\ &= -\,\mathrm{Ad}\,k(Z,Z')\,\overline{\xi}.\end{aligned}$$

The holomorphic quantization.

For any unitary character χ of K_0 there is a Hermitian holomorphic line bundle L over $\mathcal{D}$ whose curvature is the Kähler form of an invariant Hermitian metric on $\mathcal{D}$. If χ also denotes the holomorphic extension to K then the Hermitian metric has Kähler potential $\log\chi(k(Z,Z))$ and L has a zero-free holomorphic section s_0 with

$$|s_0(Z)|^2 = \chi(k(Z,Z))^{-1}.$$

For χ sufficiently positive s_0 is square-integrable and the representation U of G_0 on the space $\mathcal{H}$ of square-integrable sections of L is one of Harish-Chandra's holomorphic discrete series [7,8]. s_0 is a highest weight vector for the extremal K-type so is a smooth vector for the representation. We form the coherent states e_q and see that $e_{s_0(0)}$ transforms the same way as s_0 and so they must be equal up to a multiple. This means that the coherent states are also smooth vectors of the representation. Further, since the quantization is homogeneous, ϵ will be constant. Thus

$$\langle e_{s_0(Z')}, e_{s_0(Z)}\rangle = \epsilon\,\chi(k(Z,Z')).$$

Lemma 4. *Up to a constant (determined by the normalization of Haar measure on G_0) ϵ is the formal degree d_U of the discrete series representation U.*

Proof. d_U is the positive constant such that

$$\int_{G_0} |\langle U_g v, w\rangle|^2\, dg = d_U^{-1}\|v\|^2\|w\|^2 \qquad \forall v,w\in\mathcal{H}.$$

If we take v to be the coherent state based at the identity coset eK_0 then multiplying the left-hand side of this equation by $|s_0(eK_0)|^2$ it becomes

$$\int_{G_0} |(U_{g^{-1}}w)(eK)|^2\,dg = \int_{G_0} |w(gK)|^2\,dg = c\|w\|^2$$

where c depends on the normalization of Haar measure (relative to the Riemannian volume on $\mathcal{D}$). However the right-hand side becomes

$$d_U^{-1}\epsilon\|w\|^2$$

and a comparison of the two formulas gives

$$d_U = c\epsilon.$$

Corollary. *ϵ is a polynomial function of the differential $d\chi$ of the character χ.*

Proof. This is a consequence of Harish-Chandra's formula for the formal degree d_U. See, for example, [12, §10.2.4].

The two-point function ψ is given by

$$\psi(Z,Z') = \frac{|\chi(k(Z,Z'))|^2}{\chi(k(Z,Z))\chi(k(Z',Z'))}$$

Lemma 5. *The quantization is regular in the sense of* [6].

Proof. Since we already know ϵ is constant, showing regularity amounts to establishing the property that ψ takes the value 1 only on the diagonal. We recall that ψ is G_0-invariant under simultaneous transformation of both variables, and everywhere bounded by 1. Suppose we have a pair of points where ψ has the value 1, then we can assume that one of these points is the origin in $\mathfrak{m}_+$ so that it is enough to consider solutions of $\psi(Z,0) = 1$. The stabilizer K_0 of 0 is still available to move Z around and so can move it into the polydisk $\mathcal{D}_0$. Since ψ is natural with respect to holomorphic isometries, then its restriction to $\mathcal{D}_0$ will be the product of the corresponding 2-point functions for 1-dimensional disks where we know the result to be true from [6].

Polynomial differential operators and symbols.

We let $\mathcal{A}$ denote the algebra of holomorphic differential operators on functions on $\mathcal{D}$ with polynomial coefficients. We filter $\mathcal{A}$ by both the orders of the differentiation and the degrees of the coefficients: $\mathcal{A}_{p,q}$ denotes the subspace of operators of order at most p with coefficients of degree at most q. Obviously, the composition of operators gives a map

$$\mathcal{A}_{p,q} \times \mathcal{A}_{p',q'} \to \mathcal{A}_{p+p',q+q'}.$$

The global trivialization by s_0 of the holomorphic line bundle L corresponding with the character χ allows us to transport the above operators to act on sections of L by sending $D \in \mathcal{A}$ to D^χ where

$$D^\chi(fs_0) = (Df)s_0.$$

Let $\mathcal{A}(\chi)$ denote the resulting algebra of operators on sections of L and $\mathcal{A}_{p,q}(\chi)$ the corresponding subspaces.

