MODULAR FORMS OVER NUMBER FIELDS

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Introduction

These notes were originally written by the first author (LD) as a short survey of the algorithm he developed in [6, 7] in order to compute with Hilbert modular forms over real quadratic fields. Unlike classical modular forms, computational results on Hilbert modular forms are still very limited. One goal is to create a database of Hilbert modular forms over real quadratic fields in order to compensate for that lack of numerical results. In originally writing these notes for the MSRI graduate summer school, he hoped to motivate students to join this long term project. The notes have been revised for a joint course given in October–December 2009 by both authors at the University of Warwick for the Taught Course Centre, a collaboration between the Mathematics Departments at the Universities of Bath, Bristol, Imperial, Oxford and Warwick, funded by EPSRC, to provide graduate courses in a range of topics to students at these five universities.

Most of the course follows its predecessor closely and concerns Hilbert modular forms. However, since some of the general theory has wider application to the situation where the ground field is a more general number field, some of the presentation will be done in greater generality. This will also allow (time permitting) the second author (JC) to say something about the imaginary quadratic case.

The first section will briefly discuss classical modular forms from an adèlic viewpoint. When the base field is a general number field, with a possibly non-trivial class group and infinite unit group, the adèlic approach has many advantages. On the other hand, since we wish to be able to carry out explicit computations, we will always seek to make our constructions as concrete as possible.

Our approach to the computation of Hilbert modular forms is via the Eichler-Shimizu or Jacquet-Langlands correspondence. So after giving the definitions and the basic properties of Hilbert modular forms in Section 2 and Section **??** and discussing some applications of them in Section **??**, we state this correspondence in Section 4. We then present our algorithm and make few comments on how to implement it. In the last section, we explain how one can compute the elliptic curve corresponding to a Hilbert normalized eigenform that has rational Fourier expansion. The computations in that section are based on [8] and uses results from Oda [16]. Namely, we use the 2-cycles constructed by Oda in order to compute the periods of a given cusp form. In our application, however, we use the stronger assumption made in Conjecture 4.

As well as the motivating question of studying the modularity of elliptic curves over number fields, the first author (LD) has also successfully applied Hilbert modular forms to settle three of the four outstanding cases of Gross's conjecture, that for each prime p there exists a number field which is Galois with non-solvable Galois group and ramified only at p. An old construction of Serre uses the Galois representations attached to classical elliptic modular forms to prove this for all $p \geq 11$, but that construction does not apply for $p \leq 7$. Using Hilbert modular forms, and the Galois representations attached to them, the cases $p = 2, 3$ and 5 have also now been settled.

Prerequisites: (1) Algebraic number theory, including some familiarity with the adèle ring $\mathbb{A}_K$ and the idele group $\mathbb{A}_K^\times$ of a number field K. (2) classical modular forms for congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$.

1. From classical modular forms to adèlic modular forms

When dealing with number fields other than $\mathbb{Q}$, it is usual to deal with modular forms from an adèlic viewpoint, for various reasons: it enables one to handle non-trivial unit groups and class groups in a more efficient way. We start by sketching how to reinterpret classical modular forms, first defined as certain complex-valued functions on the upper half-plane $\mathfrak{H}$, first as functions on $\mathrm{SL}_2(\mathbb{R})$ and then as functions on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. One further step would take us to consider representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, but we will not consider automorphic representations here. We here follow Chapters 2 and 3 of Gelbart's *Automorphic Forms on Adele Groups.*

1.1. **From functions on $\mathfrak{H}$ to functions on** $\mathrm{SL}_2(\mathbb{R})$**. . . .** Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \Gamma_0(N)$ (or some other congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$). We have the so-called Iwasawa decomposition $G = BK = NAK$, where

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}_{>0} \right\}; \qquad N = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{R} \right\};$$

$$K = \mathrm{SO}(2) = \left\{ r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}; \qquad B = NA.$$

So $B = NA$ acts transitively on $\mathfrak{H}$:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} (i) = x + yi,$$

and K is the stabiliser in G of i, so we can identify $\mathfrak{H}$ with the coset space G/K. Hence we can use (x, y, θ) as coordinates on G, for $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, writing

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} r(\theta),$$

so that $z = x + yi = g(i)$ and $\theta = \arg(d + ci)$.

Recall that $S_k(\Gamma)$ denotes the space of holomorphic cusp forms of weight k for Γ. To each $f \in S_k(\Gamma)$ we define $\phi_f : G \to \mathbb{C}$ by

$$\phi_f(g) = f(g(i)) \cdot j(g, i)^{-k},$$

where for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathbb{C}$ we set $j(g, z) = (cz + d)\det(g)^{-\frac{1}{2}}$. This gives a linear injective map from $S_k(\Gamma)$ to the space of all functions $\phi : G \to \mathbb{C}$ satisfying the following conditions:

(1) $\phi(\gamma g) = \phi(g) \qquad \forall \gamma \in \Gamma$;
(2) $\phi(gr(\theta) = e^{-ik\theta}\phi(g) \qquad \forall r(\theta) \in K$;
(3) ϕ is *bounded*, in the sense that

$$\int_{\Gamma\backslash G} |\phi(g)|^2 \, dg < \infty$$

(if $\phi = \phi_f$ then this integral equals $\int_\mathcal{F} |f(z)|^2 y^{k-2} dxdy$, where $\mathcal{F}$ is a fundamental region for Γ, which is just the Peterssen norm of f up to a constant factor), and *cuspidal*, in the sense that for all $g \in G$ and $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\int_0^1 \phi(\sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g) dx = 0,$$

where h is the width of the cusp $\sigma(\infty)$; when $\phi = \phi_f$ the latter integral is

$$\int_0^1 (f|_k \, \sigma)(hx + z) dx$$

(independent of $z \in \mathfrak{H}$), which is precisely the 0th Fourier coefficient of $f|_k \, \sigma$, that is, the constant term in the expansion of f at the cusp $\sigma(\infty)$.

These three conditions are not sufficient to characterise the image of $S_k(\Gamma)$: there is nothing yet about the holomorphicity of f, and the conditions listed so far would also hold for ϕ_f if f was a non-holomorphic modular form (such as Maass form, for example). For that we need a further analytic condition. On the space $L^2(\Gamma\backslash G)$ of square-integrable functions on $\Gamma\backslash G$, there is a Laplace operator Δ, which is given in terms of the coordinates (x, y, θ) by

$$\Delta - -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - y\left(\frac{\partial^2}{\partial x\partial\theta}\right),$$

and one can check that $\phi = \phi_f$ is an eigenfunction for this:

(4) $\Delta\phi = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi$.

Now the image of $S_k(\Gamma)$ is characterised by the four conditions (1)–(4):

Theorem 1. *The map $f \mapsto \phi_f$ is an isomorphism between $S_k(\Gamma)$ and the set of all functions $\phi : G \to \mathbb{C}$ satisfying conditions (1), (2), (3) and (4). The inverse map is $\phi \mapsto f$ where*

$$f(z) = \phi(g)j(g, i)^k \qquad \textit{for any } g \in G \textit{ such that } g(i) = z \in \mathfrak{H}.$$

One can consider more general "Γ-automorphic forms for G": (2) is generalised to the condition that the right translates of ϕ by $k \in K$ span a finite-dimensional space (but not necessarily 1-dimensional), and we can allow eigenvalues of Δ other than the special ones in (4).

1.2. … **to functions on** $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$**.** Recall the adèle ring of $\mathbb{Q}$ is $\mathbb{A} = \mathbb{A}_\mathbb{Q} = \prod'_{p\le\infty}\mathbb{Q}_p$, where for $p < \infty$ $\mathbb{Q}_p$ is the p-adic field and $\mathbb{Q}_\infty = \mathbb{R}$, and the direct product is *restricted*, meaning that for almost all p the component is in $\mathbb{Z}_p$. We embed $\mathbb{Q} \hookrightarrow \mathbb{A}_\mathbb{Q}$ diagonally. Now set

$$G_\mathbb{A} = \mathrm{GL}_2(\mathbb{A}) = \prod_{p\le\infty}{}' \mathrm{GL}_2(\mathbb{Q}_p),$$

restricted with respect to the compact subgroups $K_p \le \mathrm{GL}_2(\mathbb{Q}_p)$, namely $K_\infty = \mathrm{O}(2)$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ for $p < \infty$. The centre of $G_\mathbb{A}$ is

$$\mathcal{Z}_\mathbb{A} = \left\{\begin{pmatrix} t & 0 \\ 0 & t\end{pmatrix} \mid t \in \mathbb{A}^\times\right\}$$

which is isomorphic to the idele group $\mathbb{A}^\times$, and again $G_\mathbb{Q} = \mathrm{GL}_2(\mathbb{Q})$ embeds in $G_\mathbb{A}$ on the diagonal. Note that $\mathcal{Z}_\mathbb{Q}\backslash\mathcal{Z}_\mathbb{A} \cong \mathbb{Q}^\times\backslash\mathbb{A}^\times$, the idele class group of $\mathbb{Q}$.

Recall that $\mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}^\times_{>0}\prod_p\mathbb{Z}_p$, an identity which follows from the fact that $\mathbb{Q}$ has class number 1 and unit group $\{\pm1\}$. The corresponding statement for GL_2 is the *strong approximation* relation, which we give here in a more general form:

$$G_\mathbb{A} = G_\mathbb{Q}G^+_\infty K'_0$$

where $G_\infty = \mathrm{GL}_2(\mathbb{R})$, $G^+_\infty = \mathrm{GL}^+_2(\mathbb{R})$ (the subgroup of matrices with positive determinant), and K'_0 is a subgroup of $\prod_{p<\infty}K_p$ of the form $K'_0 = \prod_{p<\infty}K'_p$ where for each p, K'_p is an open subgroup of K_p, equal to K_p for almost all p, such that the determinant map $K'_p \to \mathbb{Z}_p^\times$ is surjective for all p. The main example of such a subgroup is $K_0(N) = \prod_{p<\infty}K^N_p$ for N a positive integer, where

$$K^N_p = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K_p = \mathrm{GL}_2(\mathbb{Z}_p) \mid c \in N\mathbb{Z}_p\right\}.$$

Now for $f \in S_k(N)$ we define $\phi_f : G_\mathbb{A} \to \mathbb{C}$ by

$$\phi_f(g) = f(g_\infty(i)) \cdot j(g_\infty, i)^{-k}$$

where for $g \in G$ we use any decomposition $g = \gamma g_\infty k_0$ with $\gamma \in G_\mathbb{Q}$, $g_\infty \in G^+_\infty$ and $k_0 \in K_0(N)$. One may check that the value of $\phi_f(g)$ is independent of this decomposition of g, using the fact that

$$G_\mathbb{Q} \cap G^+_\infty K_0(N) = \Gamma_0(N).$$

Note also that if we restrict this ϕ_f to $\mathrm{SL}_2(\mathbb{R}) \subset G^+_\infty$ then we get back the function we called ϕ_f in the previous subsection.

