Symmetric symplectic spaces with Ricci-type curvature

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Abstract

We determine the isomorphism classes of symmetric symplectic manifolds of dimension at least 4 which are connected, simply-connected and have a curvature tensor which has only one non-vanishing irreducible component – the Ricci tensor.

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Moshé Flato has been a close and wonderful friend and an inspiration for us for more than twenty years. This contribution is dedicated to him, always present in our hearts.

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1 Introduction

On any symplectic manifold (M,ω) the space of symplectic connections (linear connections ∇ with vanishing torsion and such that $\nabla\omega = 0$) is infinite dimensional. In order to select a smaller family of symplectic connections, a variational principle was introduced in [2]. This principle has Euler-Lagrange equations

$$(\nabla_X r)(Y,Z) + (\nabla_Y r)(Z,X) + (\nabla_Z r)(X,Y) = 0 \tag{1}$$

for all vector fields X, Y, Z; r denotes the Ricci tensor of ∇

$$r(X,Y) = \mathrm{Tr}(Z \mapsto R(X,Z)Y).$$

In [2] the case where $\dim M = 2$ was examined in complete detail so we shall assume throughout the $\dim M \geq 2$.

It was observed in [3] that the field equations (1) are identically satisfied if one assumes that the irreducible component of the curvature, denoted there by W (see also [5]), vanishes

$$W = 0. \tag{2}$$

The tensor W is the symplectic analogue of the Weyl or conformal curvature of a Riemannian connection. The vanishing of W (equation (2)) is equivalent to the requirement that the curvature tensor R of ∇ is expressed in terms of its Ricci tensor by

$$\begin{aligned} R(X,Y)Z \;=\; & \frac{1}{2(n+1)}\Big[2\omega(X,Y)AZ + \omega(X,Z)AY - \omega(Y,AZ)X \\ & -\omega(Y,Z)AX + \omega(X,AZ)Y\Big] \end{aligned} \tag{3}$$

where $\dim M = 2n$, $n \geq 2$, where X, Y, Z are vector fields and where A is the Ricci tensor viewed as an endomorphism of the tangent bundle using ω:

$$r(X,Y) = \omega(X, AY). \tag{4}$$

The Ricci tensor is symmetric so A is an infinitesimal symplectic endomorphism of each tangent space.

Equations (2) (or (3)) imply the existence of a 1-form u on (M,ω) such that

$$(\nabla_X r)(Y,Z) = \omega(X,Y)u(Z) + \omega(X,Z)u(Y). \tag{5}$$

If $u = 0$, then $\nabla r = 0$ and since R is expressed in terms of r (3), ∇ is locally symmetric.

The condition $W = 0$ also appears as the integrability condition for the almost complex structure naturally defined from a symplectic connection on (M,ω) on the manifold $\mathcal{J}(M)$ of almost complex stuctures on M which are compatible with ω.

In this note we prove, amongst other things, the following two results.

Theorem 1 *Let* $(M,\omega) = (M_1,\omega_1)\times(M_2,\omega_2)$ *be symplectic manifolds of dimension greater than zero and* $\nabla = \nabla_1 + \nabla_2$ *be a symplectic connection. If* $W^{\nabla} = 0$ *then* $\nabla, \nabla_1, \nabla_2$ *are flat.*

Theorem 2 *Let* (M,ω,s) *be a connected, simply-connected, symmetric symplectic space of dimension* $2n(\geq 4)$*; let* ∇ *be its canonical invariant symplectic connection and let* r *be its Ricci tensor; let* A *be the corresponding endomorphism*

$$\omega(X, AY) = r(X,Y).$$

Assume $W^{\nabla} = 0$*. Then*

$$A^2 = \lambda\,\mathrm{Id}$$

for some real number λ*.*

If $\lambda \neq 0$*, the transvection group* G *of* (M,ω,s) *is semisimple and, up to coverings,* $M = G/K$ *with either* $G = SL(n+1,\mathbb{R})$ *and* $K = GL(n,\mathbb{R})$ *or* $G = SU(p+1,q)$ *and* $K = U(p,q)$ *where* $\dim M = 2n$*,* $p+q=n$*.*

