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The Articulation Principle for making long-term sense of mathematical expressions by how they are spoken and heard: Two case studies

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Abstract: The Articulation Principle explains how mathematical expressions can be given a clear and unambiguous meaning by the way in which they are spoken and heard. Leaving short pauses in speech, sub-expressions can be interpreted as operations to be carried out in time or as mental objects that can be manipulated at a more sophisticated level. This offers a fundamental foundation for the growth of meaningful mathematical thinking at all levels from young children to the wide array of adult mathematics. This paper sets the Articulation Principle in a wider long-term learning framework and provides empirical evidence for its use in meaningful interpretation of mathematical expressions. In this paper, we investigate how two teachers who have learned to operate routinely with expressions react when they are presented with expressions spoken and written in different ways. This reveals how the experience enables them to give meaning to mathematical expressions that previously had only been learned by rote. The Articulation Principle has the advantage that it can be introduced at any level to help teachers and learners make sense of symbolism in a manner appropriate to their experience and level of development. It has wide ranging implications in addressing the development of meaningful long-term curricula in the challenge of "math wars" between communities of practice with differing beliefs and needs in a complex society.

Keywords: Articulation principle, Mathematical expressions, Order of precedence, Sense making, Supportive and problematic conceptions, Three worlds of mathematics

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1 Introduction

The interpretation of mathematical expressions is traditionally handled by introducing conventions, such as multiplication takes precedence over addition, and mnemonics, such as Please Excuse My Dear Aunty Sally (PEMDAS), to specify the order of precedence "Parenthesis, Exponent, Multiplication, Division, Addition, Subtraction", or its English equivalent BIDMAS (Brackets, Index, Division, Multiplication, Addition, Subtraction). A range of variants are used around the world.

These mnemonics, which are formulated to assist students to remember the order of precedence, are interpreted in a variety of ways that may cause conceptual difficulties. One reason is because they are seen as being formulated in terms of arbitrary conventions that need to be learned by rote (Hewitt, 2012). Learning by rote without conceptual meaning can involve individual learners interpreting the mnemonics in unintended ways. For instance, a mnemonic such as PEMDAS may lead to the belief that the operations must be performed in that given order, so that the order MD means multiplication must be given priority over division and AS gives priority of addition over subtraction (Cardone, 2015; Glidden. 2008; Papadopoulos, 2015). There is also a tendency to read an expression such as $2 + 2 \times 2$ in the order from left to right to give $2 + 2$ is 4 and then 4×2 is 8. Discussions with classroom teachers (Dupree, 2016; Jeon, 2012; Zazkis, 2018) reveal a wide range of student interpretations of mnemonics that do not agree with conventional expectations. Instead of learning arbitrary rules of the order of operations, the National Council of Teachers of Mathematics (NCTM, 2009, p. 4) proposes making sense through "developing understanding of a situation, context, or concept by connecting it with existing knowledge or previous experience". This proposal opens many areas of discussion to deal with different situations, contexts and concepts, which invariably leads to a complex array of detail. A fundamental problem is to find a basic idea that underlies the interpretation of mathematical expressions that makes sense for the majority of teachers and learners in the whole population. This can be found in terms of the way in which expressions are articulated when they are spoken and heard. For example, the phrase "ten take away four plus two" can be articulated as "ten take away four [pause] plus two", leaving a small gap in speech to allow it to be interpreted as "six … plus two", giving eight. It may also be spoken as "ten take away … four plus two" and interpreted as "ten … take away six", which is four. This is formulated in the general statement of the *Articulation Principle* that "the meaning of an expression depends on the manner in which it is articulated" (Tall, 2019, p. 7). This principle is already widely grasped in an intuitive sense, but it is rarely made explicit in teaching and learning. Tall (2020) proposes that it offers a meaningful foundation for a long-term learning theory that enables teachers as mentors to encourage learners to take control of their own learning. It begins by revealing that articulation gives precise meaning to simple mathematical expressions and gives a reason to use brackets as a meaningful indication of order of operations rather than as an arbitrary convention. It also includes a dual interpretation of a sub-expression such as $2 + 3$ as an *operation* "2 plus 3" and as a mental *object*, in this case the number "5". This flexibility of treating an expression or sub-expression dually as process (operation) or concept (mental object) is termed a *procept* (Gray & Tall, 1994). It continues to be a meaningful feature of mathematical expressions throughout the whole mathematics curriculum and on to the ever-broadening horizons of mathematical research (Tall, 2013).

2 Related literature

The study of the long-term development of mathematical symbolism took a major step forward in Piaget's research on the development of whole number concepts in young children, published in French in 1941 and translated into English in 1952 as *The Child's Conception of Number*, as part of a much wider theory of *The Psychology of Intelligence* (English translation: Piaget, 1950). However, in the following years, educational psychology offered a range of different theories. Skemp (1962) reviewed all the papers in major educational psychology journals since 1940 and concluded that they were inadequate for dealing with mathematics:

> … theories relating to conditioning, reinforcement learning, sign learning, perceptual learning, etc., are not adequate for the classroom. A theory is required which takes account (among other things) of the systematic development of an organised body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning: as when a knowledge of arithmetic makes possible the learning of algebra, and when this knowledge of algebra is subsequently used for the understanding of analytical geometry.
>
> The only theory yet available which does this has been put forward by Piaget (1950). He calls such a body of knowledge a 'schema'. The incorporation of new knowledge into an existing schema is called 'assimilation'; and the enlargement of a schema, which may be necessary if it is not adequate for the above purpose in its existing form, is called 'accommodation'. These three related concepts would seem to offer a basis for the kind of learning theory which is needed. (Skemp, 1962, p. 133)

In the years that followed this observation, a broader overall theory has evolved to elaborate *How Humans Learn to Think Mathematically* (Tall, 2013). This evolves through three interrelated forms of mathematical thinking:

- *conceptual embodiment* (broadly based on increasingly sophisticated forms of human perception and operation from physical to mental thought experiment),
- *operational symbolism* (based on symbolising operations and manipulating symbols),
- *formalism* (beginning with the theoretical reorganisation of experience based on carefully selected definitions and deduction and, at a higher *axiomatic formal* level, based on set-theoretic definition and logical proof).