We can make this construction more general: let $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character modulo N. This determines for each prime p a character $\psi_p : \mathbb{Z}_p^\times \to \mathbb{C}^\times$. [Decompose $(\mathbb{Z}/N\mathbb{Z})^\times$ by the Chinese Remainder theorem as a product of $(\mathbb{Z}/p^e\mathbb{Z})^\times$ and compose ψ with $\mathbb{Z}_p^\times \to (\mathbb{Z}/p^e\mathbb{Z})^\times \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$.] These in turn determine a character $\tilde{\psi} = \prod_p\psi_p : \prod_p\mathbb{Z}_p^\times \to \mathbb{C}^\times$ and hence a character on $\mathbb{A}^\times$, trivial on $\mathbb{Q}^\times\mathbb{R}_+^\times$, called a *grossencharakter*. Now if $f \in S_k(N,\psi)$ we can use the fact that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi_p(d)$ is a character on K_p^N to set

$$\phi_f(g) = f(g_\infty(i))j(g_\infty, i)^{-k}\psi(k_0)$$

where $g = \gamma g_\infty k_0$ as above.

Theorem 2. *The map $f \to \phi_f$ is an isomorphism (of finite-dimensional complex vector spaces) from $S_k(N,\psi)$ to the space of functions $\phi : G_{\mathbb{A}} \to \mathbb{C}$ satisfying*

1. $\phi(\gamma g) = \phi(g)$ *for all* $\gamma \in G_{\mathbb{Q}}$;
2. $\phi(gk_0) = \phi(g)\psi(k_0)$ *for all* $k_0 \in K_0(N)$;
3. $\phi(gr(\theta)) = e^{-ik\theta}\phi(g)$ *for all* $r(\theta) \in K_\infty^+$;
4. *The restriction of* ϕ *to* $G_\infty^+ = \mathrm{GL}_2^+(\mathbb{R})$ *satisfies* $\Delta\phi = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi$;
5. $\phi(zg) = \phi(gz) = \psi(z)\phi(g)$ *for all* $z \in \mathcal{Z}_{\mathbb{A}}$;
6. ϕ *is slowly increasing (definition omitted);*
7. ϕ *is cuspidal, meaning that* $\int_{\mathbb{Q}\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)dx = 0$ *for all* g.

Note how the grossencharakter ψ appears in two of these conditions: we have a character modulo N and a central character, and these must be compatible.

By generalising these conditions (as before) we arrive at the definition of an *adèlic automorphic form for* GL_2 (over $\mathbb{Q}$). The generalization involves the following changes:

- no change to (1), (6), (7);
- in (5) we can replace ψ with any grossencharakter;
- (2), (3) are replaced by a condition of right-K-finiteness for $K = K_\infty \prod_{p<\infty} K_p'$;
- in (4) we require the restriction of ϕ to G_∞ to be smooth and $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the centre of the universal enveloping algebra of G_∞.

See Gelbart for details.

Final remark: We use GL_2 and not SL_2 in the adèlic theory for a number of reasons. It has a nontrivial centre, which makes it easier to involve characters; and it makes the definition of Hecke operators easier, since these involve matrices whose determinant is not 1. We will see this again in the adèlic definition of Hilbert modular forms.

2. Hilbert modular forms and varieties

At the start of this section, F will denote a completely general number field. Later, we will restrict to the case of a totally real field.

2.1. Basic notions and notation.

We fix a number field F of degree g.

Let Σ_F be the set of all real and complex places of F. We write $\Sigma_F = \Sigma_F^{\mathbb{R}} \cup \Sigma_F^{\mathbb{C}}$ where $\Sigma_F^{\mathbb{R}}$ is the set of real places, identified with the set of real embeddings $\sigma : F \hookrightarrow \mathbb{R}$, and $\Sigma_F^{\mathbb{C}}$ is the set of complex places, identified with a set of pairwise non-conjugate complex embeddings $\sigma : F \hookrightarrow \mathbb{C}$. For each $\sigma \in \Sigma_F$ we set F_σ to be the corresponding completion, so $F_\sigma = \mathbb{R}$ for $\sigma \in \Sigma_F^{\mathbb{R}}$ and $F_\sigma = \mathbb{C}$ for $\sigma \in \Sigma_F^{\mathbb{C}}$. We have $g = [F : \mathbb{Q}] = \#\Sigma_F^{\mathbb{R}} + 2\#\Sigma_F^{\mathbb{C}}$, and $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \oplus_{\sigma\in\Sigma_F} F_\sigma$. For $\sigma \in \Sigma_F$ and $a \in F$ we denote by a^σ the image $\sigma(a) \in F_\sigma$. F is *totally real* when $\Sigma_F = \Sigma_F^{\mathbb{R}}$ (and $\Sigma_F^{\mathbb{C}} = \emptyset$), and *totally complex* if $\Sigma_F = \Sigma_F^{\mathbb{C}}$ (and $\Sigma_F^{\mathbb{R}} = \emptyset$).

We say that an element $a \in F$ is *totally positive* if $a^\sigma > 0$ for all $\sigma \in \Sigma_F^{\mathbb{R}}$, and denote this condition by $a \gg 0$. (This agrees with the usual notion when F is totally real, and is a vacuous condition when F is totally complex.)

Let $\mathcal{O}_F$ be the ring of integers of F, and $\mathfrak{d}$ its different. For a prime $\mathfrak{p}$ of F, we denote by $F_{\mathfrak{p}}$ and $\mathcal{O}_{F,\mathfrak{p}}$ the completions of F and $\mathcal{O}_F$, respectively, at $\mathfrak{p}$.

Let $\mathbb{A}$ be the ring of adèles of F:

$$\mathbb{A} = \mathbb{A}_F = \prod_{\sigma\in\Sigma_F} F_\sigma \times {\prod_{\mathfrak{p}}}' F_{\mathfrak{p}}$$

and denote its finite part by $\mathbb{A}_f = \prod_{\mathfrak{p}}' F_{\mathfrak{p}}$, restricted with respect to the $\mathcal{O}_{F,\mathfrak{p}}$.

Let $\mathcal{I}_F$ denote the group of fractional ideals of F. Recall that the *class group* of F, denoted $\mathcal{C}\ell_F$, is the quotient of $\mathcal{I}_F$ by the subgroup of principal fractional ideals, and is finite of order $h = h_F$, the *class number* of F. The *strict class group* of F, denoted $\mathcal{C}\ell_F^+$, is the quotient of $\mathcal{I}_F$ by the subgroup of principal fractional ideals which have a totally positive generator; it is finite of order $h^+ = h_F^+$, the *strict class number* of F.

We fix an integral ideal $\mathfrak{n}$ of F, called the *level.*

2.2. **The algebraic group G.** Let $G := \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$ be the algebraic group obtained by restriction of scalars from F to $\mathbb{Q}$. Equivalently, one defines G as the $\mathbb{Q}$-algebraic group whose set of A-rational points, for any $\mathbb{Q}$-algebra A, is given by $G(A) = \mathrm{GL}_2(A \otimes_{\mathbb{Q}} F)$. In the rest of these lecture notes, we will sometime denote this set of A-rational points by G_A.

In particular, we have $G_{\mathbb{Q}} = \mathrm{GL}_2(F)$, and we have a canonical identification

$$G_{\mathbb{R}} = \mathrm{GL}_2(\mathbb{R} \otimes_{\mathbb{Q}} F) = \prod_{\sigma \in \Sigma_F} \mathrm{GL}_2(F_\sigma),$$

and we set $G_{\mathbb{R}}^+$ to be the connected component of the identity in $g_{\mathbb{R}}$, which is subgroup of elements with "totally positive determinant", that is,

$$\begin{aligned} G_{\mathbb{R}}^+ &= \{g = (g_\sigma)_{\sigma \in \Sigma_F} \mid \det(g_\sigma) > 0 \quad \forall \sigma \in \Sigma_F^{\mathbb{R}}\} \\ &= \prod_{\sigma \in \Sigma_F^{\mathbb{R}}} \mathrm{GL}_2^+(\mathbb{R}) \times \prod_{\sigma \in \Sigma_F^{\mathbb{C}}} \mathrm{GL}_2(\mathbb{C}). \end{aligned}$$

Let $\sigma \in \Sigma_F$. Then σ extends canonically into an embedding of $G_{\mathbb{Q}}$ into $\mathrm{GL}_2(F_\sigma)$ by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} =: \gamma^\sigma.$$

By putting these embeddings together, we get an embedding $G_{\mathbb{Q}} \hookrightarrow G_{\mathbb{R}}$, $\gamma \mapsto (\gamma^\sigma)_{\sigma \in \Sigma_F}$. We identify $G_{\mathbb{Q}}$ with its image inside $G_{\mathbb{R}}$, and we let $G_{\mathbb{Q}}^+ = G_{\mathbb{Q}} \cap G_{\mathbb{R}}^+ = \{g \in \mathrm{GL}_2(F) \mid \det(g) \gg 0\}$.

2.3. **The action of $G_{\mathbb{Q}}^+$ on $\mathcal{O}_F$-lattices in F^2.** By a *lattice* (or *$\mathcal{O}_F$-lattice*) we mean an $\mathcal{O}_F$-submodule L of F^2 of rank 2 such that $FL = L \otimes_{\mathcal{O}_F} F = F^2$. Elements of F^2 will be viewed as row vectors, with matrices therefore acting naturally on the right; since we prefer a left action we will define the action of $G_{\mathbb{Q}} = \mathrm{GL}_2(F)$ on F^2 via

$$g \cdot (u, v) = (u, v)g^{-1} \qquad \text{for } (u, v) \in F^2,\ g \in G_{\mathbb{Q}}.$$

This induces an action of $G_{\mathbb{Q}}$ on lattices. (We may also view L and $g \cdot L$ as the same lattice with a different basis on the ambient space F^2, where g is the change of basis matrix.)