If $\lambda = 0$ *and* $\mathrm{Rank}(A) > 1$*, the transvection group* G *of* (M,ω,s) *is neither solvable nor semisimple. The radical of* G *is 2-step unipotent if* $\mathrm{Rank}(A) < n$ *and abelian in* $\mathrm{Rank}(A) = n$*. If* $\lambda = 0$ *and* $\mathrm{Rank}(A) = 1$*, the transvection group* G *of* (M,ω,s) *is solvable.*

2 Proof of Theorem 1

Let $(M,\omega) = (M_1,\omega_1)\times(M_2,\omega_2)$ be symplectic manifolds and $\nabla = \nabla_1+\nabla_2$ be a symplectic connection. Then $R(X,Y)Z = R_1(X_1,Y_1)Z_1 + R_2(X_2,Y_2)Z_2$ where $X = X_1+X_2$, $Y = Y_1+Y_2$, $Z = Z_1+Z_2$ and suffices indicate components tangent to M_1 and M_2, respectively. Then also $r(X,Y) = r_1(X_1,Y_1) + r_2(X_2,Y_2)$. On the other hand, the relation between W, W_1 and W_2 involves cross terms $C(X,Y)Z$:

$$W(X,Y)Z = W_1(X_1,Y_1)Z_1 + W_2(X_2,Y_2)Z_2 + C(X,Y)Z.$$

These can be read off equation (3). Then $W=0$ implies $W_1 = 0$, $W_2 = 0$ and $C = 0$. We have

$$C(X_1,Y_1)Z_2 = \frac{1}{2(n+1)}\left[-2\omega(X_1,Y_1)A_2Z_2\right]$$

so $A_2 = 0$ and interchanging 1 and 2 we see also $A_1 = 0$. Thus $r_1 = 0$ and $r_2 = 0$, and hence $R_1 = 0$ and $R_2 = 0$.

3 Some facts about symmetric symplectic spaces

Affine symmetric spaces are studied in Loos [4], symplectic symmetric spaces are studied in Bieliavsky [1].

Definition 3 A **symmetric symplectic manifold** is a triple (M, ω, s) where M is a smooth connected manifold, where ω is a smooth symplectic form on M and where s is a smooth map $M \times M \to M$, $(x, y) \mapsto s_x(y)$, such that:

(i) for each x in M, s_x is an involutive symplectic diffeomorphism of (M, ω) (called the symmetry at x) and x is an isolated fixed point of s_x,

(ii) $s_x s_y s_x = s_{s_x(y)}$ for all x, y in M.

The **transvection group** G of (M, ω, s) is the group generated by products of an even number of symmetries.

We recall below some general facts about symmetric spaces ([4], [1]).

(1) (M, ω, s) has a unique connection ∇ such that $\nabla\omega = 0$ and such that each symmetry s_x is an affine transformation of (M, ∇). Observe that $s_{x*x} = -\mathrm{Id}_{T_xM}$ because $(s_{x*x})^2 = \mathrm{Id}_{T_xM}$ and x is an isolated fixed point of s_x. Since $\omega_x(\nabla_X Y, Z) = \frac{1}{2}(\omega_x(\nabla_X Y, Z) + (s_x{}^*\omega)_x(\nabla_X Y, Z))$, the connection is given by

$$\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x(\omega(Y + s_x \cdot Y, Z)) \tag{6}$$

for $x \in M$, where X, Y, Z are vector fields on M and $(s_x \cdot Y)_y = s_{x*} Y_{s_x(y)}$. This connection ∇ has no torsion and is thus a symplectic connection. The symmetry s_x coincides with the geodesic symmetry around x, since an affinity is determined by its 1-jet at a point.

(2) The automorphism group $\mathrm{Aut} = \mathrm{Aut}(M, \omega, s)$ of (M, ω, s) is the set of symplectic automorphisms φ of (M, ω) such $\varphi \circ s_x = s_{\varphi(x)} \circ \varphi$, $\forall x \in M$. It is the intersection of the affine group of (M, ∇) and the symplectic diffeomorphism group of (M, ω). It is thus a Lie group containing the transvection group so acts transitively on M (since any two points in M can be joined by a broken geodesic).