The Articulation Principle forms an essential link from conceptual embodiment to operational symbolism. It is relevant at all levels of development, from the difficulties young children encounter with word problems, through the transitions between number systems and notations, from counting numbers, to fractions, to signed numbers, decimal representation, rational and irrational numbers, real numbers, complex numbers, from arithmetic to algebra and more advanced forms of symbolic calculus, vector algebra and abstract algebra.

Since the publication of *How Humans Learn to Think Mathematically*, new developments have broadened the theory to include information about the physical structure of the human brain, how it interprets text and continuous motion, how perception and action are linked to emotional and physical reactions. There are also other essential cultural aspects relating to conflicting approaches and beliefs of differing communities of practice where the interpretation of one community (e.g. pure mathematics) may involve aspects that are inappropriate for others (e.g. engineering, economics, biology, teachers of young children) (Tall, 2019).

This paper focuses on the Articulation Principle as a foundation for giving mathematical expressions a clear and unambiguous meaning. It links embodiment (through speaking and hearing) to meaningful symbolism (Chin

& Jiew, 2018; Tall, 2013) as part of a broader long-term framework for making mathematics meaningful (Tall, 2019, 2020).

2.1 A long-term theoretical framework

"There is nothing so practical as a good theory" (Skemp, 1989, p. 27).

In the book *Mathematics in the primary school*, Richard Skemp (1989) sought a long-term learning curriculum in which learners build on their previous experience, expanding their knowledge where it fits with what they already know and reconstructing ideas to take account of new situations. The third author of this paper had already completed a doctoral thesis in pure mathematics with Field's medallist Michael Atiyah, and completed a second doctorate in mathematics education with Skemp in 1986. Subsequently, this led to building theoretical constructs from practical experience in partnership with others, including the notion of *concept image*, based on the construct of Shlomo Vinner (Tall & Vinner, 1981), the dual use of symbolism as process and concept as *procept* with Eddie Gray (Gray & Tall, 1994), *Advanced Mathematical Thinking* (with the working group of PME, ed. Tall, 1991) and textbooks on university mathematics with Ian Stewart (e.g. *Foundations of Mathematics*, 2nd edition, Stewart & Tall, 2014). He also supervised international PhD students researching the learning of children and students from early childhood to post-graduate level where each thesis added a new aspect to the fuller picture, detailed in Tall (2008). The framework, summarised in *How Humans Learn to Think Mathematically* (Tall, 2013), applies not only to different individuals learning over a lifetime but also to the historical development over generations in different cultures.

Chin (2013) focused on successive stages of development in trigonometry, from triangle trigonometry taught to teenagers aged 14-16, through circle trigonometry with arbitrary angles and functions visualised graphically at ages 16-19, and on to formal trigonometry at university in real or complex analysis. It reveals how teachers and learners make sense of the various ideas within and between stages. For example, triangle trigonometry interprets the visual concept imagery of a right-angled triangle to move flexibly between the symbolic process of calculating the ratio of sides to the symbolic concept of the ratio as a number. Circle trigonometry focuses on varying the angle to study trigonometric functions visually as graphs of general angles measured in radians and to develop their symbolic properties. Formal trigonometry moves on to more general ideas such as series as infinite processes relating to limit concepts and real and complex relationships between trigonometric and exponential functions. At each stage, the Articulation Principle plays a meaningful role in giving precise meaning to expressions consistent with accepted mathematical conventions.

Over the longer term, making sense of spoken expressions involves working in a particular context for a time, say counting and performing simple whole number arithmetic, then shifting to new contexts introducing new ideas which may involve expansion (assimilation) or reconstruction (accommodation). Chin noted that the transition to a new context may develop a new conception with a mixture of supportive and problematic aspects where supportive aspects worked in the old context and continue to work in the new but problematic aspects that worked in the old context fail to work in the new.

In introducing fractions, for instance, new concepts and procedures come into play such as the notion of equivalent fractions with new procedures for addition and multiplication. Old experiences with whole numbers such as "multiplication gives a bigger number" no longer work and rote learning of the procedures without making sense can alienate the learner. Yet there are other properties that continue to be supportive. For example, Piaget's Principle of Conservation of Number states that the number of elements in a collection of objects is independent of the way it is counted. This continues to be supportive through successive number contexts, including fractions, signed numbers, finite and infinite decimal representations, real numbers, and complex numbers.

Tall (2020) proposes a supportive approach to deal explicitly with problematic changes of meaning. This entails the curriculum designer and teacher being aware of changes in meaning and to explicitly encourage the learner to be aware of ideas that remain supportive through several changes of context to give them confidence in building on them. Then, when problematic ideas arise, they may be explicitly addressed to encourage the learner to reconstruct their ideas to fit the new context. This led to a major principle formulated to enable learners to take control of their own learning, guided by teachers acting as mentors. It was named in honour of his grandson Simon, then aged 11, who was responsible for observing how mathematical expressions could be spoken in different ways to give different meanings.

> *The Simon Principle*: The teacher should be aware of those ideas that remain supportive through several changes of context, to give confidence to the learner, and to make explicit those ideas that are problematic so that they can be addressed meaningfully (Tall, 2020, p. 2).

We propose that teachers apply the Articulation Principle in their classrooms so that students can make better sense of the meaning of operations in arithmetic and algebra. It can be introduced at any stage to improve understanding and to build more coherent relationships as mathematics becomes more sophisticated. Spoken articulation offers a way to help students begin to make sense of the sequence of operations in a mathematical expression, mentally and spontaneously.

2.2 Previous studies of spoken expressions

When learners first encounter mathematical concepts, there is a huge difference between their natural language and mathematical terminology. "It's like hearing a foreign language" (Kotsopoulos, 2007, p. 301). Language is an essential means of communication, both between individuals and within the mind (Lakoff & Johnson, 1980; Lakoff & Núñez, 2000; Pimm,1987; Sfard et al., 1998). In this regard, language is used to communicate mathematical meanings as it is readily "seen and heard" (Chapman, 1997, p. 156) and works with graphing, ordering and so on. Thus, language is essential not only for communication but also influences human thought (Freudenthal, 1991). Our interest is in the increasing sophistication of symbolism to support the long-term learning and meaning throughout the curriculum.