In this non-compact situation elements of $\mathcal{A}(\chi)$ do not define bounded operators on the Hilbert space $\mathcal{H}$, but the fact that the coherent states are smooth vectors of the holomorphic discrete series representation and that polynomials are bounded on $\mathcal{D}$ means that each operator in $\mathcal{A}(\chi)$ maps the coherent states into $\mathcal{H}$ so that it makes sense to speak of the symbols of these operators.

Lemma 6. *The analytically continued symbol $\widehat{D^\chi}(Z, Z')$ of an operator D^χ in $\mathcal{A}_{p,q}(\chi)$ is a polynomial in Z and $m_-(Z, Z')$ of bidegree p, q.*

Proof. This follows from Lemma 1 and the following observation:

$$\begin{aligned}\widehat{D^\chi}(Z, Z')s_0(Z) &= \frac{\langle D^\chi e_{s_0(Z')}, e_{s_0(Z)}\rangle}{\langle e_{s_0(Z')}, e_{s_0(Z)}\rangle} s_0(Z)\\ &= \frac{(D^\chi e_{s_0(Z')})(Z)}{\langle e_{s_0(Z')}, e_{s_0(Z)}\rangle}\end{aligned}$$

so if we set $f_{Z'} = \langle e_{s_0(Z')}, e_{s_0(Z)}\rangle/\epsilon$ then $e_{s_0(Z')} = f_{Z'}\,\epsilon\, s_0$ and hence

$$\widehat{D^\chi}(Z, Z') = \frac{(Df_{Z'})(Z)}{f_{Z'}(Z)}.$$

Thus to calculate the symbol of D^χ we apply D to the Z argument of $f_{Z'}(Z) = \chi(k(Z, Z'))$. Lemma 3 (i) tells us that applying one derivative gives us components of $m_-(Z, Z')$, and Lemma 3 (ii) that each successive derivative of this raises the degree in $m_-(Z, Z')$ by one.

Corollary. *The space of symbols of the operators in $\mathcal{A}_{k,l}(\chi)$ is the space of polynomials in Z and $m_-(Z, Z')$ of bidegree k, l so, in particular, is independent of χ.*

Proof. We know that the operation of taking symbols is an injective map, and the dimension of $\mathcal{A}_{p,q}(\chi)$ is the same as that of $\mathcal{A}_{p,q}$ which is also the same as the dimension of the space of polynomials since $\mathcal{D}$ and $\mathfrak{m}_-$ have the same dimension.

Denote by $\mathcal{E}_{p,q}$ the space of polynomials in Z and $m_-(Z, Z')$ of bidegree p, q and by $\mathcal{E}$ the union of these spaces. $\mathcal{E}$ is an algebra under pointwise multiplication – the algebra of symbols – which we shall utilize in the rest of the paper.

Remark. We have shown more in the proof of Lemma 6 than we have so far claimed, since the process outlined in the proof shows that the symbol of an operator in $\mathcal{A}_{p,q}(\chi)$ is a polynomial of degree q in the differential $d\chi$ of χ. So if we take a symbol in $\mathcal{E}_{p,q}$ then it is the symbol of an operator in $\mathcal{A}_{p,q}(\chi)$. Taking two such operators and composing them corresponds with the composition of two polynomial operators in $\mathcal{A}_{p,q}$ and so can be expressed in terms of a basis for $\mathcal{A}_{2p,2q}$ as a rational function of $d\chi$. In other words the Berezin product $f * g$ of two symbols f, g in $\mathcal{E}_{p,q}$ is a symbol in $\mathcal{E}_{2p,2q}$ depending rationally on $d\chi$.

The $*$-product.

Let us now try to follow the method of [6] to construct a formal deformation of the algebra $C^\infty(\mathcal{D})$. We do so by first constructing it on the subalgebra $\mathcal{E}$.

We consider the powers L^k of the line bundle L which correspond with the powers χ^k of χ. These powers have differentials $kd\chi$, so the Berezin product $f *_k g$ of two symbols f, g in $\mathcal{E}_{p,q}$ is a rational function of k by the results of the previous section.