Let L be an $\mathcal{O}_F$-lattice. As it is a torsion-free, projective $\mathcal{O}_F$-module, it has a *class* (the *Steinitz class*) in the class group $\mathcal{C}\ell_F$, which we denote $[L]$. One definition of $[L]$ is the class of the projective rank one $\mathcal{O}_F$-module $\Lambda^2_{\mathcal{O}_F} L$, the *determinantal ideal* of L. (Every projective rank one $\mathcal{O}_F$-module is isomorphic to a fractional ideal).

The class $[L]$ is trivial if and only of L is free (isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F$). In general, $L_1 \cong L_2$ (as $\mathcal{O}_F$-modules) if and only if $[L_1] = [L_2]$. The action of $G_{\mathbb{Q}}$ on lattices preserves the class, since clearly the action is via $\mathcal{O}_F$-module isomorphisms; conversely, if $[L_1] = [L_2]$ then any $\mathcal{O}_F$-module isomorphism from L_1 to L_2 extends by linearity to an F-module isomorphism $F^2 \to F^2$ which is necessarily given by a matrix $g \in \mathrm{GL}_2(F) = G_{\mathbb{Q}}$. Thus the classes of lattices are precisely the $G_{\mathbb{Q}}$-orbits.

Since every lattice is an $\mathcal{O}_F$-module (where we write scalar multiplication on the left) we also have an action of the group $\mathcal{I}_F$ on the set of lattices, since $\mathfrak{a} \cdot L$ is clearly a lattice for each $\mathfrak{a} \in \mathcal{I}_F$ whenever L is. Moreover we have $[\mathfrak{a}L] = [\mathfrak{a}]^2[L]$: this follows immediately from the normal form of lattices which we now consider.

The theory of modules over Dedekind Domains [1] implies that every lattice L may be expressed as
$$L = \mathfrak{a}x \oplus \mathfrak{b}y$$
where x, y are an F-basis for F^2 and $\mathfrak{a}, \mathfrak{b}$ are fractional ideals. In other words, $L = g^{-1} \cdot (\mathfrak{a} \oplus \mathfrak{b})$ where g is the matrix with rows x, y. The Steinitz class is $[L] = [\mathfrak{ab}]$, and $L \cong \mathfrak{a} \oplus \mathfrak{b} \cong \mathcal{O}_F \oplus \mathfrak{ab}$.

It is possible to restrict attention to lattices of the form $L = \mathcal{O}_F \oplus \mathfrak{b}$, since these represent all the classes (as $\mathfrak{b}$ runs through representatives of the class group of F). Other choices of representatives are possible: for example, if the class number h_F is *odd* then every class has a representative ideal which is a square, and lattices of the form $L = \mathfrak{a} \oplus \mathfrak{a} = \mathfrak{a}(\mathcal{O}_F \oplus \mathcal{O}_F)$ (which has class $[\mathfrak{a}]^2$) represent all the classes.

2.4. **Congruence subgroups of $G_{\mathbb{Q}}^+$.** Note that we can characterise $\mathrm{GL}_2(\mathcal{O}_F)$ as the subgroup of $\mathrm{GL}_2(F)$ which stabilises the lattice $L = \mathcal{O}_F \oplus \mathcal{O}_F$; in other words, $\mathrm{GL}_2(\mathcal{O}_F) = \mathrm{Aut}(\mathcal{O}_F \oplus \mathcal{O}_F)$. More generally, for any lattice L we may define $\mathrm{Aut}(L) = \{g \in G_{\mathbb{Q}} \mid g \cdot L = L\}$. We also write $\mathrm{GL}(L) = \mathrm{Aut}(L)$, and $\mathrm{GL}^+(L) = \mathrm{Aut}(L) \cap G_{\mathbb{Q}}^+$.

Lemma 1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals and $L = \mathfrak{a} \oplus \mathfrak{b}$. Then*
$$\mathrm{Aut}(L) = \mathrm{Aut}(\mathfrak{a} \oplus \mathfrak{b}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_F & \mathfrak{a}^{-1}\mathfrak{b} \\ \mathfrak{ab}^{-1} & \mathcal{O}_F \end{pmatrix} \mid xw - zw \in \mathcal{O}_F^\times \right\}.$$

Proof. Exercise. □

We have
$$\Gamma_0(\mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \mid c \in \mathfrak{a} \right\} = \mathrm{Aut}(\mathfrak{a} \oplus \mathcal{O}_F) \cap \mathrm{GL}_2(\mathcal{O}_F)$$
and
$$\Gamma^0(\mathfrak{b}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \mid b \in \mathfrak{b} \right\} = \mathrm{Aut}(\mathcal{O}_F \oplus \mathfrak{b}) \cap \mathrm{GL}_2(\mathcal{O}_F),$$
both characterised by their action on appropriate lattices.

Since left multiplication by scalars and right multiplication by matrices commute, it is clear that $\mathrm{Aut}(\mathfrak{a}L) = \mathrm{Aut}(L)$ for any $\mathfrak{a} \in \mathcal{I}_F$. In particular, $\mathrm{Aut}(\mathfrak{a} \oplus \mathfrak{ab}) = \mathrm{Aut}(\mathcal{O}_F \oplus \mathfrak{b})$. Thus the group $\mathrm{Aut}(L)$ only depends, up to conjugacy in $G_{\mathbb{Q}}$, on the image of $[L]$ in the *genus group* $\mathcal{C}\ell_F/\mathcal{C}\ell_F^2$.

It is useful in practice (both for explicit computations, and for more theoretical results about cusp equivalence, as in the next subsection), to make various lattice isomorphisms quite explicit. First we give a generalization of the well-known two-generator theorem for ideals in a Dedekind Domain.

Lemma 2. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{I}_F$. Then there exist $x \in \mathfrak{a}^{-1}\mathfrak{c}$ and $y \in \mathfrak{b}^{-1}\mathfrak{c}$ such that $x\mathfrak{a} + y\mathfrak{b} = \mathfrak{c}$. Moreover, x may be chosen arbitrarily in $\mathfrak{a}^{-1}\mathfrak{c} \setminus \{0\}$.*

[Note that we are not dealing with lattices here, just ideal arithmetic.]

Proof. Let $x \in \mathfrak{a}^{-1}\mathfrak{c}$ be an arbitrary nonzero element. Set $\mathfrak{a}' = (x)\mathfrak{ac}^{-1} \subseteq \mathcal{O}_F$. Choose $\mathfrak{b}' \subseteq \mathcal{O}_F$ coprime to $\mathfrak{a}'$ with $[\mathfrak{b}'] = [\mathfrak{bc}^{-1}]$ (every ideal class contains ideals coprime to any given ideal!), so $\mathfrak{b}' = (y)\mathfrak{bc}^{-1}$ for some $y \in F^\times$; and since $\mathfrak{b}'$ is integral, we have $y \in \mathfrak{b}^{-1}\mathfrak{c}$. Now from $\mathfrak{a}' + \mathfrak{b}' = \mathcal{O}_F$ we obtain $(x)\mathfrak{a} + (y)\mathfrak{b} = \mathfrak{c}$ as required. □

Lemma 3. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{I}_F$ with $[\mathfrak{ab}] = [\mathfrak{cd}]$. Then $\mathfrak{a} \oplus \mathfrak{b} \cong \mathfrak{c} \oplus \mathfrak{d}$, via a matrix $g \in G_{\mathbb{Q}}$ such that $(\mathfrak{a} \oplus \mathfrak{b})g = \mathfrak{c} \oplus \mathfrak{d}$. If g' also satisfies this equality then $g(g')^{-1} \in \mathrm{Aut}(\mathfrak{a} \oplus \mathfrak{b})$.*

Proof. From $[\mathfrak{ab}] = [\mathfrak{cd}]$ we have $\mathfrak{ab}(\gamma) = \mathfrak{cd}$ for some $\gamma \in F^\times$. By the preceding lemma there exist $x \in \mathfrak{a}^{-1}\mathfrak{c}$, $y \in \mathfrak{b}^{-1}\mathfrak{c}$ such that $\mathfrak{c} = \mathfrak{a}x + \mathfrak{b}y$. Now $(\gamma) = \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{cd} = x\mathfrak{b}^{-1}\mathfrak{d} + y\mathfrak{a}^{-1}\mathfrak{d}$, so there exist u, v in $\mathfrak{a}^{-1}\mathfrak{d}$, $\mathfrak{b}^{-1}\mathfrak{d}$ respectively such that $\gamma = xv - yu$. Set $g = \begin{pmatrix} x & u \\ y & v \end{pmatrix}$. Then clearly $(\mathfrak{a} \oplus \mathfrak{b})g \subseteq \mathfrak{c} \oplus \mathfrak{d}$, and an easy check using $\det(g) = \gamma$ shows that the reverse inclusion also holds. □

[1] For the structure theory of finitely-generated modules over a Dedekind Domain developed in a concrete and explicit way, see Chapter 1 of Cohen's Advanced Topics in Computational Number Theory.

Remark 1. We deduced Lemma 3 from Lemma 2. The reverse implication may also be useful: given $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ set $\mathfrak{d} = \mathfrak{a}\mathfrak{b}\mathfrak{c}^{-1}$. Then by Lemma 3 there exists $g = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \in \mathrm{SL}_2(F)$ such that $(\mathfrak{a} \oplus \mathfrak{b})g = \mathfrak{c} \oplus \mathfrak{d}$, and in particular $\mathfrak{a}x + \mathfrak{b}y = \mathfrak{c}$.

Remark 2. As well as the bijection between $G_{\mathbb{Q}}$-orbits of $\mathcal{O}_F$-lattices and the class group $\mathcal{C}\ell_F$ given by $L \mapsto \wedge^2_{\mathcal{O}_F} L$, the same map induces a bijection between $G^+_{\mathbb{Q}}$-orbits of $\mathcal{O}_F$-lattices and the narrow class group Cl^+_F. In order to take this fact into account one must consider all groups of the forms $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b})$, as $\mathfrak{b}$ runs over a complete set of representatives of the classes in Cl^+_F, and not just the group $\mathrm{GL}^+_2(\mathcal{O}_F) = \mathrm{GL}^+(\mathcal{O}_F \oplus \mathcal{O}_F)$. This fits naturally into the adèlic treatment.

