The Evolution of Calculus: A Personal Experience 1956–2019

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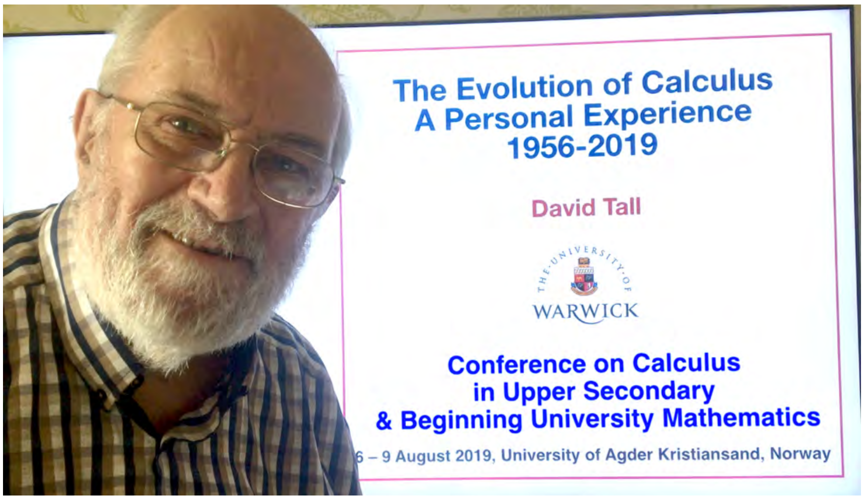

Introduction

This presentation was prepared as a 30-minute opening plenary for the *Conference on Calculus in Upper Secondary and Beginning University Mathematics* at The University of Agder, Norway, August 6 – 9, 2019. I was invited to present a personal view of the development of calculus over the last fifty years to set the scene for the conference. For health reasons I was unable to travel, so I made the video at home. It is available on YouTube at https://www.youtube.com/watch?v=eOwQlEPKCfY&feature=youtu.be

At the conference, participants will present various different approaches to the calculus. This opening plenary is designed to respect all viewpoints and to encourage participants to reflect on their personal views, to seek a broader overview of the whole enterprise. The issues are complex and no single individual can have a complete overall picture. So, although I will fulfil the request to present a personal view of the evolution of calculus, I will present evidence from a range of sources to seek an overall view. I will also add extra detail [in smaller type].

The calculus focuses on how we humans perceive, interpret and predict *change and growth*. The presentation will consider how we give meaning to dynamic change, how we interpret operational symbols as operations which then function as mental objects that can themselves be manipulated. It formulates how mathematics develops in sophistication through practical, theoretical and formal ways of thinking. This will include very recent developments in my own publications that offer simple new ways of giving meaning to symbolic expressions and new ways of interpreting dynamic change using retina displays.

I will begin with the human capacity for *embodiment* as we interact and interpret the world through practical activities and move to a higher theoretical level where visual embodiment of the limiting process stabilizes visually on the limit object. I will take an excursion to formal analysis where I will prove a simple formal theorem that shows that we can visualise infinitesimals as points on the number line. I will use the cultural theory of Raymond Wilder to interpret how social aspects colour our ways of thinking and invite you to reflect honestly on your own viewpoint. I invite you to produce evidence to challenge and improve the overall picture that I offer. We make advances by challenging our own thinking.

Opening Plenary at the *Conference on Calculus in Upper Secondary and Beginning University Mathematics*,
The University of Agder, Norway, August 6 – 9, 2019.

Introduction

This presentation reviews the rapid evolution of calculus over the last half century. It formulates simple ways to make calculus meaningful to the wide range of teachers, learners and experts that take account of the latest developments in digital technology, the workings of the human brain and the cultural aspects that affect the nature of human society.

The conference organisers invited me to offer a personal view of the evolution of calculus as it was 50 years ago and as it is now. I have decided to broaden this, to experiences since I first encountered calculus myself as a fifteen-year-old in school.

I note that the conference is focused around four themes.

A. The school-university transition with a focus on calculus

B. The fundamental theorem

C. The use of digital technology

D. Teaching & learning in various disciplines

This will be a personal journey. It will touch on ideas that have fascinated me as we live through a life-time that has seen – and continues to see – unprecedented evolution of digital technology. From such a vast enterprise, I seek an inner simplicity that gives human meaning to the mathematics of change and growth.

