CHANGING EMILY'S IMAGES

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Introduction

Symbolism has the power to dually and ambiguously represent processes to *do* and concepts to *know* (Gray & Tall, 1994). To benefit from the flexibility provided by such ambiguity the young child's conception of arithmetic must progress through several phases of compression: lengthy counting procedures which are interpretations of processes to *do* must eventually become concepts to *know*.

This is a story of an eight year old who had considerable difficulty in simple arithmetic. Though she could use real things to help her sort out mathematical combinations she had begun to feel that arithmetic should be done in her head. However, her efforts to do so did not lead to successful outcomes. Her mental approaches relied heavily upon the manipulation of imaginistic objects–analogues of the very things she was trying to move away from. It was hypothesised that if the 'procedural clutter' associated with her physical interpretation of mathematical symbols could be removed, she too could focus on the power of symbols. To do this we provided her with a graphic calculator, the 'super-calculator'. Our focus is the opportunity that the resource may give for stimulating the construction of mental imagery associated directly with arithmetical symbols as opposed to imagery that is an analogical transformation of them.

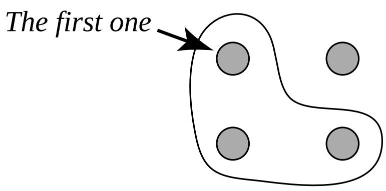
Pitta and Gray (1997) described how children at extreme levels of achievement in elementary arithmetic focus on imagery which is of different qualities. Imagery identified by 'high achievers' tended to be symbolic, used to support the production of known facts and/or the numeric transformations which produce derived facts. Imagery reported by 'low achievers' was usually based on analogical representations of physical objects. These images appear to be clear imitations of actions that could have taken place with real objects. The issue for this paper is whether an alternative 'procedure' may discourage a 'low achiever's' need to use manipulatives in the mind but stimulate the creation and construction of symbolic images that help to generate thought.

We first met Emily in February 1995. She had considerable difficulty with elementary arithmetic. Articulate and highly motivated, she was identified as one of the lowest achievers within her year group of 119 children. Test results (SEAC, 1994) placed her amongst the bottom four children. Our initial conversations with her were about the numbers 1 to 10. Her responses were dominated by descriptions of images that were analogues of physical objects. She relied extensively on active mental images to deal with elementary number combinations If these involved combinations greater than ten Emily made considerable use of her fingers.

Emily saw the numbers one to six as mental arrays of dots in the mind very much like those arranged on a die. It is very possible that extensive experience with board games supplied the basis for these. Emily explained:

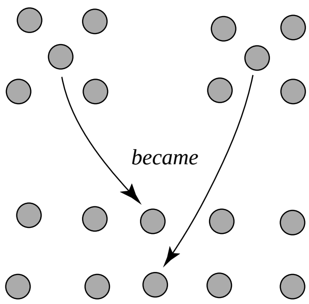
> *When I was young, when it was winter, we often played board games because we were not allowed outside. We were using dice. We were playing all of the time using dice.*

Numbers between seven and ten were mental images of fingers arranged in a linear fashion. Emily manipulated her mental images of dots, her preferred image, relatively easily. The solution to 4–3 was explained as:



As I see it there's two dots above each other and then there's.... the first one, the one below and the one next to it are being taken away and there is only one left up at the top.

Whilst 5+5 was frequently see as a transformation of two arrays of five.



Emily described how she could take away the dot from the middle of the five and put it in line with the two's. She now had a line "*with five lots of two and I can see that is ten*"

Emily recognised that there was much greater difficulty associated with her 'finger like' images. Using these meant doing two things at once, counting and concentrating on the sequence in which each finger was used:

> *I am trying to think out the answer as well as use all of my fingers–this is confusing... with the dots it is easier [than with fingers] because you don't have to keep thinking, 'No it's that one I need to move, no, its that one, or that one... [with the dots] it doesn't matter which you move.*

It seemed as if the arrangement of the dots was allowing her to immediately see the amount in the set whereas the linear arrangement of fingers forced her to count.

Emily seemed to associate counting with fingers with the use of a sequence of particular fingers. When she used mental images of the dots she implied that she could use any dots.

For relatively more difficult combinations such as ‘nine take away six’ Emily used her fingers in an indirect way by ‘feeling’ them without looking at them, touching them or moving them. But Emily did not like using her fingers:

> *I find it easier not to do it with my fingers at times because sometimes I get into a big muddle with them because I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. I can take longer to work out the sum than it does to work out the sum in my head.*

Emily appeared to recognise that there was a qualitatively difference between using perceptual items and mental representations of these items. It was not only that she believed the later was easier but to her it also made a difference between ‘*doing*’ arithmetic and ‘*thinking*’ about arithmetic:

> *I try not to use my hands much... I don’t bother looking because I am too busy thinking so... when I am not using my hands I am trying to work the sum ou*t.