Choose a base point o in M. Denote by $\widetilde{\sigma}$ the conjugation by the symmetry s_o, it is an involutive automorphism of Aut.

Let K' denote the stabilizer of o in Aut and let $A^{\widetilde{\sigma}}$ (respectively $A^{\widetilde{\sigma}}_o$) denote the group of fixed points of $\widetilde{\sigma}$ in Aut (respectively its connected component). Then $A^{\widetilde{\sigma}} \supseteq K' \supseteq A^{\widetilde{\sigma}}_o$.

Hence, if $\mathfrak{a}$ (respectively $\mathfrak{k}'$) is the Lie algebra of Aut (respectively K') and if $\sigma = \widetilde{\sigma}_{*Id}$, then $\mathfrak{k}'$ is the subalgebra of $\mathfrak{a}$ of fixed points of σ.

(3) Let $\mathfrak{p} = \{X \in \mathfrak{a} \mid \sigma(X) = -X\}$. Then $\mathfrak{a} = \mathfrak{k}' \oplus \mathfrak{p}$.

Denote by π' the projection $\mathrm{Aut} \to M$ given by $\pi'(g) = g \cdot o$. Then $\pi'_{*e}|_{\mathfrak{p}}\colon \mathfrak{p} \to T_oM$ is a linear isomorphism which identifies the tangent space T_oM with $\mathfrak{p}$.

Denote by $\mathrm{Exp}\colon T_oM \to M$ the exponential map given by the connection ∇ at the point o and by exp the exponential map from the Lie algebra $\mathfrak{a}$ to the Lie group Aut.

Observe that $s_{\mathrm{Exp}\,\frac{t}{2}X}\, s_o$, $X \in T_oM$, is an affinity in G which realises the parallel transport along $\mathrm{Exp}\, tX$, since $s_{\mathrm{Exp}\, uX*}$ for any $u \in \mathbb{R}$ maps a vector field which is parallel along the geodesic $\mathrm{Exp}\, tX$ to another such parallel vector field. Hence $s_{\mathrm{Exp}\,\frac{t}{2}\pi'_{*e}X}\, s_o = \exp tX$, $\forall X \in \mathfrak{p}$.

It follows that the transvection group G, which is stable by $\widetilde{\sigma}$, is the connected Lie subgroup of $\mathrm{Aut}(M, \omega, s)$ whose Lie algebra is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \text{where} \qquad \mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]. \tag{7}$$

Indeed, if G_1 denotes that subgroup, clearly by the above $G_1 \subset G$ and the parallel transport along a geodesic $\mathrm{Exp}\, tX$ is in G_1, but then any $x \in M$ can be written as $x = g \cdot o$ for $g \in G_1$ hence $s_x s_o = g s_o g^{-1} s_o = g\widetilde{\sigma}(g^{-1}) \in G_1$ and $G \subset G_1$.

Let K denote the stabilizer of o in G. Its Lie algebra is $\mathfrak{k}$ and $\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$. Since the Lie group G acts effectively on M, the representation of K on T_oM, $k \mapsto k_{*o}$, is faithful so $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$.

(4) Denote by π the projection $\pi\colon G \to M$ where $\pi(g) = g \cdot o$. Denote by X^* the vector field on M which is the image under π_* of the right invariant vector field on G, i.e. ${X^*}_{g\cdot o} = \frac{d}{dt} \exp tX \cdot g \cdot o|_{t=0}$. Observe that $[X^*, Y^*] = -[X, Y]^*$. Since ω is invariant under G, formula (6) yields $\omega_x(\nabla_{Y^*}X^*, Z^*) = \frac{1}{2}\omega_x([Y^*, X^* + s_x \cdot X^*], Z^*)$ so $(\nabla_{X^*}Y^*)_x = [X^*, Y^*] + \frac{1}{2}[Y^*, X^* + s_x \cdot X^*]$. But $s_{g\cdot o} \cdot X^* = g \cdot s_o \cdot g^{-1} \cdot X^* = (\mathrm{Ad}\, g \sigma(\mathrm{Ad}\, g^{-1} X))^*$ so the connection has the form

$$(\nabla_{X^*}Y^*)_{g\cdot o} = ([Y, \mathrm{Ad}\, g(\mathrm{Ad}\, g^{-1}X)_{\mathfrak{p}}])^*_{g\cdot o} \tag{8}$$

where $Z_{\mathfrak{p}}$ denotes the component in $\mathfrak{p}$ of $Z \in \mathfrak{g}$ relatively to the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and where $[\ ,\ ]$ is the bracket in $\mathfrak{g}$.