We will give a rather general definition of a "congruence subgroup" before becoming more explicit.

Definition 1. *Let Γ be a subgroup of $G^+_{\mathbb{Q}}$. We say that Γ is a* congruence subgroup *if there exists a* compact open *subgroup K of the finite idèles $G_f \leq G_{\mathbb{A}}$ such that $\Gamma = G_{\mathbb{Q}} \cap KG^+_\infty$.*

Example 1. Let $\mathfrak{c}$ be a fractional ideal of F and $\mathfrak{n}$ (as before) an integral ideal called the level. Set

$$\begin{aligned} \Gamma_0(\mathfrak{c}, \mathfrak{n}) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_F & \mathfrak{c}^{-1} \\ \mathfrak{n}\mathfrak{c} & \mathcal{O}_F \end{pmatrix} : ad - bc \in \mathcal{O}_F^{\times +} \right\} \\ &= \mathrm{GL}^+(\mathfrak{n}\mathfrak{c} \oplus \mathcal{O}_F) \cap \mathrm{GL}^+(\mathfrak{c} \oplus \mathcal{O}_F) \\ \Gamma_1(\mathfrak{c}, \mathfrak{n}) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{c}, \mathfrak{n}) : d \equiv 1 \mod \mathfrak{n} \right\}. \end{aligned}$$

Then, $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ and $\Gamma_1(\mathfrak{c}, \mathfrak{n})$ are congruence subgroups of $G^+_{\mathbb{Q}}$ (see section???). This is the only type of congruence subgroup that we will be interested in for the rest of this course.

2.5. **Cusps and cusp equivalence.** Let Γ be a congruence subgroup of $G^+_{\mathbb{Q}}$.

Definition 2. *The set of* cusps *for Γ is the set $\Gamma\backslash\mathbb{P}^1(F)$, that is, the set of Γ-orbits on $\mathbb{P}^1(F)$.*

Lemma 4. *Fix $\mathfrak{a}$, $\mathfrak{b} \in \mathcal{I}_F$, and let $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \in F^2 - \{0\}$. The following are equivalent:*

(1) *$x\mathfrak{a} + y\mathfrak{b} = x'\mathfrak{a} + y'\mathfrak{b}$;*
(2) *there exists $g \in \mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})$ such that $\begin{pmatrix} x' \\ y' \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix}$.*

Proof. Assume (1), and let $\mathfrak{c} = \mathfrak{a}x \oplus \mathfrak{b}y = \mathfrak{a}x' \oplus \mathfrak{b}y'$. Let $\mathfrak{d} = \mathfrak{a}\mathfrak{b}\mathfrak{c}^{-1}$. By Lemma 2 and its proof (with $\gamma = 1$) there exist g_1, $g_2 \in \mathrm{SL}_2(F)$ such that $\mathfrak{c} = (\mathfrak{a} \oplus \mathfrak{b})g_1 = (\mathfrak{a} \oplus \mathfrak{b})g_2$, where the first columns of g_1, g_2 are $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}$ respectively. Then $g = g_2 g_1^{-1}$ satisfies (2).

Now assume (2), set $\mathfrak{c} = \mathfrak{a}x + \mathfrak{b}y$ and $\mathfrak{d} = \mathfrak{a}\mathfrak{b}\mathfrak{c}^{-1}$. As before, there exist u, v such that $g_1 = \begin{pmatrix} x & u \\ y & v \end{pmatrix}$ satisfies $(\mathfrak{a} \oplus \mathfrak{b})g_1 = \mathfrak{c} \oplus \mathfrak{d}$. Since g satisfies $(\mathfrak{a} \oplus \mathfrak{b})g = \mathfrak{a} \oplus \mathfrak{b}$, we also have $\mathfrak{c} \oplus \mathfrak{d} = (\mathfrak{a} \oplus \mathfrak{b})gg_1 = (\mathfrak{a} \oplus \mathfrak{b})g_2$, where $g_2 = gg_1 = \begin{pmatrix} x' & u' \\ y' & v' \end{pmatrix}$, and hence $\mathfrak{c} = \mathfrak{a}x' \oplus \mathfrak{b}y'$ also. $\square$

Proposition 3. *Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_F$. There is a bijection:*

$$\mathrm{GL}^+(\mathfrak{a} \oplus \mathfrak{b})\backslash(F^2 - \{0\}) \xrightarrow{\sim} \mathcal{I}_F, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \mathfrak{c} = x\mathfrak{a} + y\mathfrak{b}$$

which induces a bijection of finite sets:

$$\mathrm{GL}^+(\mathfrak{a} \oplus \mathfrak{b})\backslash\mathbb{P}^1(F) \xrightarrow{\sim} \mathcal{C}\ell_F.$$

Proof. Surjectivity is Lemma 2 and injectivity is Lemma 4. $\square$

As a special case, for $\mathfrak{b} \in \mathcal{I}_F$ we have a bijection $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{b}\mathfrak{n})\backslash\mathbb{P}^1(F) \xrightarrow{\sim} \mathcal{C}\ell_F$.

2.6. **Classical Hilbert modular forms.** From now on we fix F to be a *totally real* field, so $\Sigma_F = \Sigma_F^{\mathbb{R}}$ and $\Sigma_F^{\mathbb{C}} = \emptyset$. Now $G_{\mathbb{R}}^+ = \prod_{\sigma\in\Sigma_F} \mathrm{GL}_2^+(\mathbb{R})$.

Let $\mathfrak{H}$ be the Poincaré upper-half plane and put $\mathfrak{H}_F = \mathfrak{H}^{\Sigma_F} = \{z = (z_\sigma)_{\sigma\in\Sigma_F}\}$. Then $G_{\mathbb{R}}^+$ acts on $\mathfrak{H}_F$ as follows. For any $\gamma = (\gamma_\sigma)_{\sigma\in\Sigma_F} \in G_{\mathbb{R}}^+$ and $z = (z_\sigma)_{\sigma\in\Sigma_F} \in \mathfrak{H}_F$,

$$\gamma_\sigma \cdot z_\sigma = \frac{a_\sigma z_\sigma + b_\sigma}{c_\sigma z_\sigma + d_\sigma}, \qquad \text{where } \gamma_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}.$$

Definition 3. *An element $\underline{k} = (k_\sigma)_\sigma \in \mathbb{Z}^{\Sigma_F}$ is called a* weight vector*. A weight vector $\underline{k}$ is called* arithmetic *if all its components $k_\sigma \geq 2$ have the same parity.*

From now on, we let $\underline{k}$ be an arithmetic weight. We let $k_0 = \max\{k_\sigma | \sigma \in \Sigma_F\}$, and define $\underline{m}$, $\underline{n}$, $\underline{t} \in \mathbb{Z}^{\Sigma_F}$ by $m_\sigma = \frac{k_0 - k_\sigma}{2}$, $n_\sigma = k_\sigma - 2$ and $\underline{t} = (1, \cdots, 1) \in \mathbb{Z}^{\Sigma_F}$.

For $z = (z_\sigma) \in \mathfrak{H}_F$ and $\gamma = (\gamma_\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{R}}^+$ set

$$j(\gamma, z) = (cz + d) = \prod_\sigma j(\gamma_\sigma, z_\sigma) = \prod_{\sigma\in\Sigma_F} (c_\sigma z_\sigma + d_\sigma);$$

and

$$j(\gamma, z)^{-\underline{k}} = \prod_\sigma j(\gamma_\sigma, z_\sigma)^{-k_\sigma}.$$

Note the multi-index product notation which we will use frequently. Now for each function $f : \mathfrak{H}_F \to \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}}^+$, we define $f|_{\underline{k}}\gamma : \mathfrak{H}_F \to \mathbb{C}$ by

$$\left(f|_{\underline{k}}\gamma\right)(z) = \det(\gamma)^{\underline{k}+\underline{m}-\underline{t}} j(\gamma, z)^{-\underline{k}} f(\gamma z).$$

It is easy to check that this does define a group action, i.e. that $f|_{\underline{k}}\gamma_1\gamma_2 = (f|_{\underline{k}}\gamma_1)|_{\underline{k}}\gamma_2$.

Definition 4. *A* classical Hilbert modular form *of level $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ and weight $\underline{k}$ is a holomorphic function $f : \mathfrak{H}_F \to \mathbb{C}$ such that $f|_{\underline{k}}\gamma = f$, for all $\gamma \in \Gamma_0(\mathfrak{c}, \mathfrak{n})$. The space of all classical Hilbert modular forms of level $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ and weight $\underline{k}$ will be denoted by $M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$.*

Periodicity and Fourier expansions. Let $f \in M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$. If $\mu \in \mathfrak{c}^{-1}$, then $\gamma = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma_0(\mathfrak{c}, \mathfrak{n})$ and, the definition implies that

$$f(z + \mu) = f(z), \quad \text{for all } z \in \mathfrak{H}_F$$

since $(f|_{\underline{k}}\gamma)(z) = f(z + \mu)$. (Here $z + \mu = (z_\sigma + \mu^\sigma)_\sigma \in \mathfrak{H}_F$.) This implies that f admits a Fourier expansion of the form

$$f(z) = \sum_{\mu\in\mathfrak{c}\mathfrak{d}^{-1}} a_\mu e^{2\pi i \,\mathrm{Tr}(\mu z)} = \sum_{\mu\in\mathfrak{c}\mathfrak{d}^{-1}} a_\mu q^\mu \tag{1}$$

where $\mathrm{Tr}(\mu z) = \sum_{\sigma\in\Sigma_F} \mu^\sigma z_\sigma$ and $q = (q_\sigma) = (e^{2\pi i z_\sigma})$ so that (using multi-index product notation again) $q^\mu = \prod_\sigma q_\sigma^{\mu^\sigma} = e^{2\pi i \,\mathrm{Tr}(\mu z)}$. The series (1) is absolutely convergent, uniformly on compact subsets of $\mathfrak{H}_F$.