Overall Emily’s experience had led her to some conclusions about simple arithmetic. First she felt it was easier to do the sum in her head and secondly, some images were better than others. It seemed to her that it was easier to see a number and remember it if it was recognised by some form of pattern like the array on a die. It was harder to think about if the representation was based upon a line of finger like objects, each being focused upon at a separate point in the counting procedure. Thirdly, arithmetic involved being seen to be ‘doing’, but this was unsettling because she was trying to ‘think’. Unfortunately however, she was not thinking with the tools her more able peers were using, the arithmetical symbols. Her tools were analogical images of real objects manipulated in accordance with her recollections of former experiences. Numerical symbols were concretised to form objects which supported the use of mental imagery that was episodic and active. Her focus was on an action which could be simplified by the nature of the representation that she gave to the objects. However, whether or not she used dots, fingers or finger like objects the intrinsic quality of the object did not change. Her perception of quantity influenced her choice of objects and the way the objects were used, so the focus turned to the nature of the action but the action was always the same—counting. Though it was evident that her procedural competence was sound, it had not supported the encapsulation of numerical processes into concepts. She was not filtering out unnecessary information and making the cognitive shift that would lead to the realisation that symbols could become objects of thought. Unless some alternative pedagogy was tried the longer term prognosis of Emily's achievement was that the qualitative difference between Emily's thinking and that of her more able peers would widen into a gulf. A graphic calculator was to be the tool for this pedagogy.

An Alternative Procedure: Focusing on Symbols

Contrary to the belief in many quarters that calculators have caused a decline in the ability of children to handle basic arithmetic has been recognised for some time that calculators can give children an insight into numerical patterns and relationships that are hard to discern if children are constrained by the use of lengthy counting procedures or the knowledge of isolated number combinations.

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4 + 5
                         9
4 + 4 + 1
                         9
3 + 2 + 4
                         9
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Graphic Calculator Display
Combinations to 9

The graphic calculator—after Ken Ruthven we called it a 'supercalculator'—seems to have an added advantage in this field. Combinations can be recorded and displayed in their entirety and equivalent outcomes from different procedures may also be seen at the same time. In addition the child can control the form of display on the screen.

For our attempt to minimise Emily's focus on counting the supercalculator offered two strengths; it provided an alternative procedure, pressing buttons, and it also had the potential to provide an alternative representation for numbers, it could display all symbols and operations at the same time. It was conjectured that this would offer an opportunity to stop counting and concentrate on numerical symbols as objects of thought. This, it was conjectured, would provide a stimulus which would support mental organisation.

A calculator provides and opportunity to create a number by pressing a button. It also permits a particular number to be created using the combination of a composite sequence of button pressing. Thus, by asking the child to create 9, this could be done by pressing 4+5=, by pressing 6+3= or it could be formed from 2+3+4 or 13–4 etc. By eliminating a counting procedure the 'alternative' procedure had the potential to create a "wholeness" about number. This may be seen at two levels; a specific one in which the focus could be on number triples, such as 4, 5, 9, and a more generic one during which it is possible to identify the relationships between numbers and simple operations, for example 9 is 4 +5, or 12–3, or 10–4+3 etc. It is unfortunately the case that many "low achievers" find it hard to switch from harder to easier methods if the first is habitual and unfamiliar. The "button pressing" procedure had the potential to overcome this difficulty since the child may not regard it as a mathematical action which should become a focus of attention and the child feels is worth memorising whereas counting is.

Emily was introduced to the super calculator in April 1995. The programme build around its use was not seen as simply another way of doing things. The calculator was not a means for completing the result of arithmetical combinations but a way of seeking different combinations that made a particular number. Emily's mental or physical procedures provide her with one route to a number. The calculator provide an infinite number of routes. Thus she started with the number and considered alternative ways to obtain it. Four phases were established to support Emily's development:

1. An opportunity to think about numbers without using the calculator.
2. The calculator is used to support thinking not simply to check answers. Emily could control the form of the numbers and seeing one combination maintained in display she could try the same numbers with a different button pressing procedure. Memory usually associated with holding quantities and carrying out counting procedures could be directed towards thinking about number combinations.
3. At the end of each activity she could consider interesting things that had been discovered during the activity.
4. She was given and opportunity to talk about individual numerals and associated combinations.

Working with nine...... 9

Making nine **1.** ....

2. .... **3.** ....

4. .... **5.** ....

Working with the calculator

Ways to make nine

1. .... **2.** ....

3. .... **4.** ....