Since any G-invariant tensor on M is parallel, the curvature tensor of (M, ∇) is parallel $(\nabla R = 0)$ and if X, Y, Z belong to $\mathfrak{p}$, one has,

$$R_o(X^*_o, Y^*_o)Z^*_o = -([[X, Y], Z])^*_o. \tag{9}$$

Definition 4 A **symmetric symplectic triple** is a triple $(\mathfrak{g}, \sigma, \Omega)$ where $\mathfrak{g}$ is a finite dimensional real Lie algebra, σ is an involutive automorphism of $\mathfrak{g}$ such that if we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\sigma = \mathrm{Id}_{\mathfrak{k}} \oplus -\mathrm{Id}_{\mathfrak{p}}$, then

- $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$;
- the action of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful

and where Ω is a non degenerate skewsymmetric 2-form on $\mathfrak{p}$, invariant by $\mathfrak{k}$ under the adjoint action.

We have seen above that to any connected symmetric symplectic manifold (M, ω, s), when one chooses a base point $o \in M$, one associates a symmetric symplectic triple $(\mathfrak{g}, \sigma, \Omega)$ with $\mathfrak{g}$ the Lie algebra of its transvection group, with σ the differential at the identity of the conjugation by the symmetry s_o and with $\Omega = \omega_o$ with the identification between T_oM and $\mathfrak{p}$.

Reciprocally, given a symmetric symplectic triple $(\mathfrak{g}, \sigma, \Omega)$, one builds a simply-connected symmetric symplectic space (M, ω, s) with $M = G/K$ where G is the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and K is its connected subgroup with Lie algebra $\mathfrak{k}$, with ω the G-invariant 2-form on M whose value at eK is given by Ω (identifying $T_{eK}M$ and $\mathfrak{p}$ via the differential of the canonical projection $\pi: G \rightarrow G/K$) and with symmetries defined by $s_{\pi(g)}\pi(g') = \pi(g\widetilde{\sigma}(g^{-1}g'))$ where $\widetilde{\sigma}$ is the automorphism of G whose differential at e is σ.

4 Proof of Theorem 2

Consider a symmetric symplectic space (M, ω, s) and assume that its canonical invariant symplectic connection ∇ has a curvature with $W = 0$.

Since $\nabla R = 0$, the Ricci tensor r and its associated endomorphism A (where $r(X, Y) = \omega(X, AY)$) are covariantly constant and hence A commutes with the curvature endomorphisms

$$AR(X, Y) = R(X, Y)A.$$

This implies, when we substitute R by its expression in terms of A into equation (3)

$$-\omega(X, Z)A^2Y + \omega(Y, Z)A^2X = \omega(Y, A^2Z)X - \omega(X, A^2Z)Y.$$

If $Y \neq 0$ is arbitrary, $Z = Y$, and we pick X so that $\omega(X, Y) = 1$, then $\omega(Y, A^2Y) = \omega(AY, AY) = 0$, so $A^2Y = \lambda_Y Y$ for some function λ_Y. Substituting back into the equation shows that $\lambda_Y = \lambda$ is independent of Y, and since A is covariant constant, λ must be constant.

Remark that if $\lambda \neq 0$ then r is a non-degenerate parallel symmetric bilinear form so ∇ is its Levi-Civita connection and (M, r, s) is a pseudo-Riemannian symmetric space.