In [6] we introduced the algebra $\mathcal{B} \subset C^\infty(M)$ of functions f which have an analytic continuation off the diagonal in $M \times \overline{M}$ so that $f(x, y)\psi(x, y)^r$ is globally defined, smooth and bounded on $K \times M$ and on $M \times K$ for each compact subset K of M for some positive power r and $\mathcal{B}_r$ denotes those for which the power r suffices. If we assume for the moment that the symbols in $\mathcal{E}_{p,q}$ are in one of the spaces $\mathcal{B}_r$ then, as proved in [6], we will have the existence also of the asymptotic expansion (and hence also its convergence). Since the Berezin product is associative for each k, the same argument as in [5] shows that its asymptotic expansion in k^{-1} is an associative formal deformation on $\mathcal{E}$ with bidifferential operators as coefficients. To see that it extends to all of $C^\infty(\mathcal{D})$ we need to see that $\mathcal{E}$ contains enough functions to determine these operators.

Suppose D is a differential operator of order r such that $Df = 0$ for all $f \in \mathcal{E}$. Then we can analytically continue such equations off the diagonal in $\mathcal{D} \times \mathcal{D}$ and suppose that we have a holomorphic differential operator D such that the same equation holds, where now f is an analytically continued symbol. D will have order r in each of Z and Z' and annihilates all polynomials in Z and $m_-(Z, Z')$. We now change variables from Z, Z' to Z, W where $W = m_-(Z, Z')$. We claim this is a diffeomorphism, and so will transform D into another differential operator of degree $2r$ in Z and W vanishing on all polynomials of Z and W. So it clearly must vanish.

To see that the change of variables is a diffeomorphism we check its differential near the diagonal $Z = Z'$. So let $\Phi(Z, Z') = (Z, m_-(Z, Z'))$ then, in an obvious notation, Φ has differential

$$\begin{pmatrix} 1 & \frac{\partial m_-(Z,Z')}{\partial Z} \\ 0 & \frac{\partial m_-(Z,Z')}{\partial Z'} \end{pmatrix}.$$

But Lemma 3 (iii) shows that m_- has invertible derivative in its second variable giving the following result.

Lemma 7. *If the symbol spaces $\mathcal{E}_{p,q}$ are contained in some $\mathcal{B}_r$ (where r may depend on p, q) then the resulting asymptotic expansion of $f *_k g$ has bidifferential operator coefficients which satisfy the cocycle conditions to define a formal product on $C^\infty(\mathcal{D})$ which is associative.*

It remains to verify the boundedness condition on the symbols. In fact we show that they are in $\mathcal{B}_0$. It is enough to show that $\operatorname{ad} m_-(Z, Z')$ is bounded on $X \times \mathcal{D}$ and $\mathcal{D} \times X$ for any compact subset X of $\mathcal{D}$ since the adjoint representation is injective. We follow the method of [10, lemmas 2.11, 2.12].

If we choose a basis η_i of $\mathfrak{k}_0$ which is orthonormal for $-B$, the Killing form of $\mathfrak{g}_0$ and a basis ξ_i for $\mathfrak{m}_+$ such that $B(\xi_i, \overline{\xi_j}) = \delta_{ij}$ then $\xi_1, \dots, \xi_n, \eta_1, \dots \eta_m, \overline{\xi_1}, \dots, \overline{\xi_n}$ is a basis for $\mathfrak{g}$ and for $Z \in \mathfrak{m}_+$

$$\operatorname{ad} Z\xi_i = 0, \qquad \operatorname{ad} Z\eta_i = \sum_j Z_{ji}\xi_j, \qquad \operatorname{ad} Z\overline{\xi_i} = \sum_j W_{ji}\eta_j$$

where

$$Z_{ji} = B(\overline{\xi_j}, \operatorname{ad} Z\eta_i), \qquad W_{ji} = -B(\eta_j, \operatorname{ad} Z\overline{\xi_i}).$$