Sketch of proof. If we fix the imaginary part $y = (y_\sigma)$ and regard $x = (x_\sigma) \mapsto f(x + iy)$ as a function of $x \in \mathbb{R}^g$, then it is smooth and periodic with respect to the $\mathbb{Z}$-lattice $\mathfrak{c}^{-1} \subset \mathbb{R}^g$, and hence has a Fourier expansion of the form

$$f(z) = \sum_\mu a_\mu(y) e^{2\pi i \,\mathrm{Tr}(\mu x)}$$

where μ runs over the dual lattice (with respect to the trace), which is $\mathfrak{c}\mathfrak{d}^{-1}$ (recall that $\mathfrak{d}$ is the different of F). Now holomorphicity (essentially the Cauchy-Riemann equations) implies that $a_\mu(y) = a_\mu e^{-2\pi \,\mathrm{Tr}(\mu y)}$ with $a_\mu \in \mathbb{C}$ independent of y. This gives (1). For more details, see Garrett (pages 4,5). □

You may recall from the classical theory that to define a holomorphic modular form f we needed to specify that f is holomorphic at the cusps, by insisting that all negative Fourier coefficients are zero. However, when $g = [F : \mathbb{Q}] > 1$, every Hilbert modular form is automatically holomorphic at all the cusps! This important basic result, known as Koecher's principle, relies on the action of the unit group $\mathcal{O}_F^*$, which has rank $g-1$.

Lemma 5 (Koecher's principle)**.** *Assume that* $[F : \mathbb{Q}] > 1$. *Then every* $f \in M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$ *is* holomorphic *at the cusp* ∞ *(and hence at all cusps) in the following sense:*

$$a_\mu \neq 0 \Rightarrow \mu = 0 \text{ or } \mu \gg 0.$$

Proof. The stabilizer $\mathrm{Stab}_{\Gamma_0(\mathfrak{c},\mathfrak{n})}(\infty)$ of the cusp at ∞ in $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ is

$$\mathrm{Stab}_{\Gamma_0(\mathfrak{c},\mathfrak{n})}(\infty) := \left\{ \gamma \in \begin{pmatrix} \mathcal{O}_F & \mathfrak{c}^{-1} \\ 0 & \mathcal{O}_F \end{pmatrix} : \det(\gamma) \in \mathcal{O}_F^{\times +} \right\}.$$

Modulo its center, it is the semi-direct product of $\mathcal{O}_F^{\times +}$ and $\mathfrak{c}^{-1}$. In particular, for every $\varepsilon \in \mathcal{O}_F^{\times +}$, $\gamma(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Stab}_{\Gamma_0(\mathfrak{c},\,\mathfrak{n})}(\infty)$, which means that $f|_{\underline{k}}\gamma(\varepsilon) = f$. By equating the Fourier expansion of both sides of this equality, it follows that

$$a_{\varepsilon\mu} = \varepsilon^{\underline{k}+\underline{m}-\underline{t}} a_\mu, \quad \text{for all } \mu \in \mathfrak{c}\mathfrak{d}^{-1},$$

where we again use multi-index product notation: $\varepsilon^{\underline{k}} = \prod_{\sigma \in \Sigma_F} (\varepsilon^\sigma)^{k_\sigma}$.

Now, let us assume that there is a non-zero $\mu_0 \in \mathfrak{c}\mathfrak{d}^{-1}$, not totally positive, such that $a_{\mu_0} \neq 0$. Choose σ_0 such that $\mu_0^{\sigma_0} < 0$. By Dirichlet's Units Theorem, there exists $\varepsilon \in \mathcal{O}_F^{\times +}$ such that

$$\varepsilon^{\sigma_0} > 1 \quad \text{and} \quad 0 < \varepsilon^\sigma < 1, \text{ for all}\, \sigma \neq \sigma_0.$$

We now consider the subseries of $f(z) = \sum_{\mu \in \mathfrak{c}\mathfrak{d}^{-1}} a_\mu e^{2\pi i \,\mathrm{Tr}(\mu z)}$ indexed by μ in the set $\{\mu_0 \varepsilon^m,\ m \in \mathbb{N}\}$, in which we substitute $z = \underline{i} = (i, i, \ldots, i)$. We have

$$a_\mu e^{2\pi i \,\mathrm{Tr}(\mu i)} = a_\mu e^{-2\pi \,\mathrm{Tr}(\mu)} = a_{\mu_0 \varepsilon^m} e^{-2\pi \,\mathrm{Tr}(\mu_0 \varepsilon^m)} = \varepsilon^{m(\underline{k}+\underline{m}-\underline{t})} a_{\mu_0} e^{-2\pi \,\mathrm{Tr}(\mu_0 \varepsilon^m)}.$$

But

$$e^{-2\pi \,\mathrm{Tr}(\mu_0 \varepsilon^m)} = e^{-2\pi \mu_0^{\sigma_0} (\varepsilon^{\sigma_0})^m} \cdot \prod_{\sigma \neq \sigma_0} e^{-2\pi \mu_0^\sigma (\varepsilon^\sigma)^m},$$

and the second factor $\to 1$ as $m \to \infty$ (since the exponents $\to 0$) while the first factor $\to \infty$ (since $\mu_o^{\sigma_0} < 0$ and $\varepsilon^{\sigma_0} > 1$). Hence this subseries diverges, contradiction. $\square$

Definition 5. *We say that* $f \in M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$ *is a* cusp form *if the constant term* a_0 *in the Fourier expansion of* $f|_{\underline{k}}\gamma$ *is equal to* 0 *for all* $\gamma \in G_{\mathbb{Q}}^+$ *(i.e., if* f *vanishes at all cusps). We will denote by* $S_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$ *the space of cusp forms of weight* $\underline{k}$ *and level* $\Gamma_0(\mathfrak{c}, \mathfrak{n})$*.*

Thus every $f \in M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$ has a Fourier expansion of the form

$$f(z) = a_0 + \sum_{\substack{\mu \in \mathfrak{c}\mathfrak{d}^{-1} \\ \mu \gg 0}} a_\mu q^\mu$$

with $a_0 = 0$ for $f \in S_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$.

Corollary 4. *Every Hilbert modular form is a cups form, i.e.* $S_{\underline{k}}(\mathfrak{c}, \mathfrak{n}) = M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$*, unless* $k_\sigma = k_{\sigma'}$ *for all* $\sigma, \sigma' \in \Sigma_F$ *(the "parallel weight" case).*

Proof. Let $f \in M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$. If at some cusp, the expansion of $f|_{\underline{k}}\gamma$ has constant term $a_0 \neq 0$, then from $a_0 = \varepsilon^{\underline{k}+\underline{m}-\underline{t}} a_0$ for all $\varepsilon \in \mathcal{O}_F^{\times +}$ it follows that we must have $\varepsilon^{\underline{k}} = 1$ for all $\varepsilon \in \mathcal{O}_F^{\times +}$. But this is possible only if all the k_σ are equal, by Dirichlet's Unit Theorem. $\square$

Proposition 5. *(i)* $M_{\underline{k}}(\mathfrak{c}, \mathfrak{n}) = 0$ *unless* $k_\sigma \geq 0$ *for all* $\sigma \in \Sigma_F$*.*
(ii) $M_0(\mathfrak{c}, \mathfrak{n}) = \mathbb{C}$ *and* $S_0(\mathfrak{c}, \mathfrak{n}) = 0$*.*

Proof. **Part 1.** Assume that $0 \neq f \in M_{\underline{k}}(\Gamma)$. Then, for any $\gamma \in G_{\mathbb{Q}}^{+}$, $z \mapsto f|_{\underline{k}}\gamma(z, \ldots, z)$, $z \in \mathfrak{H}$, is an elliptic modular form of weight $\sum_{\sigma\in\Sigma_F} k_\sigma$, which is non-zero for some γ (since the image of the diagonal under $G_{\mathbb{Q}}^{+}$ is dense in $\mathfrak{H}_F$). Now the classical theory implies that $\sum_{\sigma\in\Sigma_F} k_\sigma \geq 0$, and $\sum_{\sigma\in\Sigma_F} k_\sigma = 0$ if and only if f is constant.

Part 2. Suppose that $\sum_{\sigma\in\Sigma_F} k_\sigma > 0$ and $k_{\sigma_0} = 0$ for some $\sigma_0 \in \Sigma_F$. Then every $f \in M_{\underline{k}}(\mathfrak{c}, \mathfrak{n})$ is a cusp form, since the weight is not parallel. Let

$$g(z) = y^{\underline{k}+\underline{m}-\underline{t}} f(z) = \left(\prod_{\sigma\neq\sigma_0} y_\sigma^{k_\sigma+m_\sigma-1}\right) f(z), \text{ where } z = (x_\sigma + iy_\sigma)_{\sigma\in\Sigma_F}.$$

Then g is holomorphic as a function of z_{σ_0}, and is invariant under Γ. Moreover g vanishes at cusps (since f does). Therefore g is bounded on $\mathfrak{H}$; and by Liouville's theorem, g must be constant, still as a function of z_{σ_0}.

Let $\Sigma_0 = \Sigma_F - \{\sigma_0\}$ and consider the projection map $\pi_{\Sigma_0} : \mathfrak{H}_F \to \mathfrak{H}^{\Sigma_0}$. Define a function h on $\mathfrak{H}^{\Sigma_0}$ by setting $h(z') = g(z)$ for $z' \in \mathfrak{H}^{\Sigma_0}$, where $z \in \mathfrak{H}_F$ is chosen arbitrarily with $\pi_{\Sigma_0}(z) = z'$. The value is independent of this choice by what was just proved, so h is a well-defined Γ-invariant function, and $|h|$ is bounded. It must be constant, and even zero, since the action Γ on $\mathfrak{H}^{\Sigma_0}$ is not discontinuous. (Indeed, the image of $G_{\mathbb{Q}}$ is dense in $\prod_{\sigma\in\Sigma_0} \mathrm{GL}_2(\mathbb{R})$). This shows that f is zero.

Part 3. Assume that $\sum_{\sigma\in\Sigma_F} k_\sigma > 0$ and $k_{\sigma_0} < 0$ for some $\sigma_0 \in \Sigma_F$. Let $G_{4,\mathfrak{c}}$ be the Eisenstein series of parallel weight 4 defined below, and put $g = f^4 G_{4,\mathfrak{c}}^{-k_{\sigma_0}}$. We obtain a non-zero form of weight $\underline{k}'$ with $k'_{\sigma_0} = 0$. This contradicts Part 2.