In Australia, the National Numeracy Review Report (COAG, 2008) states that mathematical symbols and expressions hinder children from understanding concepts in mathematics. In the expression $4 + 3 \times 2$, for example, the operation 3×2 needs to be performed first because "multiplication takes precedence over addition" as a convention. However, reading $4 + 3 \times 2$ as "four plus three times two" in standard left to right order gives a different result. This introduces a conflict between natural reading from left to right and the conventions for interpreting mathematical expressions.

Mathematics teachers may tend to write mathematical expressions on a whiteboard without focusing on the ways to speak or read them. This may fail to encourage students to develop the capacity to access mathematical meaning through spoken language. The study of Ellerton and Clements (1991) suggests that teachers should encourage students to speak and read mathematics.

Different individuals may interpret various spoken mathematical expressions in different ways. For instance, the spoken expression, "one divided by two plus three" may be interpreted as either $\frac{1}{2+3}$ or $\frac{1}{2}+3$. Gellenbeck and Stefik (2009) performed a study to examine if insertion of pauses in spoken mathematical expressions could reduce ambiguity between similar algebraic expressions. Their study involved 16 students who were around 22 years old majoring in computer science. The participants were given two, side-by-side, algebraic expressions and were required to rate each expression on a Likert scale from zero to ten on how well they conceived the expression corresponded to the audio. The spoken expressions were either with or without pauses. Their findings revealed that the use of pauses for spoken expressions dramatically improved participants' ability to distinguish between similar algebraic expressions. The study of Gellenbeck and Stefik (2009) claims that there is no empirical study that has been conducted on the use of pauses as a causal mechanism for interpreting mathematical expressions and no theoretical framework has been formulated to explain or predict this phenomenon. Our purpose here is to provide a theory and offer empirical evidence in support.

3 The case study

Realising the importance of verbalising mathematics and reading mathematical expressions, this paper specifically seeks to answer two research questions:

1. Does spoken articulation assist the sense making of sequences of operations in arithmetic expressions?
2. How does the Articulation Principle support the interpretation of arithmetic expressions in combination with the flexible use of symbolism as process and concept?

3.1 Methodology

The study was designed using an in-depth interview (Yin, 2003). Instead of allowing theories to emerge from qualitative data, the study collected data to test pre-existing theories (Inglis, 2006). A quasi-judicial method of data analysis was employed. This is to evaluate the quality of evidence, taking account of multiple sources (Bromley, 1986). It seeks not only evidence that supports the theory but more importantly, examines data that contradict the theory, analysed through systematic and rigorous discussions among the research team members with independent input from another experienced researcher in the field. The issue of generalisation from single case-studies (or several case-studies) is seen in terms of the validity of the analysis, not in terms of the representativeness of the case (Inglis, 2006).

3.2 Data collection and participants

Data were collected in Malaysian secondary schools where two mathematics teachers (given pseudonyms, Eric and Fabian) from two different schools, participated on a voluntary basis. Both teachers were familiar with the conventions for mathematical expressions, such as "multiplication takes precedence over addition", which they had been taught in the primary school curriculum. The choice of Eric as a participant was due to his own preference to help students to make sense of mathematics. He wanted to improve his teaching practices yet struggled to identify what and how to improve. Fabian, on the other hand, was able to articulate his goals for many tasks in a more comprehensive manner which offered the possibility of a deeper analysis.

Each participant took part in a clinical interview conducted in English, lasting approximately half an hour. The interviews were audio-recorded and transcribed for subsequent analysis. Although the data were interpreted using the Articulation Principle, the term itself was not used in the interviews, which were carried out using the familiar language of the mathematics classroom.

After an initial introduction, the interviewer spoke the expressions "the square root of four times four" in a simple even tone and asked the participants to write the expression down. The interviewer then wrote a pair of expressions that would be spoken in similar ways. These were $\sqrt{4\times 4}$, $\sqrt{4}\times 4$. The participants were required to read out these expressions. The process repeated with another spoken expression "one over two plus one" and another written pair of expressions, $\frac{1}{2}+1$ and $\frac{1}{2+1}$.

The next part researched in greater depth how the participants made sense of spoken expressions. First the interviewer revisited the spoken expressions "the square root of four times four" and "one over two plus one", now spoken with different articulations. These corresponded to the written symbolic expressions, $\sqrt{4\times 4}$, $\sqrt{4}\times 4$ and $\frac{1}{2+1}$, $\frac{1}{2}+1$ but the symbolism was not referred to by the interviewer. Then two new spoken expressions were investigated: "negative five squared" and "two plus three times four". This was to seek how the participants gave meaning to expressions through articulation and how they handled the dual meaning of an expression or sub-expression as mental process or a mental object.

4 Results and analysis

In the following, we report excerpts from the interviews and our analysis of the use of spoken articulation in making sense of arithmetic expressions. We present the results based on a case-by-case basis. In the following excerpts, "I" refers to the interviewer and statements in square brackets indicate the nature of the actions of the person speaking.

4.1 The case of Eric

As in Excerpt 1, Eric was shown with the written expressions $\sqrt{4\times 4}$ and $\sqrt{4}\times 4$ and required to read them out loud. He used the same sequence of words and was unable to distinguish them verbally. He was aware that there was a problem but had not yet formulated what the problem was. He also read $\frac{1}{2}+1$ and $\frac{1}{2+1}$ in the same way, even though he realised that these two expressions yield two different answers.