Since B is invariant, it follows that $W_{ji} = Z_{ij}$ and hence the matrix of $\operatorname{ad} Z$ in this basis has the form

$$\begin{pmatrix} 0 & {}^tW & 0 \\ 0 & 0 & W \\ 0 & 0 & 0 \end{pmatrix}.$$

Exponentiating we have

$$\operatorname{Ad}\exp Z = \begin{pmatrix} 1 & {}^tW & \frac12 {}^tW\,W \\ 0 & 1 & W \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly

$$\operatorname{Ad}\exp -\overline{Z'} = \begin{pmatrix} 1 & 0 & 0 \\ -\overline{W'} & 1 & 0 \\ \frac12 {}^t\overline{W'}\,\overline{W'} & -{}^t\overline{W'} & 1 \end{pmatrix}$$

so

$$\operatorname{Ad}\exp -\overline{Z'}\exp Z$$
$$= \begin{pmatrix} 1 & {}^tW & \frac12 {}^tW\,W \\ -\overline{W'} & 1-\overline{W'}\,{}^tW & W-\frac12\overline{W'}\,{}^tW\,W \\ \frac12 {}^t\overline{W'}\,\overline{W'} & -{}^t\overline{W'}+\frac12 {}^t\overline{W'}\,\overline{W'}\,{}^tW & 1-{}^t\overline{W'}W+\frac14 {}^t\overline{W'}\,\overline{W'}\,{}^tWW \end{pmatrix}.$$

If $Z, Z' \in \mathcal{D}$ then

$$\operatorname{ad} m_+(Z,Z') = \begin{pmatrix} 0 & {}^tU & 0 \\ 0 & 0 & U \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{ad} m_-(Z,Z') = \begin{pmatrix} 0 & 0 & 0 \\ \overline{V} & 0 & 0 \\ 0 & {}^t\overline{V} & 0 \end{pmatrix},$$

$$\operatorname{Ad} k(Z,Z') = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

then

$$\operatorname{Ad}\exp m_+(Z,Z')k(Z,Z')\exp m_-(Z,Z')$$
$$= \begin{pmatrix} A+{}^tUB\overline{V}+\frac14 {}^tUUC\,{}^t\overline{V}\,\overline{V} & {}^tUB+\frac12 {}^tUUC\,{}^t\overline{V} & \frac12 {}^tUUC \\ B\,\overline{V}+\frac12 UC\,{}^t\overline{V}\,\overline{V} & B+UC\,{}^t\overline{V} & UC \\ \frac12 C\,{}^t\overline{V}\,\overline{V} & C\,{}^t\overline{V} & C \end{pmatrix}$$

and thus

$$C = 1-{}^t\overline{W'}W+\tfrac14 {}^t\overline{W'}\,\overline{W'}\,{}^tWW, \qquad C\,{}^t\overline{V} = -{}^t\overline{W'}+\tfrac12 {}^t\overline{W'}\,\overline{W'}\,{}^tW.$$

It follows that the boundedness of V (and hence m_-) is determined by that of C^{-1}. Observe that C is unchanged if we replace Z by rZ and Z' by $(1/r)Z'$. If $X \subset \mathcal{D}$ is compact we know from the corollary to lemma 2 that $X \subset r\mathcal{D}$ for some $0 < r < 1$ and hence that C^{-1} is defined on $X \times (1/r)\mathcal{D}$ which contains the compact set $X \times \overline{\mathcal{D}}$. Hence C^{-1} is bounded on $X \times \mathcal{D}$. C is symmetrical in Z and Z' so we have the boundedness we need to prove the following theorem:

Theorem. *Let $\mathcal{D}$ be a bounded symmetric domain and $\mathcal{E}$ the functions on $\mathcal{D}$ which are polynomials in Z and $m_-(Z,Z)$, then for f and g in $\mathcal{E}$ the Berezin product $f *_k g$ has an asymptotic expansion in powers of k^{-1} which converges to a rational function of k. The coefficients of the asymptotic expansion are bidifferential operators which define an invariant and covariant $*$-product on $C^\infty(\mathcal{D})$.*

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(Michel Cahen and Simone Gutt) Département de Mathématiques, Université Libre de Bruxelles, Campus Plaine CP 218, 1050 Bruxelles, Belgique

E-mail address: Simone Gutt: sgutt@ulb.ac.be

(Simone Gutt) Département de Mathématiques, Université de Metz, Ile du Saulcy, F-57045 Metz Cedex 01, France

E-mail address: gutt@poncelet.univ-metz.fr

(John Rawnsley) Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

E-mail address: jhr@maths.warwick.ac.uk