Hence, unless f is constant, it has weight $\underline{k}$ with all $k_\sigma > 0$. □

Example 2. *Eisenstein series.* To each $k \in \mathbb{Z}$ with $k > 2$ and each ideal class $\mathcal{C}$, we define an element $G_{k,\mathcal{C}}(z) \in M_{\underline{k}}(\mathfrak{a}, \mathcal{O}_F)$, for any ideal $\mathfrak{a} \subset \mathcal{O}_F$, as follows. (Here $\underline{k} = (k, k, \ldots, k)$.) Choose a representative $\mathfrak{c} \in \mathcal{C}$, and put

$$G_{k,\mathcal{C}}(z) = \mathbf{N}(\mathfrak{c})^k \sum_{(c,d)\in\mathbf{P}^1(\mathfrak{ac}\times\mathfrak{c})} \mathbf{N}(cz+d)^{-k},$$

where $\mathbf{P}^1(\mathfrak{ac}\times\mathfrak{c}) = \{(c,\, d) \in \mathfrak{ac}\times\mathfrak{c} | (c,\, d) \neq (0,\, 0)\}/\mathcal{O}_F^\times$.

Proposition 6. *This expression does not depend on $\mathfrak{c} \in \mathcal{C}$, and defines a modular form of weight $\underline{k} = (k, \cdots, k)$ with respect to $\Gamma_0(\mathfrak{a}, \mathcal{O}_F)$, called the Eisenstein series of weight k and class $\mathcal{C}$ with respect to $\Gamma_0(\mathfrak{a}, \mathcal{O}_F)$.*

As in the classical case, one can find the expansion at ∞ by making use of Poisson summation (see van der Geer [10, Chap. 1, sec. 6] for details.

2.7. **Classical Hilbert modular varieties.** Recall that in the classical theory we extend the upper half-plane $\mathfrak{H}$ to $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$, and for a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ the quotient $X_\Gamma = \Gamma\backslash\mathfrak{H}^*$ is the *modular curve* associated to Γ, which has the structure of a Riemann surface and of an algebraic curve. For suitable Γ (such as $\Gamma_0(N)$ or $\Gamma_1(N)$) it is a moduli space, parametrizing elliptic curves with some extra level structure (for example, elliptic curves with a given point of order N or with a given cyclic subgroup of order N).

Let $\mathfrak{H}_F^* = \mathfrak{H}_F \cup \mathbf{P}^1(F)$, which we endow with the following topology, called the *Satake topology*:

- At any point $z \in \mathfrak{H}_F$, a basis of open neighborhoods is given by the usual (product) topology.
- At the cusp $\infty \in \mathbb{P}^1(F)$, open neighbourhoods are the sets W_c for $c > 0$ defined by

$$W_c = \{z \in \mathfrak{H}_F | \prod_{\sigma\in\Sigma_F} \mathrm{im}(z_\sigma) > c\} \cup \{\infty\}.$$

- For any cusp σ, choose $\gamma \in G_{\mathbb{Q}}$ such that $\sigma = \gamma\infty$. A basis of open neighborhoods for σ is given by $\{\gamma W_c \mid c > 0\}$.

For a congruence subgroup Γ, we let $X_\Gamma = \Gamma\backslash\mathfrak{H}_F^*$. This is a compact Hausdorff topological space, which also carries the structure of an analytic space, called the *Hilbert modular variety* attached to Γ. When $[F:\mathbb{Q}]>1$, the space X_Γ is highly singular.

One can show that the Hilbert modular variety X_Γ is quasi-projective and has a minimal arithmetic compactification. This uses the fact that X_Γ admits a moduli theoretic interpretation. See Goren [11] for such a moduli theoretic interpretation (which concerns abelian schemes with real multiplication), and see Bailey-Borel [1], Rapoport [18] and Dimitriov [9] for the various compactifications.

Remark 3. The motivation for Definitions 3 and 4 is the following. The map

$$\gamma \mapsto \det(\gamma)^{-\underline{k}/2} j(\gamma,\, z)^{\underline{k}},\ z\in\mathfrak{H}_F,\ \gamma\in\Gamma,$$

is a 1-cocyle which defines an analytic line bundle $\underline{\omega}^{\underline{k}}$ over $Y_\Gamma = \Gamma\backslash\mathfrak{H}_F$. In the classical setting, the space of modular forms $M_{\underline{k}}(\Gamma)$ is the set of global sections of this line bundle. For arithmetic and number theoretic applications however, one is only interested in the forms that appear in the cohomology of X_Γ with coefficients in an algebraic local system *i.e.* given by algebraic representations of the group G. Such representations are of the form

$$\bigotimes_{\sigma\in\Sigma_F} \det{}^{m_\sigma}\otimes \mathrm{Sym}^{n_\sigma},$$

and only define local systems on X_Γ if the center of Γ acts trivially (see [12, 9]).

2.8. **Adelic Hilbert modular forms and varieties.** We recall that $G_{\mathbb{R}}^+$ acts transitively on $\mathfrak{H}_F$ by linear fractional transformations, and that the stabilizer of $\underline{i}=(i,\ldots,i)$ is given by $K_\infty^+ = (\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}))^{\Sigma_F}$. This action extends uniquely to $G_{\mathbb{R}}$ as follows. On the factor $\mathfrak{H}$ indexed by $\sigma\in\Sigma_F$, we let the element $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ act by $z_\sigma\mapsto -\bar{z}_\sigma$.

Let K be a compact open subgroup of the finite adèles G_f. This will play the role of a "generalised level". Our main (and essentially only) example of such subgroups are $K_0(\mathfrak{n})$ for integral ideals $\mathfrak{n}\subseteq\mathcal{O}_F$. This is defined as in the introductory section over $\mathbb{Q}$. Explicitly, let $\hat{\mathbb{Z}}=\prod_p \mathbb{Z}_p$ and $\hat{\mathcal{O}}_F=\prod_{\mathfrak{p}}\mathcal{O}_{F,\mathfrak{p}}$. Then $G(\hat{\mathbb{Z}})=\mathrm{GL}_2(\hat{\mathcal{O}}_F)$, and $K_0(\mathfrak{n})$ is the subgroup of elements $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ such that $c_{\mathfrak{p}}\in\mathfrak{n}\mathcal{O}_{F,\mathfrak{p}}$ for all $\mathfrak{p}$. (Here, $\mathfrak{p}$ runs over all primes of $\mathcal{O}_F$, and the condition is trivially satisfies when $\mathfrak{p}\nmid\mathfrak{n}$.) Note that the determinant map $K_0(\mathfrak{n})\to\hat{\mathcal{O}}_F$ is surjective; one might also consider choices of compact open subgroups K for which this condition fails, but matters are a little simpler when it holds.

Lemma 6. *Let K be a compact open subgroup of G_f. Then, there is a bijection*

$$G_{\mathbb{Q}}\backslash G_{\mathbb{A}}/KG_\infty^+ \xrightarrow{\sim} F^\times\backslash\mathbb{A}_F^\times/\det(K)\mathbb{R}_+^{\Sigma_F}.$$

In particular, when $\det(K)=\hat{\mathcal{O}}_F^\times$, this double coset space has the same cardinality as the narrow class group Cl_F^+.

Proof. This follows from Strong Approximation. □

We will now define an *adèlic Hilbert Modular Form of weight $\underline{k}$ and level K*, and discuss the connection between these and the classical ones considered previously. When $K=K_0(\mathfrak{n})$ these will also be said to have *level* $\mathfrak{n}$.

Definition 6. *An* adèlic Hilbert Modular Form *of weight $\underline{k}$ and level $\mathfrak{n}$ is a function $f: G_{\mathbb{A}}\to\mathbb{C}$ satisfying the following conditions:*

- (i) *$f(\gamma g u)=f(g)$ for all $\gamma\in G_{\mathbb{Q}}$, $u\in K$ and $g\in G_{\mathbb{A}}$. (So f is well-defined on the double coset space $G_{\mathbb{Q}}\backslash G_{\mathbb{A}}/K$.)*
- (ii) *$f(gu)=\det(u)^{\underline{k}+\underline{m}-\underline{t}} j(u,\,\underline{i})^{-\underline{k}} f(g)$ for all $u\in K_\infty^+$ and $g\in G_{\mathbb{A}}$.*

For all $x\in G_f$, define $f_x:\mathfrak{H}_F\to\mathbb{C}$ by $z\mapsto \det(g)^{\underline{t}-\underline{m}-\underline{k}} j(g,\,\underline{i})^{\underline{k}} f(xg)$, where we choose $g\in G_\infty^+$ such that $z=g\cdot\underline{i}$. By (ii), f_x does not depend on the choice of g.

(iii) *For all* $x \in G_f$, f_x *is holomorphic. (When* $F = \mathbb{Q}$*, an extra holomorphy condition at cusps is needed).*

(iv) *In addition, when* $\int_{U(\mathbb{A})/U(\mathbb{Q})} f(ux)du = 0$ *for all* $x \in G_{\mathbb{A}}$ *and all additive Haar measures* du *on* $U(\mathbb{A})$*, where* U *is the unipotent radical of* $G_{\mathbb{Q}}$*, we say that* f *is an* adèlic cusp form.

We will denote the space of all Hilbert modular forms (resp. cusp forms) of weight $\underline{k}$ and level K by $M_{\underline{k}}(K;\mathbb{C})$ (resp. $S_{\underline{k}}(K;\mathbb{C})$), or in the case $K = K_0(\mathfrak{n})$ by $M_{\underline{k}}(\mathfrak{n};\mathbb{C})$ (resp. $S_{\underline{k}}(\mathfrak{n};\mathbb{C})$). In the rest of this section, we will discuss the connection between classical and adèlic Hilbert modular forms.

We fix $\mathfrak{n}$ to be an integral ideal of F and $K = K_0(\mathfrak{n})$.