Excerpt 1

1	I	Please read the written expression. [Pointing at $\sqrt{4\times 4}$]
2	Eric	Square root of four times four.
3	I	How would you read this mathematical expression? [Writing $\sqrt{4}\times 4$]

4	Eric	It's the same way, I think. Square root of four times four.
5	I	Are you reading it similar to this one? [Pointing at $\sqrt{4 \times 4}$]
6	Eric	Hmm … I think so. Yes, I read it the same way, but I might as well write it down to avoid confusion. Now I see the problem.
7	I	Please read this written expression. [Pointing at $\frac{1}{2} + 1$]
8	Eric	One over two plus one.
9	I	How would you read this mathematical expression? [Writing $\frac{1}{2+1}$]
10	Eric	One over two plus one.
11	I	Do you think that these two expressions have the same answer? [Pointing at $\frac{1}{2+1}$ and $\frac{1}{2} + 1$]
12	Eric	No.

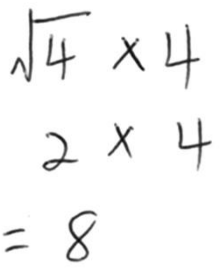
The expressions were spoken by the interviewer in different articulations. When the expression was spoken with a pause, Eric separated out the parts before and after the pause, as illustrated in Excerpt 2. On hearing the expression "the square root of four [pause] times four", articulated with a slight pause, he immediately wrote down the expression $\sqrt{4} \times 4$, square-rooted the four first to get 2, then he performed 2×4 to get 8.

When the interviewer then asked for "the square root of [pause] four times four", he calculated four times four first and then square-rooted the product to get the result, 4. On this occasion, he performed the process in his head without writing the calculation down.

Eric processed both interpretations instinctively and fluently. Looking more closely at the evidence, he sees the sub-expression $\sqrt{4}$ of the expression $\sqrt{4} \times 4$ as a process, saying "I will square root the four first", and then evaluates it as the value "2". This exhibits the notion of procept (Gray and Tall, 1994) in which a symbol can be interpreted flexibly either as a process (an operation) or as a concept (a mental object). Eric was able to switch fluently between a process and a concept as if they are two different ways of thinking of the same thing.

Excerpt 2

13	I	What is the square root of four [pause] times four?
14	Eric	Eight.
15	I	How did you get your answer?
16	Eric	When you say it like that, for me, instinctively I will square root the four first. Square root of four is two, two times four is eight. [Writing on the paper:]



17	I	What is the square root of [pause] four times four?
18	Eric	Four, I think ... I did four times four first then square-rooted the product.

Similarly, in Excerpt 3, when given two different ways of articulating “one over two plus one”, he saw two different meanings and wrote down two different forms of symbolism. He stated that the difference in meaning could be explained “if you say it slowly”. He had an implicit awareness of the role of articulation without explicitly saying what it is. He was able to sequence the operations correctly because he was able to identify the sub-expressions that needed to be performed first. For instance, he interpreted “one over [pause] two plus one”, spontaneously to have two separate sub-expressions, “one” and “two plus one”. The sub-expression “two plus one” is again interpreted flexibly as either a process of addition or a concept of sum.

Excerpt 3

19	I	What is one over [pause] two plus one?
20	Eric	If you say it slowly, the two plus one is the denominator. [Writing on the paper:]
		$\frac{1}{2+1}$
21	I	What is your answer?
22	Eric	One over three.
23	I	What is one over two [pause] plus one?
24	Eric	This is my expression. [Writing on the paper:]
		$\frac{1}{2} + 1$
25	I	What is your answer?
26	Eric	One and a half.

To further unfold how spoken articulation aids in interpreting expressions, the next part of the interview was carried out only in spoken form. There were no written expressions in view and the interviewer asked questions based on the interview protocol without showing anything to each participant.

When the interviewer spoke different mathematical expressions using articulation to separate different sub-expressions, Eric was able to sequence the operations correctly. Based on Excerpt 4, when Eric heard “negative [pause] five squared”, he sensed “five squared” as a sub-expression and imagined it having a bracket round it. He performed the operation to square five first before forming the negative. He also interpreted the expression “two plus three times four” by first performing the operation 3×4 to get 12, then adding $2 + 12$ to get 14. Now Eric was implicitly linking the use of articulation to imagine the symbolic use of brackets.

Excerpt 4

27	I	What is negative [pause] five squared?

28	Eric	Negative twenty-five. When you squared the number five you will get twenty-five, a positive answer. It depends on whether or not we have a bracket here. [As Eric spoke, he moved his index finger on the table as if he was writing a minus sign followed by a bracket round the number five.] But in this case, I think this is a negative twenty-five.
29	I	What is negative five [pause] squared?
30	Eric	For me, right now I'm thinking the answer is twenty-five… I squared negative five.
31	I	What is two plus [pause] three times four?
32	Eric	This is fourteen. [Writing on the paper:] $2+3\times 4=14$
33	I	How did you get your answer?
34	Eric	I multiply first then do addition. Three times four is twelve, plus two is fourteen.

In Excerpt 5, the interviewer next sought to explore if Eric noticed the differences between spoken expressions without the insertion of pauses and those spoken expressions with the insertion of pauses. Eric explicitly stated that he "pictured the expression in his mind" to "see" imaginary brackets and that certain spaces "acted like a space button in a computer to separate things". He used his previous learning to "see" imaginary brackets to distinguish the two different meanings of "negative five squared".

Excerpt 5

35	I	Did you notice any difference when I read the expressions?
36	Eric	Yes, the way you read the expressions somehow affected my thinking, how I pictured the expressions in my mind. It is actually like a space button in a computer. It separates things.
37	I	How does it affect your thinking about the expression?
38	Eric	It's a way to imagine, a way of saying the expression that will help us to convey the message without writing the expression down.

4.2 The case of Fabian

Fabian's response was broadly similar to Eric's. However, he expressed an Aha! moment when he realised that he would speak $\sqrt{4\times 4}$ and $\sqrt{4}\times 4$ using exactly the same words that do not distinguish between their different meanings (see Excerpt 6). An *Aha!* moment arises in someone's mind spontaneously when earlier unlinked planes of thought are in a flash brought together (Tall, 2021), resulting in sudden spark of insights. This experience gave an inspiration to Fabian and developed his thinking about the expressions. In the case of $\frac{1}{2}+1$ and $\frac{1}{2+1}$, he realised the same phenomenon, saying "Same thing!"