Let me begin by taking you back to my first encounter with calculus as a fifteen-year-old schoolboy. When I began to learn calculus, my mathematics master, Mr J. H. Butler, a graduate of Oxford University, wrote:

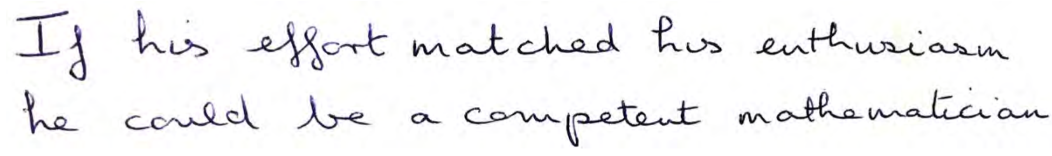

Figure 1: First report as a calculus student

He was concerned that I spent too much time on a variety of activities and did not spend all my time on mathematics. This is still true and it is why I wish to refer to a range of other disciplines in my presentation, because we gain a greater insight into the ways we make sense of the calculus by considering the wider picture.

I really enjoyed working from the Calculus book by school teachers Durell and Robson, published in the thirties.

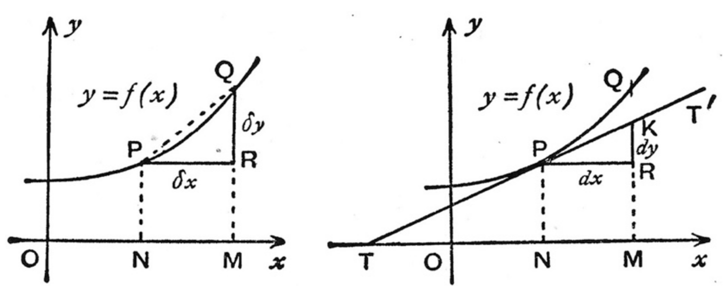

Fig 2: Introducing the derivative in the book by Durell and Robson (1933)

It defined the derivative as the limit of δ*y* over δ*x* as δ*x* tends to zero. Much later, for any real number *dx*, it defined *dy* to be $f'(x)$ times *dx*, then, for *dx* non-zero, $f'(x)$ equals the quotient *dy*/*dx*.

This is based on the book by the Cambridge mathematician, Robert Woodhouse, published in 1803, which introduced the Leibniz notation to England after more than a century using mainly Newton's approach. It spread to English speaking nations around the world, including the American Mathematical Society which based its organisation on that of the London Mathematical Society in the 1880s.

Note the problematic meaning of the limit where δy over δx could only be calculated for δx non-zero while the limit value involved putting δx equal to zero.

There were also multiple meanings of symbolism, such as $f'(x)$ being both a quotient and the symbol *d*/*dx* being an operation:

$$f'(x) = \frac{dy}{dx} = \left(\frac{d}{dx}\right)y$$

It was even an operation that could be repeated to give

$$\left(\frac{d}{dx}\right)^2 y = \frac{d^2y}{dx^2}.$$

Then there was the meaning of *dx* in *dy*/*dx* and in the integral $\int y\,dx$, where it is spoken as *the integral of y '**with respect to x**'*.

Many problematic meanings still remain.

Problematic meanings in calculus today

Calculus is still based on an informal limit concept. For instance, the US Advanced Placement Calculus speaks of four 'big ideas':

- **Limits**
- **Derivatives**
- **Integrals and the Fundamental Theorem**

with an alternative curriculum including

- **Series**

The detailed curriculum focuses on concepts that can be *tested* on multi-choice tests.

Meanwhile, mathematics education research, is full of misconceptions that occur in learning.

Digital technology offers new facilities:

- for performing fast and accurate numerical calculations
- for the manipulation of symbols
- for drawing dynamic pictures that offer visual and conceptual *meaning*

Graphic Calculus

My own work in the 1980s involved developing software called *Graphic Calculus* to visualise concepts.

One program allowed the user to magnify part of the graph, to see that, close up, familiar graphs looked less curved and — under high magnification — they looked *locally straight*.

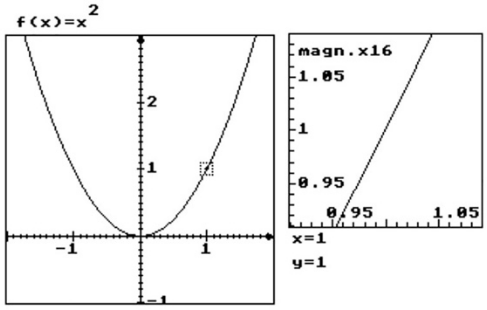

Figure 3: Magnify

The derivative now is not introduced as a limit, **it is the slope of the graph itself.**
It is possible to look along the graph to ***see*** its changing slope.
It is also possible to trace it with your hand to ***feel*** it.

It has human ***meaning***.

Another program plots successive lines of slope ($f(x + c) – f(x)) / c$ on the graph, for variable x and small fixed c, to plot points that lie on the graph that I termed *the practical slope function* for variable x. (In the UK we called it the 'gradient function'.)