Let G be the transvection group of our symmetric symplectic space. Choose a base point $o \in M$ and let $(\mathfrak{g}, \sigma, \Omega)$ be the symmetric triple associated to (M, ω, s). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of the Lie algebra of G into the $+1$ and -1 eigenspaces of σ. Then $\Omega(X, Y) = \omega_o(X^*_o, Y^*_o)$ and with a slight abuse of notations we denote by R the map $R: \mathfrak{p} \times \mathfrak{p} \rightarrow \mathrm{End}(\mathfrak{p})$ so that $(R(X,Y)Z)^*_o = R_o(X^*_o, Y^*_o)Z^*_o$ and by A the map $A\colon \mathfrak{p} \to \mathfrak{p}$ so that $(A(X))^*_o = A_o(X^*_o)$. Since $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$, we view $\mathfrak{k}$ as a subset of $\operatorname{End}(\mathfrak{p})$; by formula (9),

$$\mathfrak{k} = \{R(X,Y) \in \operatorname{End}(\mathfrak{p}) \mid X, Y \in \mathfrak{p}\} \tag{10}$$

and the brackets on $\mathfrak{g} \subset \mathfrak{p} \oplus \operatorname{End}(\mathfrak{p})$ are

$$[(C,X),(D,Y)] = ([C,D] - R(X,Y), CY - DX) \tag{11}$$

where $C, D \in \mathfrak{k} \subset \operatorname{End}(\mathfrak{p})$, and $X, Y \in \mathfrak{p}$.

Define the 1-form on $\mathfrak{p}$ corresponding to a vector $X \in \mathfrak{p}$ by $\underline{X} = i(X)\Omega$. Formula (3) giving the curvature when $W = 0$ is equivalent to

$$R(X,Y) = k\,(2\Omega(X,Y)A + AY \otimes \underline{X} - AX \otimes \underline{Y} + X \otimes \underline{AY} - Y \otimes \underline{AX})$$

where $k = 1/(m+2)$ if $m = \dim M = 2n$. Note that, for a symplectic symmetric space built from a Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ whose bracket of $\mathfrak{p}$ into $\mathfrak{k}$ is given by this formula, then the canonical connection will have curvature given by this formula and hence W will vanish.

Define $B = Y \otimes \underline{X} - X \otimes \underline{Y}$. Clearly B satisfies $\Omega(U, BV) = \Omega(BU, V)$ and any antisymplectic endomorphism of $\mathfrak{p}$ can be written as a sum of such operators. Then

$$R(X,Y) = k(\operatorname{Tr}(B)A + AB + BA)$$

and, if we put $B' = k(B + \frac{1}{2}\operatorname{Tr}(B)I)$, the RHS becomes $C = AB' + B'A$.

Lemma 5 *For any λ,*

$$\mathfrak{k} = \{C = AB + BA \mid B \in \operatorname{End}(\mathfrak{p}) \text{ and } \Omega(X, BY) = \Omega(BX, Y)\}.$$

If $\lambda \neq 0$ then $\mathfrak{k}$ is the set of endomorphisms $C \in \operatorname{End}(\mathfrak{p})$ which are infinitesimally symplectic and commute with A.

Proof The first part follows from the considerations above and the fact that the map $B \mapsto B + \frac{1}{2}\operatorname{Tr}(B)$ is a bijection on the space of antisymplectic endomorphisms of $\mathfrak{p}$. C commutes with A since $AC = \lambda B' + AB'A = CA$. Also $\Omega(X, CY) = -\Omega(AX, B'Y) + \Omega(B'X, AY) = -\Omega(B'AX, Y) - \Omega(AB'X, Y) = -\Omega(CX, Y)$.

Conversely, if $\lambda \neq 0$, given C commuting with A and such that $\Omega(X, CY) = -\Omega(CX, Y)$, let $B = \frac{1}{2}\lambda^{-1}AC$; then

$$BA + AB = \frac{1}{2}\lambda^{-1}2\lambda C = C.$$

□

4.1 Case $\lambda > 0$

Write $\lambda = a^2$, $a > 0$. Then $\mathfrak{p} = V^+ \oplus V^-$ where $V^\pm = \{X \in \mathfrak{p} \mid AX = \pm aX\}$. Let $P^\pm$ be the projection onto $V^\pm$. Then $A = a(P^+ - P^-)$. Clearly

$$\begin{aligned} \omega(V^+, V^+) &= \omega(V^-, V^-) = 0 \\ R(V^+, V^+) &= R(V^-, V^-) = 0 \\ R(X, Y) &= 2ka(\Omega(X, Y)(P^+ - P^-) - Y \otimes \underline{X} - X \otimes \underline{Y}), \end{aligned}$$