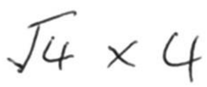
Excerpt 6

1	I	Please read the written expression. [Pointing at $\sqrt{4 \times 4}$]
2	Fabian	Square root of four times four.
3	I	How would you read this mathematical expression? [Writing $\sqrt{4} \times 4$]
4	Fabian	Square root … haha … same thing. Square root of four times four.
5	I	Do you mean that these two expressions have the same answer? [Pointing at $\sqrt{4} \times 4$ and $\sqrt{4 \times 4}$]
6	Fabian	No, they have different answers.
7	I	Please read this written expression. [Pointing at $\frac{1}{2} + 1$]
8	Fabian	One over two plus one.
9	I	How would you read this mathematical expression? [Writing the expression, $\frac{1}{2+1}$]
10	Fabian	One over two plus one. Same thing!

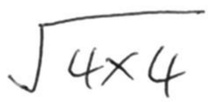
The expressions were then spoken by the interviewer in different articulations. As illustrated in Excerpt 7, Fabian suddenly has an Aha! moment when he sensed the required link.

Excerpt 7

11	I	What is the square root of four [pause] times four?
12	Fabian	Hmm … Which square root of four times four are you referring to?
13	I	Let me repeat the expression once again. What is the square root of four [pause] times four?
14	Fabian	Oh! I got it! Ask me again!
15	I	What is the square root of four [pause] times four?
16	Fabian	Eight. Square root of four is two, times four, then is eight. [Writing on the paper:]



17	I	What is the square root of [pause] four times four?
18	Fabian	Four. Four times four is 16, the square root of 16 is four. [Writing on the paper:]



According to Excerpt 8, Fabian distinguished the different ways of speaking an expression in terms of "slowing down" to indicate a gap in speaking, as had Eric. He went a step further by explicitly referring sub-expressions as "objects".

Excerpt 8

19	I	What is one over [pause] two plus one?
20	Fabian	One over three. Two plus one is three, so one over three. [Writing on the paper:]

		$\frac{1}{2+1}$
21	I	What is one over two [pause] plus one?
22	Fabian	One and a half. [Writing on the paper:]
		$\frac{1}{2}+1$
23	I	How did you get two different answers and expressions?
24	Fabian	You read the expressions differently. The way you speak differentiates the operations. You separate the operations obviously. For example, you read "one over two … plus one", you are separating the half and one. Just like one object plus one object.
25	I	How would you interpret this way of separation?
26	Fabian	When there is an operation, you will slow down to indicate the separation. Whenever you slow down, I will instantly separate what you have read with what you are going to read.

To further examine how spoken articulation helps to interpret expressions, the interviewer spoke different mathematical expressions using articulation. As illustrated in Excerpt 9, Fabian explained how he imagined brackets in his mind to interpret the expressions in a precise manner, noticing each sub-expression as process or object to flexibly interpret the expressions correctly according to the standard conventions. As for the expression "two plus [pause] three times four", he double-underlined the expression 3×4 to emphasise it as a single item.

Excerpt 9

27	I	What is negative [pause] five squared?
28	Fabian	Negative twenty-five. I think of a bracket for the five squared and the negative is outside the bracket.
29	I	What is two plus [pause] three times four?
30	Fabian	Fourteen. [Writing on the paper:]
		$2+\underline{\underline{3\times 4}}$
31	I	How did you get your answer?
32	Fabian	Do multiplication before the addition… Hmm … If you read the expression without slowing down, I might just add two and three to get five, then multiply four to get twenty. But you read "two plus … three times four", so I know that I need to do three times four first.

4.3 Summary of data and analysis

The research presented here has addressed the two research questions. The findings confirm

- that the spoken articulation assists the participants in making sense of the sequence of operations in arithmetic expressions,
- that the articulation of expressions supports the meaningful interpretation of arithmetic expressions and the participants interpret sub-expressions flexibly as process or concept.

This confirms the insight available in clarifying mathematical expressions by speaking them in such a way that sub-objects are separated by articulating brief pauses.

Previous studies (e.g. Gellenbeck & Stefik, 2009) have identified the ambiguity in spoken mathematical expressions and show that the insertion of pauses could dramatically improve participants' ability to disambiguate two similar algebraic expressions. Looking more closely at the responses of the participants in this study illustrates another more fundamental idea. Articulating the expression "the square root of four times four" as "the square root of four … times four" breaks the expression into two parts, "the square root of four" times "four". The sub-expression "square root of four" is interpreted fluently as an operation to find the square root of four and also as the number resulting from that operation, which is 2. The "square root of four" is dually an operation (or process) and a mental object (or concept). The same happens with all the other expressions, for example "negative five squared" is either "negative … five squared" (the negative of five squared), which is –25, or "negative five … squared" (the square of negative five) which is +25.

Not only do both participants read sub-expressions dually as concepts in their own right that can be calculated as arithmetic processes, they also speak of mentally processing them in their mind to "see" brackets to clarify the meanings. The sub-expressions can then be handled mentally as processes to be evaluated or as mental objects produced by the process.

5 Looking to the future

As the number of operations in an expression increases, the use of the Articulation Principle becomes increasingly complicated. It can give a meaning to relatively simple expressions, such as the two different meanings of $2 + 3 \times 4$, one reading left to right as $2 + 3 \ldots \times 4$ to give 5×4, which is 20, the other as $2 \ldots + 3 \times 4$ to give $2 + 12$, which is 14. But it soon becomes complicated as it deals with more complex expressions.

For this reason, it is sensible to adopt conventions to make the symbolism simpler. For example, the convention that multiplication takes precedence over addition allows $2 + (3 \times 4)$ to be written as $2 + 3 \times 4$, while it is still necessary to keep the brackets in $(2 + 3) \times 4$.