Now let $\mathfrak{c}_\lambda$, $\lambda = 1, \ldots, h^+$, be a complete set of representatives of the narrow ideal classes of F. For each $\lambda = 1, \ldots, h^+$, take $x_\lambda \in G_{\mathbb{A}}$, so that $\alpha_\lambda = \det(x_\lambda)$ generates the ideal $\mathfrak{c}_\lambda$. The simplest choice of x_λ is to take $x_\lambda = \begin{pmatrix} \alpha_\lambda & 0 \\ 0 & 1 \end{pmatrix}$ where α_λ is a local generator of the ideal $\mathfrak{c}_\lambda$ (meaning $\alpha_\lambda \mathcal{O}_{F,\mathfrak{p}} = \mathfrak{c}_\lambda \mathcal{O}_{F,\mathfrak{p}}$ for all primes $\mathfrak{p}$). Then, by Lemma 6, we have the finite disjoint union

$$G_{\mathbb{A}} = \coprod_{\lambda=1}^{h^+} G_{\mathbb{Q}} x_\lambda K_0(\mathfrak{n}) G_\infty^+,$$

since the x_λ represent the h^+ double cosets $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / K_0(\mathfrak{n}) G_\infty^+$.

Define $\Gamma_\lambda = \Gamma_0(\mathfrak{c}_\lambda, \mathfrak{n})$; it is not hard to show (exercise!) that (with the choice of x_λ given above)

$$\Gamma_\lambda = x_\lambda K_0(\mathfrak{n}) G_\infty^+ x_\lambda^{-1} \cap G_{\mathbb{Q}}.$$

Now to each adèlic Hilbert modular form f, we associate the h^+-tuple $(f_1, \ldots, f_{h^+}) \in \oplus_{\lambda=1}^{h^+} M_{\underline{k}}(\mathfrak{c}_\lambda, \mathfrak{n})$, where $f_\lambda = f_{x_\lambda}$ is given by Definition 6, so that

$$f_\lambda(z) = \det(g)^{\underline{t}-\underline{m}-\underline{k}} j(g,i)^{\underline{k}} f(x_\lambda g)$$

where $g \in G_\infty^+$ is chose so that $g(i) = z$.

Proposition 7. *The map*

$$\begin{array}{rcl} M_{\underline{k}}(\mathfrak{n}) & \to & \bigoplus_{\lambda=1}^{h^+} M_{\underline{k}}(\mathfrak{c}_\lambda, \mathfrak{n}) \\ f & \mapsto & (f_1, \ldots, f_{h^+}) \end{array}$$

is an isomorphism of complex vector spaces, which restricts to an isomorphism $S_{\underline{k}}(\mathfrak{n}) \equiv \bigoplus_{\lambda=1}^{h^+} S_{\underline{k}}(\mathfrak{c}_\lambda, \mathfrak{n})$.

Proof. The map is well-defined and $\mathbb{C}$-linear; its inverse maps (f_λ) to the $\mathbb{C}$-valued function f on $G_{\mathbb{A}}$ defined by

$$f(\gamma x_\lambda g) = f_\lambda|_{\underline{k}} g_\infty(\underline{i}), \quad \gamma \in G_{\mathbb{Q}}, \;\; g \in K_0(\mathfrak{n}) G_\infty^+.$$

□

In the rest of these notes, we will often use Proposition 7 to identify f with the h^+-tuple $(f_1, \ldots, f_{h^+})$.

Remark 4. The choice of x_λ given above forces us to use the "twisted" groups Γ_λ, and hence h^+-tuples of classical HMFs on $\mathfrak{H}_F$, each for a different group Γ_λ. There are, however, other possibilities. For example, if h^+ is odd, then every ideal class may be represented by the square of an ideal, say $\mathfrak{c}_\lambda^2$; taking α_λ to be an idèle generating $\mathfrak{c}_\lambda$ as before, we may now take $x_\lambda = \begin{pmatrix} \alpha_\lambda & 0 \\ 0 & \alpha_\lambda \end{pmatrix}$ which is *central* (being scalar). Now instead of the Γ_λ used above we have instead

$$\Gamma_\lambda = x_\lambda K_0(\mathfrak{n}) G_\infty^+ x_\lambda^{-1} \cap G_{\mathbb{Q}} = K_0(\mathfrak{n}) G_\infty^+ \cap G_{\mathbb{Q}} = \Gamma_0(\mathcal{O}_F, \mathfrak{n}) = \Gamma_0(\mathfrak{n}),$$

independent of λ! Our h^+-tuple of HMFs are now all on the same, untwisted group, namely $\Gamma_0(\mathfrak{n})$ itself.

More generally the number of different Γ_λ one needs to consider can be reduced from $h^+ = \#\mathcal{C}\ell_F$ to $\#\mathcal{C}\ell_F/\mathcal{C}\ell_F^2$. However, for simplicity of notation we will for the rest of these notes return to the original choices, where all h^+ of the Γ_λ are distinct.

Exercise: see how the component functions f_λ, and the groups Γ_λ, change if we either replace the $\mathfrak{c}_\lambda$ with different ideal class representatives, of the x_λ with different elements whose determinants generate $\mathfrak{c}_\lambda$, or both.

We define the *adèlic Hilbert modular variety* $Y_0(n)$ by

$$Y_0(\mathfrak{n}) = G_\mathbb{Q}\backslash G_\mathbb{A}/K_0(\mathfrak{n})K_\infty^+.$$

Recall that there is an identification $G_\infty^+/K_\infty^+ \cong \mathfrak{H}_F$ (via $[F:\mathbb{Q}]$ copies of $\mathrm{GL}_2^+(\mathbb{R})/\,\mathrm{O}(2) \cong \mathfrak{H}$); hence we see that

$$Y_0(\mathfrak{n}) = \coprod_{i=1}^{h^+} \Gamma_0(\mathfrak{c}_i,\, \mathfrak{n})\backslash\mathfrak{H}_F = \coprod_{i=1}^{h^+} Y_{\Gamma_0(\mathfrak{c}_i,\mathfrak{n})}.$$

We compactify $Y_0(\mathfrak{n})$ by adding a finite number of points (cusps) to each connected component. This gives

$$X_0(\mathfrak{n}) = \coprod_{i=1}^{h^+} \Gamma_0(\mathfrak{c}_i,\, \mathfrak{n})\backslash\mathfrak{H}_F^* = \coprod_{i=1}^{h^+} X_{\Gamma_0(\mathfrak{c}_i,\mathfrak{n})}.$$

Again, we remark that when h^+ is odd, we may replace this with a disjoint union of h^+ identical spaces $X_{\Gamma_0(\mathfrak{n})}$.

2.9. **Fourier coefficients of an adèlic modular form.** We continue to keep the level $\mathfrak{n}$ (an integral ideal of $\mathcal{O}_F$) and the compact open subgroup $K = K_0(\mathfrak{n})$ fixed.

Let $f \in S_{\underline{k}}(\mathfrak{n})$ be a Hilbert modular cusp form. We recall that, by Koecher's principle, $f = (f_1, \ldots, f_{h^+})$ is holomorphic at all cusps, where the component functions f_j are defined by

$$f_j(z) = \det(g)^{\underline{t}-\underline{n}-\underline{k}}\gamma(g,\underline{i})^{\underline{k}}f(x_jg)$$

(where $g \in G_\infty^+$ satisfies $g(\underline{i}) = z$). Each f_j admits a Fourier expansion

$$f_j(z) = \sum_{0\ll\mu\in\mathfrak{c}_j\mathfrak{d}^{-1}} a_\mu(f_j)e^{2\pi i\,\mathrm{Tr}(\mu z)}.$$

For any integral ideal $\mathfrak{m}$, let $j \in \{1,\ldots,h^+\}$ be such that $\mathfrak{m}$ is in the narrow ideal class of $\mathfrak{c}_j^{-1}\mathfrak{d}$, which we write $\mathfrak{m} \sim \mathfrak{c}_j^{-1}\mathfrak{d}$. Choose a totally positive element $\mu \in F$ such that $\mathfrak{m} = (\mu)\mathfrak{c}_j^{-1}\mathfrak{d}$, and define

$$c(f,\mathfrak{m}) := \mu^{\underline{m}}a_\mu(f_j).$$

It is not hard to see that the coefficient $c(f,\mathfrak{m})$ is independent of the choice of μ; with more work, to see exactly how the f_j change when we use different representatives c_j for the ideal classes, one can show that $c(f,\mathfrak{m})$ does not depend on the choice of the $\mathfrak{c}_j$ either. We may therefore call it the *Fourier coefficient* of the form f associated to the integral ideal $\mathfrak{m}$.

2.10. **The Hecke action.** Let $\mathfrak{p}$ be a prime ideal and $\varpi_\mathfrak{p}$ a uniformizer of $\mathfrak{p}$. Write the double coset as a finite disjoint union of single cosets:

$$K\begin{pmatrix}1 & 0\\ 0 & \varpi_\mathfrak{p}\end{pmatrix}K = \coprod_j u_jK;$$

the number of single coset terms is either $N(\mathfrak{p})+1$ (if $\mathfrak{p}\nmid\mathfrak{n}$) or $N(\mathfrak{p})$ (if $\mathfrak{p}\mid\mathfrak{n}$). For each $f \in M_{\underline{k}}(\mathfrak{n})$, define $T_\mathfrak{p}f$ by setting

$$T_\mathfrak{p}f(x) := \sum_j f(xu_j)$$

for $x \in G_\mathbb{A}$. This gives a well defined linear map $T_\mathfrak{p} : M_{\underline{k}}(\mathfrak{n}) \to M_{\underline{k}}(\mathfrak{n})$ which preserves the space of cusp forms $S_{\underline{k}}(\mathfrak{n})$. We call $T_\mathfrak{p}$ the *Hecke operator at the prime* $\mathfrak{p}$.