for $X \in V^+$, $Y \in V^-$. It follows that $V^\pm$ are Lagrangian subspaces of $\mathfrak{p}$. Identifying V^- with $(V^+)^*$ via $Y \mapsto \underline{Y}|_{V^+}$ and renaming V^+ as V, we have identified $\mathfrak{p}$ with $V \oplus V^*$ with its standard symplectic structure $\Omega(X + \xi, X' + \xi') = -\langle X, \xi' \rangle + \langle X', \xi \rangle$, and A acts as $+a$ on V, $-a$ on V^*. With this notation the curvature has the form

$$R(X, \xi) = 2ak(-\langle X, \xi \rangle(\mathrm{Id}_V - \mathrm{Id}_{V^*}) + \xi \otimes X - X \otimes \xi).$$

The symplectic centraliser of A can then be identified with $\mathrm{End}(V) = \mathfrak{gl}(V)$, identifying the element in $\mathrm{End}(\mathfrak{p}) = \mathrm{End}(V \oplus V^*)$ given by

$$\begin{pmatrix} C & 0 \\ 0 & -{}^tC \end{pmatrix}$$

with the element $C \in \mathfrak{gl}(V)$.

So $\mathfrak{k} = \mathfrak{gl}(V)$ and as a vector space $\mathfrak{g} = \mathfrak{gl}(V) \oplus V \oplus V^*$ with the brackets

$$\begin{aligned} [(C, X, \xi), (C', X', \xi')] \quad = \quad & ([C, C'] + 2ka(\langle X, \xi' \rangle - \langle X', \xi \rangle)I \\ & +2kaX \otimes \xi' - 2kaX' \otimes \xi, \\ & CX' - C'X, -{}^tC\xi' + {}^tC'\xi). \end{aligned}$$

The map $j \colon \mathfrak{g} \to \mathfrak{sl}(V \oplus \mathbb{R})$ given by

$$j(C, X, \xi) = \begin{pmatrix} C - 2k\,\mathrm{Tr}(C)I & sX \\ s\,{}^t\xi & -2k\,\mathrm{Tr}(C) \end{pmatrix}$$

has the brackets above provided $s^2 = 2ka$.

Thus when $\lambda > 0$, $M = G/K$ where $G = SL(n+1, \mathbb{R})$, $K = GL(n, \mathbb{R})$. The involution σ is given by

$$\sigma \begin{pmatrix} C & v \\ \xi & -\mathrm{Tr}(C) \end{pmatrix} = \begin{pmatrix} C & -v \\ -\xi & -\mathrm{Tr}(C) \end{pmatrix}$$

and, writing (X, ξ) for $\begin{pmatrix} 0 & X \\ \xi & 0 \end{pmatrix}$, the symplectic form is given by

$$\Omega\left((X, \xi), (X', \xi')\right) = -\langle X, \xi' \rangle + \langle X', \xi \rangle.$$

The curvature of the canonical connection on this symplectic symmetric space at the base point eK is

$$R((X,\xi),(X',\xi'))(X'',\xi'') = (X''(\langle X',\xi\rangle - \langle X,\xi'\rangle) - X\langle X'',\xi'\rangle$$
$$+X'\langle X'',\xi\rangle\,,\xi'\langle X,\xi''\rangle - \xi\langle X',\xi''\rangle - \xi''(\langle X',\xi\rangle - \langle X,\xi'\rangle))$$

$$r((X,\xi),(X',\xi')) = (n+1)(\langle X,\xi'\rangle + \langle X',\xi\rangle)$$

$$A(x,\xi) = (n+1)(x,-\xi)$$

and formula (3) holds so R is of Ricci-type.

4.2 Case $\lambda < 0$

We write $\lambda = -b^2$ where $b < 0$. If we put $J = b^{-1}A$ then J defines a complex structure on the vector space $\mathfrak{p}$. We write V for $\mathfrak{p}$ viewed as an n-dimensional complex vector space. V has a (pseudo-)Hermitean structure given by

$$\langle X,Y\rangle = \Omega(X,JY) + i\Omega(X,Y)$$

which is $\mathbb{C}$-linear in the second variable. The infinitesimally symplectic transformations which commute with A, or equivalently J, are the complex linear transformations of V which are skew-Hermitean with respect to this Hermitean structure. Thus $\mathfrak{k}$ is the (pseudo-) unitary Lie algebra $\mathfrak{u}(V,\langle\,,\rangle)$.