Tall (2019) takes these ideas through the whole of symbolic mathematics, in which the duality of symbolism as process or concept plays a fundamental role. Essentially, an expression such as $2 + 3$ may be seen as the operation of adding 2 and 3 or as the single object which is the sum $2 + 3$. The aim is for the individual to flexibly imagine the two possibilities. It can be illustrated visually by putting boxes round the objects. Then

$\boxed{2} + \boxed{3}$ is the operation of adding the objects 2 and 3

while

$\boxed{2 + 3}$ is the object which is the sum of 2 and 3.

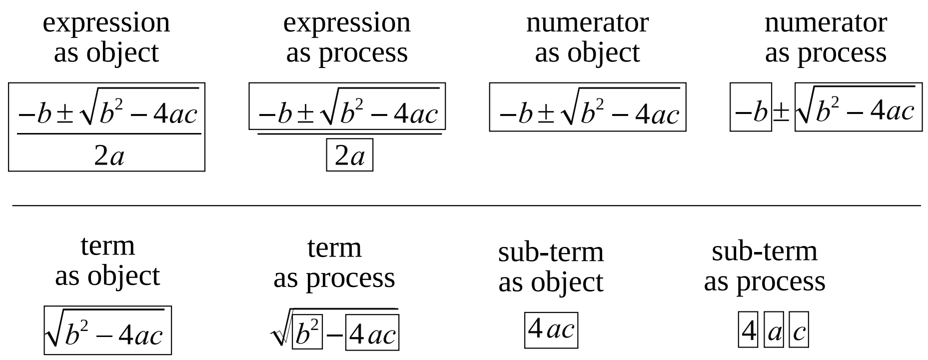
This notation can be used to visually represent the structure of more general mathematical expressions as part of a long-term development in sophistication. It is a general technique which applies to more sophisticated expressions such as

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This may be seen as a whole as an object, then as a process of division. It has sub-objects which are the numerator and denominator, each of which may be further seen as objects, then processes, recursively moving down the hierarchy to see terms, then sub-terms as object and process (Figure 1).

Figure 1

Sub-expressions as Operation or Object (Tall, 2019)



This use of boxes to see the duality of process and object is helpful in simple cases. It can be simplified by using boxes sparingly, only putting them in explicitly for more complicated sub-expressions. This can be seen for a written expression such as $2x^2 + 3x + 7$ by imagining it as the sum of three terms:

$$\boxed{3x^2} + \boxed{2x} + \boxed{5}$$

which can be added together in any order by Piaget's principle of conservation. These terms can be further broken down as

$$3\boxed{x^2} + 2\boxed{x} + \boxed{5}$$

to see it as 3 lots of x^2 plus 2 lots of x plus 5.

The ultimate purpose is to be able to interpret mathematical expressions in precise flexible ways. While successful individuals may do this implicitly, a wider range of students may be helped by explicitly introducing the Articulation Principle and encouraging them to focus on long-term supportive principles that give them confidence to address new aspects that may initially be problematic.

In practice, the increasing sophistication of expressions over the long term will be introduced in appropriate stages. The first stage involves counting objects and fingers and building simple arithmetic of whole numbers, to formulate the principle that addition of a list of numbers is independent of the manner in which the sum is calculated. This includes not only different ways of counting, but different layouts, such as 3 lots of 2 giving the same number as 2 lots of 3, extending flexibility from addition of whole numbers to multiplication. Subsequent stages are studied successively in a long-term framework which may be enhanced by curriculum designers and teachers who are aware of long-term supportive concepts that give confidence to the learner and encourages them to explicitly address problematic changes that require reconstruction of knowledge to develop new ways of thinking.

5.1 Implication for teaching and research

The framework offers practical ways to re-think how we as individuals make sense of mathematics and how we can help others to progress in their mathematical thinking. The Articulation Principle can be introduced at any level to open up discussion about the meaning of mathematical expressions. It can be used with young children or with adults with learning difficulties. It can be helpful to teachers to encourage long-term strategies to grasp the meanings of operations in arithmetic and algebra and it can help experts and curriculum designers to plan for long-term success. By making the Articulation Principle explicit, it can highlight the long-term principles for the operations of arithmetic that lead more naturally to the underlying principles of algebra and offer a supportive basis to build confidence throughout the whole curriculum.

Many of the problems raised in the literature can be linked to the interpretation of symbolic expressions. For example, children first encounter simple operations such as $2 + 3 = 5$ with a process on the left giving a result on the right. In algebra an equation such as $2x + 1 = 7$ with a process on the left and a number on the right can be "undone" by reversing the process. Meanwhile an equation with expressions on both sides is better understood as having an object on either side expressed in different ways and is solved by "doing the same thing to both sides" (Tall et al., 2014).

Over the long-term, there are successive transitions as different processes give the same object: counting a set in any way gives the same number, equivalent fractions become a single rational number, algebraic equivalences become the same function, equivalent Cauchy sequences give the same real number. While this is usually interpreted symbolically using the concept of equivalence, it is more meaningfully sensed as different symbolic representations of the same object. When these ideas are embodied on a number line and as graphs in the plane, equivalent fractions are seen as a single point, algebraically equivalent expressions and trigonometric identities give the same graph, and, more generally, infinite limiting processes stabilise visibly on their limit object.

Dienes (1960) advocated working with several different embodiments from which children can construct generalities for themselves. Here we suggest that the curriculum should be planned in such a way that the teacher is aware of the foundational supportive principles to make them explicit for the learners to use them to build confidence to address problematic aspects that impede transition to new contexts.

Embodied ideas of space and shape evolve in sophistication from practical perception and action to theoretical definition and deduction in Euclidean geometry. In the transition to calculus, a problematic conflict arises between embodiment and symbolism. Circle geometry describes a tangent as a line that touches a circle in a single point. This precise definition enables a tangent to a circle to be drawn by constructing a radius from the centre of the circle to the point in question and drawing the tangent at right angles. The definition fails to work for a tangent to a more general curve computed by symbolic differentiation. It can be resolved using embodiment to imagine zooming in on the curve at a point to see the curve as "locally straight" and give human meaning to the derivative as the slope of the curve itself (Tall, 1985).