More generally, for any integral ideal $\mathfrak{m}$, let μ be an idèle generator of $\mathfrak{m}$ and write

$$K \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} K = \coprod_j u_j K;$$

then $T_{\mathfrak{m}} f(x) = \sum_j f(xu_j)$. The Hecke operators $T_{\mathfrak{m}}$, as $\mathfrak{m}$ runs through all integral ideals in $\mathcal{O}_F$, generate a finite rank commutative $\mathbb{Z}$-subalgebra of $\mathrm{End}(S_{\underline{k}}(\mathfrak{n}))$ which we call the Hecke algebra of level $\mathfrak{n}$ and denote by $\mathbb{T}_{\underline{k}}(\mathfrak{n})$. The Hecke algebra $\mathbb{T}_{\underline{k}}(\mathfrak{n})$ is generated by the $T_{\mathfrak{p}}$ for prime $\mathfrak{p}$, and the operators $T_{\mathfrak{m}}$ satisfy multiplicative relations just as in the classical case. $\mathbb{T}_{\underline{k}}(\mathfrak{n})$ is a commutative algebra which is "almost self-adjoint" (see below) with respect to the Petersson inner product on $S_{\underline{k}}(\mathfrak{n})$. As a result, it is diagonalizable and admits a common basis of eigenvectors.

Definition 7. *Let f be a Hilbert modular cusp form. We say that f is an* eigenform *if it is a common eigenvector for the Hecke algebra. A* normalized eigenform *is an eigenform f such that $c(f, \mathcal{O}_F) = 1$.*

Thanks to Shimura [19], we have the following result.

Theorem 8 (Shimura)**.** *Let $f \in S_{\underline{k}}(\mathfrak{n})$ be a normalized eigenform. Then for each integral ideal $\mathfrak{m}$, the Fourier coefficient $c(f, \mathfrak{m})$ is an* algebraic integer *which satisfies the relation:*

$$T_{\mathfrak{m}} f = c(f, \mathfrak{m}) f.$$

Moreover, if $K_f = \mathbb{Q}(c(f, \mathfrak{m}), \mathfrak{m} \subseteq \mathcal{O}_F)$ is the field generated by all the coefficients $c(f, \mathfrak{m})$, then K_f is a number field which is either totally real or a CM field.

The following result is known as the Weak Multiplicity One Theorem.

Theorem 9. *Every normalised eigenform is uniquely determined by its Fourier coefficients. In other words, if f and g are two normalized eigenforms and*

$$c(f, \mathfrak{p}) = c(g, \mathfrak{p}), \quad \textit{for all } \mathfrak{p} \nmid \mathfrak{n},$$

then $f = g$.

2.10.1. *Action of the class group.* Since we have been working with $K_0(\mathfrak{n})$, the adèlic modular forms defined here are generalizations of classical forms for the congruence subgroup $\Gamma_0(\mathfrak{n})$. One would guess, therefore, that there are not characters in the picture: to define modular forms with character would surely involve switching from $K_0(\mathfrak{n})$ to $K_1(\mathfrak{n})$ and introducing the action of $(\mathcal{O}_F/\mathfrak{n})^*$ to split the space into subspaces indexed by the characters of this finite group. However the correct generalization from the classical elliptic modular setup involves introducing not just Dirichlet characters (induced from characters of $(\mathcal{O}_F/\mathfrak{n})^*$ as above), but Hecke characters, which are characters of the idèle class group. These involve characters of the ideal class group as well as Dirichlet characters modulo $\mathfrak{m}$ for some integral ideal $\mathfrak{m}$ which should divide the level $\mathfrak{n}$. Even at level $\mathfrak{n} = (1)$, therefore, there are "unramified characters" where the modulus is $\mathfrak{m} = (1) = \mathcal{O}_F$, which are simply characters of the ideal class group.

Without going into details (there will be some in the second half of the course), at any level $\mathfrak{n}$ there is an action of the (narrow) ideal class group $\mathcal{C}\ell_F^+$ on the space of Hilbert modular forms of level $\mathfrak{n}$. Hence the space of cusp forms $S_{\underline{k}}(\mathfrak{n})$ decomposes as a direct sum of spaces $S_{\underline{k}}(\mathfrak{n}, \chi)$ as χ runs over the characters of $\mathcal{C}\ell_F^+$. Now, the Hecke operators are not quite self-adjoint: the relation is

$$\langle T_{\mathfrak{p}} f, g \rangle = \chi(\mathfrak{p}) \langle f, T_{\mathfrak{p}} g \rangle .$$

[We have not defined the Petersson inner product in the HMF situation: see the bibliography for the details. It is just given as a $[F : \mathbb{Q}]$-dimensional integral over a fundamental region just as in the case $F = \mathbb{Q}$.]

2.11. **The subspace of newforms.** Let $\mathfrak{n}_1$ be a divisor of $\mathfrak{n}$ and $\mathfrak{n}_2$ a divisor of of $\mathfrak{n}\mathfrak{n}_1^{-1}$. For each $g \in S_{\underline{k}}(\mathfrak{n}_1)$, there is a unique form denoted $g|\mathfrak{n}_2 \in S_{\underline{k}}(\mathfrak{n})$ with Fourier coefficients satisfying

$$c(g|\mathfrak{n}_2,\, \mathfrak{m}) = c(g,\, \mathfrak{m}\mathfrak{n}_2^{-1})$$

for all integral ideals $\mathfrak{m}$. (Here we use the convention that $c(-, \mathfrak{m}) = 0$ if $\mathfrak{m}$ is a fractional ideal which is not integral.) This gives a linear map

$$S_{\underline{k}}(\mathfrak{n}_1) \to S_{\underline{k}}(\mathfrak{n}),\ g \mapsto g|\mathfrak{n}_2.$$

We define $S_{\underline{k}}^{\mathrm{old}}(\mathfrak{n})$ to be the subspace of $S_{\underline{k}}(\mathfrak{n})$ generated by the image of all these linear maps, as $\mathfrak{n}_1$ and $\mathfrak{n}_2$ vary. This space is stable under the action of $\mathbb{T}_{\underline{k}}(\mathfrak{n})$. We define the *newspace* $S_{\underline{k}}^{\mathrm{new}}(\mathfrak{n})$ to be the orthogonal complement of $S_{\underline{k}}^{\mathrm{old}}(\mathfrak{n})$ with respect to the Petersson inner product. Since the $\mathbb{T}_{\underline{k}}(\mathfrak{n})$ is almost self-adjoint with respect to the Petersson inner product, the decomposition $S_{\underline{k}}(\mathfrak{n}) = S_{\underline{k}}^{\mathrm{old}}(\mathfrak{n}) \oplus S_{\underline{k}}^{\mathrm{new}}(\mathfrak{n})$ is preserved by the Hecke action. A normalized eigenform lying in $S_{\underline{k}}^{\mathrm{new}}(\mathfrak{n})$ is called a *newform*. The set of newforms forms a basis for $S_{\underline{k}}^{\mathrm{new}}(\mathfrak{n})$.

Let f be a newform. The *L-series* attached to f is defined by

$$L(f,\, s) := \sum_{\mathfrak{m} \subseteq \mathcal{O}_F} \frac{c(f, \mathfrak{m})}{\mathrm{N}(\mathfrak{m})^s}$$

for $s \in \mathbb{C}$. It extends to an entire function which admits an Euler product expansion.

2.11.1. *The action of Hecke operators on component functions and Fourier coefficients.* In order prove that $T_{\mathfrak{m}} f = c(f, \mathfrak{m}) f$ for a normalized eigenform, we need to reformulate the action of $T_{\mathfrak{m}}$ to see how it applies to the "component functions" of $f = (f_1, \ldots, f_{h^+})$. We will not do this in detail here, but remark that the j-component of $T_{\mathfrak{p}} f$ is expressed in terms of the j'-component of f, where $\mathfrak{c}_j \sim \mathfrak{p}\mathfrak{c}_{j'}$. Clearly, the $T_{\mathfrak{p}}$ for principal primes $\mathfrak{p}$ (and more generally the $T_{\mathfrak{m}}$ for principal $\mathfrak{m}$) are easier to deal with: they are also easier to implement when computing with Hilbert modular forms explicitly. It is perhaps remarkable that knowledge of the action of just the principal $T_{\mathfrak{m}}$ on the "principal component" f_1 (assuming the components to be labelled so that $j = 1$ corresponds to the principal class) is enough to recover the complete system of Hecke eigenvalues of a newform. This has useful practical consequences.

2.12. **Galois representations attached to Hilbert modular forms.** Let f be a Hilbert newform. By work of several people, one can associated a compatible family of p-adic Galois representations. This result underscores the importance of Hilbert modular forms in Number Theory and Arithmetic.

Theorem 10 (Déligne, Carayol, Taylor, Blasius-Rogawski, Saito)**.** *Let f be a Hilbert newform of weight $\underline{k}$ and level $\mathfrak{n}$. Let K_f be the field of coefficients of f. Let $\ell \geq 2$ be a rational prime, λ a prime of K_f that lies above ℓ and $K_{f,\,\lambda}$ the completion of K_f at λ. There there is an absolute irreducible, totally odd Galois representation*

$$\rho_{f,\,\lambda} :\ \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{K}_{f,\,\lambda})$$

such that, for any prime $\mathfrak{p} \nmid \ell\mathfrak{n}$,

$$\mathrm{tr}(\rho_{f,\,\lambda}(\mathrm{Frob}_{\mathfrak{p}})) = c(f, \mathfrak{p}) \textit{ and } \det(\rho_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})) = \mathbf{N}\mathfrak{p}^{k_0 - 1}.$$

When $F = \mathbb{Q}$ and $k = 2$, one can show that the representation $\rho_{f,\,\lambda}$ is realizable in the Tate module of some simple factor of the Jacobian of the modular curve $X_0(\mathfrak{n})$. This result is known as the Eichler-Shimura construction [14, Chap. XII]. In the general case, one has a analogous conjecture which can be equivalently stated as follows.

Conjecture 1 (Eichler-Shimura)**.** *Let f be a Hilbert newform of level $\mathfrak{n}$ and parallel weight 2 over a totally real field F. Let K_f be the number field generated by the Fourier coefficients of f. Then there exists an abelian variety A_f defined over F, with good reduction outside of $\mathfrak{n}$, such that $K_f \hookrightarrow \mathrm{End}(A_f) \otimes \mathbb{Q}$ and*

$$L(A_f,\, s) = \prod_{\sigma \in \mathrm{Hom}(K_f,\, \mathbb{C})} L(f^{\sigma},\, s),$$

where f^σ is obtained by letting σ act on the Fourier coefficients of f.

For a discussion of the known cases of Conjecture 1, we refer to Blasius-Rogawski [2] and Zhang [22]. In those cases, the abelian variety A_f is constructed a factor of the Jacobian of some Shimura curve of level $\mathfrak{n}$.

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