The curvature has the form

$$R(X,Y) = kb\,(2\Omega(X,Y)J + Y\otimes\langle X,.\rangle - X\otimes\langle Y,.\rangle)\,.$$

Then $\mathfrak{g} = \mathfrak{u}(V,\langle\,,\rangle)\oplus V$ with bracket

$$\begin{aligned}[(C,X),(C',X')] &= ([C,C'] + kb(X\otimes\langle X',.\rangle - X'\otimes\langle X,.\rangle \\ &\qquad -2\Omega(X,X')J), CX' - C'X).\end{aligned}$$

and $\mathfrak{g}$ can be identified with $\mathfrak{su}(V\oplus\mathbb{C},\langle\langle\,,\rangle\rangle)$ via

$$j(C,X) = \begin{pmatrix} C - 2k\,\mathrm{Tr}(C)I & sX \\ -\overline{s}\langle X,.\rangle & -2k\,\mathrm{Tr}(C)\end{pmatrix}$$

with

$$\langle\langle(v,r),(w,t)\rangle\rangle = \langle v,w\rangle + \overline{r}t$$

provided

$$s\overline{s} = -kb.$$

Hence when $\lambda < 0$ then $M = G/K$ with $\mathfrak{g} = \mathfrak{su}(p+1, q)$, $p + q = n$, $\mathfrak{k} = \mathfrak{u}(p, q)$,

$$\sigma \begin{pmatrix} C & v \\ -\langle v, . \rangle & -\operatorname{Tr}(C) \end{pmatrix} = \begin{pmatrix} C & -v \\ \langle v, . \rangle & -\operatorname{Tr}(C) \end{pmatrix}$$

and

$$\Omega(v, w) = \operatorname{Im}\langle\langle v, w \rangle\rangle.$$

The curvature of the canonical connection on this symmetric symplectic space at eK is

$$R(v, w)z = v\langle w, z\rangle - w\langle v, z\rangle + z(-\langle v, w\rangle + \langle w, v\rangle)$$

$$r(v, z) = -2(n+1)\langle v, z\rangle$$

$$A(v) = -2(n+1)iv$$

and formula (3) holds so R is of Ricci-type.

4.3 Case $\lambda = 0$

In this case A is nilpotent since $A^2 = 0$. Let $Z = \operatorname{Image} A$ and $\widetilde{Z} = \operatorname{Ker} A$. Then $Z \subset \widetilde{Z}$, and Z and $\widetilde{Z}$ are symplectic orthogonals of each other. If V denotes a complement for Z in $\widetilde{Z}$, then the restriction of Ω to V is non-degenerate. Z is contained in the Ω-orthogonal of V; let Z' be a complement so that $V^\perp = Z \oplus Z'$. $V^\perp$ is a symplectic subspace and Z is maximal isotropic so we can also suppose that Z' is maximal isotropic. Ω gives a duality of Z with Z'.

In other words, we have written $\mathfrak{p}$ as $Z \oplus Z^* \oplus V$ where $Z \oplus Z^*$ has its standard symplectic structure and V is a symplectic vector space. A is non-zero only on Z^* and maps it isomorphically onto Z, and as such it is symmetric. In block form, the symplectic structure Ω is given by

$$\begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & J' \end{pmatrix}$$

and A by

$$\begin{pmatrix} 0 & A' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where A', by a suitable choice of basis is diagonal with ± 1 on the diagonal. An easy calculation shows that matrices of the form $AB + BA$ with $\Omega(X, BY) = \Omega(BX, Y)$ have the form

$$\begin{pmatrix} K & L & -{}^tMJ' \\ 0 & -{}^tK & 0 \\ 0 & M & 0 \end{pmatrix}$$

where ${}^tKA' + A'K = 0$, ${}^tL = L$. The matrices with $K = 0$ form an ideal which is 2-step nilpotent (abelian when $\operatorname{Rank} A = n = \frac{1}{2}\dim M$) and the matrices with $L = M = 0$ a subalgebra isomorphic to $\mathfrak{so}(p,q)$, where $p+q = r = \operatorname{Rank} A$, p the number of $+$'s and q the number of $-$'s in A' (hence (p,q) is the signature of the non degenerate symmetric bilinear form naturally induced on $\mathfrak{p}/\operatorname{Ker} A$ by the Ricci tensor $\Omega(X, AY)$).