In the 1980s, dynamic computer graphics were introduced to give embodied meaning to calculus concepts, together with powerful numeric computation and symbolic manipulation. Now the Articulation Principle and related supportive principles address the interpretation of symbolism to link meaningful conceptual embodiment to meaningful operational symbolism.

A serious problem in the current curriculum is the fragmentation of the whole system dealing with learning by developing expertise in separate parts of the whole: pre-school, early learning, kindergarten, primary, secondary, high school, college, adult learning, university, post-graduate, special needs, gifted and talented, and so on. All of these are essential, but they need to be seen as part of a greater whole, so that different communities of practice are aware of a bigger picture. What happens currently is that learning is broken into stages, with tests to decide who passes on from one stage to another. This can lead to a desire to pass the examination by rote learning, especially when there are problematic aspects involving a change in meaning. Over the longer term, cumulative changes that occur without making meaningful connections are likely to make mathematics more complicated. Long-term success may be enhanced by meaningful connections that compress complex operations into mental objects that can be manipulated in simpler ways in more sophisticated situations.

5.2 Long-term principles for meaningful learning

The Articulation Principle offers an unambiguous meaning to spoken mathematical expressions. Teachers who are aware of it can encourage learners to make long-term strategies explicit to help students to grasp the changing meanings of symbolic operations throughout the whole mathematics curriculum. These include extensions of Piaget's principle of conservation of number and general properties such as the principles that the sum of a list of numbers and the product of a list of numbers are both independent of how they are calculated. They are the foundation of the equal precedence of addition and subtraction and of multiplication and division in the rule P-E-MD-AS or its English equivalent B-I-DM-AS.

Meaningful long-term learning takes account of supportive and problematic aspects of increasingly sophisticated thinking in both embodiment and symbolism. This is enshrined in the earlier-mentioned Simon Principle, which underlies the whole framework as "The teacher should be aware of those ideas that remain supportive through several changes of context, to give confidence to the learner, and to make explicit those ideas that are problematic so that they can be addressed meaningfully" (Tall, 2020, p. 25).

This broader theory uses ideas that are already intuitively familiar, but this does not mean that there is nothing new in their explicit use. Writing about the historical evolution of calculus, which also applies to the evolution of mathematical thinking, Edwards (1979) remarked:

> What is involved here is the difference between the mere discovery of an important fact and the recognition that it is important—that is, that it provides the basis for further progress. In mathematics, the recognition of the significance of a concept ordinarily involves its embodiment in new terminology or notation that facilitates its interpretation in investigation. (p. 189)

He went on to quote Hadamard (1947), who wrote "The creation of a word or notation for a class of ideas may be, and often is, a scientific fact of great importance, because it means connecting these ideas together in our subsequent thought" (p. 38).

The creation of the naming of the Articulation Principle opens up the possibility of further progress and subsequent thought in integrating the long-term teaching and learning of mathematics as it grows in sophistication throughout the lifetime of different individuals.

6 Future research and development

This specific study involving just two students with similar educational backgrounds clearly has its limitations. However, the reader can test out the practical value of the Articulation Principle. Simply ask a few individuals in different circumstances from young children to adults, "What is $2 + 2 \times 2$?" You will find that some say "8" because they perform the operations in the order spoken, while some say "6" because they remember the rule that "multiplication takes precedence over addition". Now, speak the problem using two different articulations to see whether they make sense of the difference. Further research is possible into the Articulation Principle in different contexts.

Of far greater importance is the wider study of the use of fundamental principles in long-term learning and the transition from one context to another as new concepts are introduced and previously supportive concepts may become problematic. The Articulation Principle is unusual in that it can be introduced at any stage, in any curriculum once the learners have encountered the operations of counting and simple arithmetic. This may be with young children at the beginning of their encounter with arithmetic, adults who have severe difficulties with mathematics, or individuals at any stage of learning mathematics. If used sensitively, it can offer *meaning* to symbolic expressions in arithmetic and algebra. It offers a precise and accurate interpretation of simple expressions, in particular it offers a *reason* why brackets should be used rather than rote-learning arbitrary conventions.

While many approaches to teaching and learning mathematics focus on positive organisation of the curriculum or research into the errors and misconceptions that arise, the Articulation Principle is part of a much broader long-term approach that balances the positive nature of supportive ideas and the negative possibilities of problematic ideas. This balance shifts the context of learning to wider possibilities, just as the introduction of a balance between positive and negative numbers expands the possibilities of the number system. It offers the possibility of seeing old problems in new ways.

The current US curriculum, for example, lacks the meaning of the Articulation Principle in facing the known problems that arise in expressing and solving word problems in arithmetic. The Articulation Principle encourages a focus on the meaning of expressions, giving new ways of interpreting the known problems of transitioning from arithmetic to algebra. Extending the symbolism to calculus, it offers new insights into the transition between the symbolic idea of the process of tending to a limit and the concept of limit as an object. In the wider relationship between embodiment and symbolism, using dynamic interactive software to zoom in on the graph of a function, it opens up the link between the embodied insight of "local straightness" and the process of differentiation, to see the symbolic derivative as the slope of the graph itself. Meanwhile, the US curriculum for the calculus (College Board, 2016) lists only items that can be tested, with no mention of the *meaning* of the calculus through local straightness. Instead of building up from the student's experience, it builds down from the formal definition by phrasing it in an informal manner that is known to be problematic.

The Articulation Principle opens the door to question the very basis of current teaching practice and mathematical education research, not only in operational symbolism, but in the wider framework of the long-term development of mathematical thinking.

References

Bromley, D. B. (1986). *The case-study method in psychology and related disciplines*. Chichester: John Wiley & Sons.