5. .... **6.** ....

Ways to make nine starting with 5

1. .... **2.** ....

3. .... **4.** ....

5. .... **6.** ....

Ways to make nine starting with 10

1. .... **2.** ....

3. .... **4.** ....

5. .... **6.** ....

An interesting thing I have discovered ....

....

....

....

To accompany her work a specially personalised booklet was designed with each page following a pattern similar to that in the adjacent figure.

The programme called for Emily to try to complete a page of her booklet each week. Each week she discussed her work with the programme designers. During this time she was asked to talk about her numbers without access to the calculator or to her written responses.

Programme Development

Initially Emily had to overcome some reluctance to use the calculator. This stemmed largely from her perception of what others may think. However, by the end of the first week she had established that there were many ways in which she could make nine, the first number in the booklet. There were of course standard addition combinations such as 4 + 5, 3 + 6 etc. but she also provided others, 4 + 4 +1, 3 + 4 + 2, and using the starting points of 5 and 10 she now provided solutions such as, 5+1+1+2, 5 + 5 – 1, 5 + 6 – 2, 10 – 1. Emily admitted that she wouldn't have thought of these sorts of combinations earlier but her outstanding discovery for the week was that she had found out that she could add larger numbers and then take away; "*I didn't know that you could add larger numbers and then take away. I didn't know you could go up and down.*"

As she worked through the programme written evidence of Emily's use of standard triples during the non-calculator phase tended to decline. It became noticeable that for the first four numerals in her sequence, 9, 7, 8, and 6 she gave at most two but then she provided other 'non standard' combinations. When working with 7 for example she provided 10+10+10–20–3, with eight she provided 99–91 and 34–32+6. Working with the calculator she provided written evidence of combinations such as 90–80–4=6, 2+9+1–6=6, 30–15–9=6, 40–30–5=5, 10+30–30–2=8, 5+20–19=6.

It soon became evident that Emily's understanding of the relationship between numbers was beginning to change. She began to see a different framework for working with numbers:

> *Well,... before I would have found it harder with nine, but...um...its not that hard because I know that ten is really easy so nine is really easy because you just take away one from ten...*
>
> *It was easier to take away from eight than I thought it would be. Before I found it a bit hard with the other numbers. I thought eight would be a bit hard. But in the end it wasn't as hard as I thought it would be.*
>
> *I have discovered it is much easier to use multiplication in sums*

Inevitably pattern became a feature of Emily's "discovery" and it was common for Emily's written work to extensively include any numbers up to 100 and at times she included numbers over 100 in her combinations. She was beginning to realise that:

> *It is a lot easier to work with big numbers than I thought... I thought that big numbers would be very hard because they are so big... but it isn't. It is just the same as low numbers.*

It was evident from our discussions that Emily was now talking about numbers as objects. During all of the interviews that followed work with the calculator only on one occasion did she volunteer information about her dots. However was left until a series of follow-up interviews in January 1996 for us to begin to obtain some evidence that her imagery may be changing. When asked to think about numbers that make seven Emily's first comment was:

> *I just see the symbol 7 flashing in my mind waiting as if I was about to add it up...*

During our investigations into children's imagery no other low achiever had associated the word 'flashing' with symbolism to describe imagery (Pitta & Gray, 1997). The word had dominated descriptions of imagery by high achievers. Other numbers were also associated with this notion of flashing and when directly asked to talk about what she could see when she heard the word "Four" Emily responded by saying

> *4 flashes through my mind, and then I see, two two's like on a dice, 2 + 2, 100-96, four pounds...*

Discussion

In contrast to interaction with concrete objects which requires the individual to interpret what is going on, interaction with the super calculator offers a system in which the individual could build and test concepts first by observing and then by predicting and testing what happens. The form of presentation could be directly controlled by the child. What was becoming clear from our interactions with Emily was that she was building a different range of meanings associated with numbers and numerical symbolism – she was beginning to build a new image, a symbolic one that could stand on its own or be part of the options that would give flexibility. It seems as if her imagery was beginning to be associated with the notion of 'thought generator' rather than being seen as essential to thought.

Super calculators can carry out the evaluation of numerical expressions whilst the child can concentrate on the meaning of symbolism that remains evident throughout. If arithmetical activity focuses on the process of evaluation and the meaning of the symbolism it can offer a way into arithmetic that helps those children who are experiencing difficulty develop a more powerful understanding of symbols. However, belated emphasis on the ambiguous meaning of symbolism, when the greater proportion of previous experience has emphasised procedural and manipulative aspects, is embraced with difficulty. We may need to reappraise our purpose in emphasising counting procedures with the "low achievers". It may be too late once the die is cast.

References

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