The bracket of $\mathfrak{p}$ into $\mathfrak{k}$ is given, using formulas (11) and (3) by

$$\left[\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}\right] = -k \begin{pmatrix} \tilde{K} = A'(v'^t v - v^t v') & \tilde{L} & -t\tilde{M}J') \\ 0 & -{}^t\tilde{K} & 0 \\ 0 & \tilde{M} & 0 \end{pmatrix}$$

where $\tilde{L} = A'B + {}^tBA' + 2(\operatorname{Tr} B + {}^twJ'w')A'$ with $B = v^tu' - v'^tu$ and $\tilde{M} = -{}^t(A'(v'^tw - v^tw'))$.

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \{(K, L, M, u, v, w) \mid K \in \mathfrak{so}(p,q), L \in \operatorname{Mat}(r \times r, \mathbb{R}), {}^tL = L, M \in \operatorname{Mat}(2n-2r \times r, \mathbb{R}), u \in Z = \mathbb{R}^r, v \in Z^*, w \in W = \mathbb{R}^{2n-2r}\}$. The brackets are given, with obvious notations, by

$$[(K, L, M), (K', L', M')] = ([K, K'], L'', -M^tK' + M'^tK)$$

where $L'' = KL' - L^tK' - K'L + L'^tK - {}^tMJ'M' + {}^tM'J'M$,

$$[(K, L, M), (u, v, w)] = (Ku + Lv - {}^tMJ'w, -{}^tKv, Mv),$$

$$[(u, v, w), (u', v', w')] = (-kA'(v'^tv - v^tv'), -k\tilde{L}, k\,{}^t(A'(v'^tw - v^tw')))$$

where $\tilde{L}$ is defined as above.

We can combine $\mathfrak{so}(p,q)$ with Z^* to give $\mathfrak{so}(p, q+1)$ via

$$(K, v) \mapsto \begin{pmatrix} K & -k^{1/2}A'v \\ -k^{1/2}\,{}^tv & 0 \end{pmatrix}.$$

The subset $\mathfrak{r} = \{(0, L, M, u, 0, w) \in \mathfrak{g}\}$ is a 2-step nilpotent ideal of $\mathfrak{g}$ (abelian when $r = n$ i.e. when the rank of the Ricci tensor is half the dimension of the manifold). Hence, when $p + q = r > 1$, $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and the semisimple Levi factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p, q+1)$.

5 Some corollaries

Corollary 6 *Let* (M_i, ω_i, s_i)*,* $i = 1, 2$ *be symmetric symplectic spaces of the same dimension* $2n$ *with* $W_i = 0$ *with semisimple transvection groups* G_i*. Then* $G_1^{\mathbb{C}} = G_2^{\mathbb{C}}$*.*

Proof $SL(n+1, \mathbb{R})$ and $SU(p+1, q)$ both have $SL(n+1, \mathbb{C})$ as complexification. □

Corollary 7 *Let* (M, ω, s) *be a compact, simply-connected symmetric symplectic space of dimension* $2n$ *such that* $W = 0$ *then* (M, ω, s) *is* $\mathbb{P}_n(\mathbb{C})$*.*

Proof This follows immediately from the list in Theorem 2. The only case where G/K is compact is when $G = SU(n+1)$ and $K = U(n)$. □

In dimension 4 we have the following list of possibilities (up to coverings) for M:

- $SL(3, \mathbb{R})/GL(2, \mathbb{R})$;
- $SU(1, 2)/U(2)$;
- $SU(2, 1)/U(1, 1)$;
- $SU(3)/U(2)$;
- $\lambda = 0$ cases corresponding to:
  - Rank $A = 1$, $p = 0$ or $p = 1$;
  - Rank $A = 2$, $p = 0$, $p = 1$ or $p = 2$.

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