Cardone, T. (2015, February 16). Internalizing the order of operations. In *Mathematics Teaching in Middle School*. https://www.nctm.org/Publications/Mathematics-Teaching-in-Middle-School/Blog/Internalizing-the-Order-of-Operations/

Chapman, A. (1997). Towards a model of language shift in mathematics learning. *Mathematics Education Research Journal*, *9*(2), 152-173. https://doi.org/10.1007/BF03217309

Chin, K. E. (2013). *Making sense of mathematics: Supportive and problematic conceptions with special reference to trigonometry* [Doctoral dissertation, University of Warwick]. http://wrap.warwick.ac.uk/58403/

Chin, K. E., & Jiew F. F. (2018). A framework of making sense of mathematics. In F-J. Hsieh (Ed.), *Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education*, Vol. 1, pp. 309-319, Taipei, Taiwan: EARCOME. https://www.researchgate.net/publication/326995429_A_Framework_of_Making_Sense_of_Mathematics.

College Board. (2016). *AP calculus AB and AP calculus BC: Including the curriculum framewor*k. College Board. https://secure-media.collegeboard.org/digitalServices/pdf/ap/ap-calculus-ab-and-bc-course-and-exam-description.pdf

Council of Australian Governments (COAG). (2008). *National numeracy review report*. Canberra: Commonwealth of Australia. http://www.coag.gov.au/reports/docs/national_numeracy_ review.pdf

Dienes, Z. (1960). *Building up mathematics*. Hutchinson Educational.

Dupree, K. M. (2016). Questioning the order of operations. *Mathematics Teaching in the Middle School*, *22*(3), 152-159. https://doi.org/10.5951/mathteacmiddscho.22.3.0152

Edwards, C. H. (1979). *The historical development of the calculus*. Springer-Verlag. https://doi.org/10.1007/978-1-4612-6230-5

Ellerton, N. F., & Clements, M. A. (1991). *Mathematics in language: A review of language factors in mathematics learning.* Deakin University Press.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Springer.

Gellenbeck, E., & Stefik, A. (2009). Evaluating prosodic cues as a means to disambiguate algebraic expressions: an empirical study. ASSETS'09: P*roceedings of the 11th International ACM SIGACCESS Conference on Computers and Accessibility*, pp. 139-146. https://doi.org/10.1145/1639642.1639668

Glidden, P. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics, 108*(4), 130–136. https://doi.org/10.1111/j.1949-8594.2008.tb17819.x

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic.

Journal for Research in Mathematics Education, *26*(2), 115-141. https://doi.org/10.2307/749505

Hadamard, J. (1947). Newton and the infinitesimal calculus. In *Newton tercentenary celebrations*. The Royal Society.

Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: Are commonly experienced difficulties avoidable. *Educational Studies in Mathematics*, 81(2), 139–159. https://doi.org/10.1007/s10649-012-9394-x

Inglis, M. (2006). *Dual processes in mathematics: Reasoning about conditionals* [Doctoral dissertation, University of Warwick]. http://wrap.warwick.ac.uk/2847/1/WRAP_THESIS_Inglis_2006.pdf

Jeon, K. (2012). Reflecting on PEMDAS. *Teaching Children Mathematics, 18*(6), 370-377. https://doi.org/10.5951/teacchilmath.18.6.0370

Kotsopoulos, D. (2007). Mathematics discourse: 'It's like hearing a foreign language'. *Mathematics Teacher*, *101*(4), 301-305.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

National Council of Teachers of Mathematics (NCTM). (2009). *Focus in High School Mathematics: Reasoning and Sense Making.*

Papadopoulos, I. (2015). The rules for the order of operations: The case of an inservice teacher. *CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education*, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.324-330. hal-01281855

Piaget, J. (1941). *La genèse du nombre chez l'enfant* [*The child's conception of number*]. Routledge & Kegan Paul.

Piaget J. (1950). *The psychology of intelligence* (M. Percy, & D. E. Berlyne, Trans.). Routledge & Kegan Paul.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. Routledge.

Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics*, *18*(1), 41–51.

Skemp, R. R. (1962). The need for a schematic learning theory. *British Journal of Educational Psychology*, 32, 133–142. http://dx.doi.org/10.1111/j.2044-8279.1962.tb01748.x

Skemp, R. R. (1989). *Mathematics in the primary school*. Routledge.

Stewart, I. N., & Tall, D. O. (2014). *The foundations of mathematics* (2nd ed.). Oxford University Press.

Tall, D. O. (1985). Understanding the calculus. *Mathematics Teaching*, *110*, 49– 53.

Tall, D. O. (Ed.). (1991). *Advanced mathematical thinking*. Kluwer Academic Publishers.

Tall, D. O. (2008). A life-time's journey from definition and deduction to ambiguity and insight. *Mediterranean Journal for Research in Mathematics Education*, *7*(2), 183–196.

Tall, D. O. (2013). *How humans learn to think mathematically*. Cambridge University Press.

Tall, D. O. (2019). From biological brain to mathematical mind: The long-term evolution of mathematical thinking. In M. Danesi (Ed.), *Interdisciplinary perspectives on math cognition*, (pp. 1–28). Springer. https://doi.org/10.1007/978-3-030-22537-7_1

Tall, D. O. (2020). Long-term principles for meaningful teaching and learning of mathematics. To appear in S. Stewart (Ed.), *Mathematicians' reflections on teaching: A symbiosis with mathematics education* *theories*. Springer.

Tall, D. O. (2021). Building long-term meaning in mathematical thinking: Aha! And Uh-Huh! In B. Czarnocha & W. Baker (Eds.), *Creativity of an Aha! Moment and Mathematics Education*. Brill.

Tall, D. O., Nogueira de Lima, R., & Healy, L. (2014). Evolving a three-world framework for solving algebraic equations in the light of what a student has met before. *Journal of Mathematical Behavior*, *34*, 1-13. https://doi.org/10.1016/j.jmathb.2013.12.003

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169. https://doi.org/10.1007/BF00305619

Yin, R. K. (2003). *Case study research: Design and methods*. Sage.

Zazkis, R. (2018). 'Canada is better' – An unexpected reaction to the order of operations in arithmetic. In A. Kajander, J. Holt, & E. Chernoff (Eds.), *Teaching and Learning Secondary School Mathematics - Canadian Perspectives in an International Context* (pp. 527-546). New York, NY: Springer. https://doi.org/10.1007/978-3-319-92390-1_50