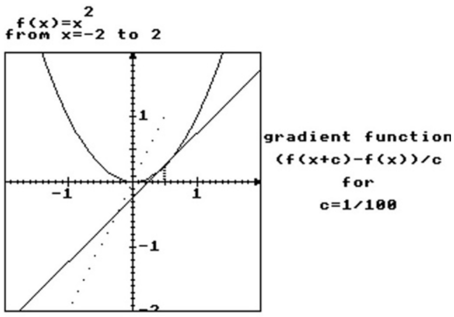

Figure 4: Plotting points on the practical slope function

In the case of $f(x) = x^2$, the practical slope function is $2x + c$. As c is taken to be small, the graph stabilises visibly to the graph of $2x$.

Local straightness gives meaning to all standard derivatives.

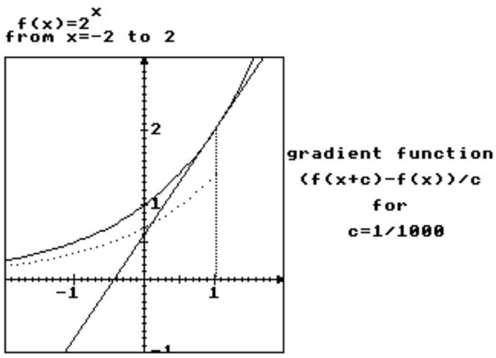

Figure 5: The practical slope function of 2^*x*

For instance, looking at the graphs of 2^*x* and 3^*x* shows that they have the same general increasing shape but the slope of 2^*x* is below the original and 3^*x* is above. Look for a number *e* between 2 and 3 where the slope function for *e*^*x* is again *e*^*x*.

Seeking a polynomial approximation links the visual information to symbolic calculation to give a *sense* of how quickly the approximation converges and allows *e* to be calculated to ten or more decimal places by hand. (Or you may use *Excel*, if you really wish to do so …)

The graphic approach gives visual meaning to the symbolic derivative. 🙂 (Wow!)
But it does not deal with the rules of differentiation. 🙁 (Uhu!)
Recent developments which I will discuss later offer new meanings to fill this gap. This will offer a simple way to understand how spoken articulation gives meaning to expressions and how to interpret the flexible meaning of sub-expressions as process or mental object.

Integration and the fundamental theorem

Now we turn to integration and the fundamental theorem. In this case, I take a function such as sin(*x*) and keep the same vertical scale while stretching the horizontal scale. In this case you can see that the graph 'pulls flat' (Figure 6).

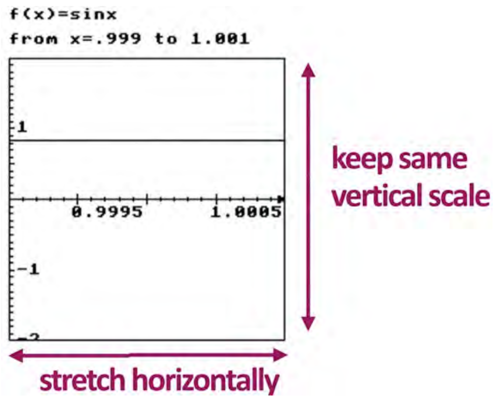

Figure 6: Stretch a (continuous) graph horizontally

Taking the vertical thickness of the pixel line as ±epsilon, the idea of 'pulling flat' embodies the formal definition of continuity at *x*:

Given any $\varepsilon > 0$, we can find a $\delta > 0$ such that, stretched in a window width $x \pm \delta$, the graph lies in the pixel line $f(x) \pm \varepsilon$.

Drawing a continuous graph on paper involves a dynamic physical process, that produces a visual object: the graph. Suitably programmed, a digital picture of a continuous graph will 'pull flat' when it is stretched horizontally.

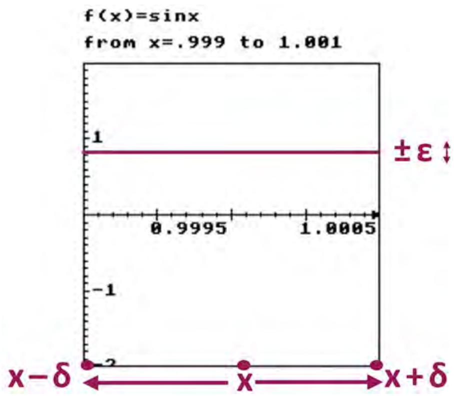

Figure 7: Dynamic definition of continuity

This offers a vital change in meaning. While a static picture in a book does not offer any insight into the fundamental relationship between practical continuity in drawing and the formal epsilon-delta definition, horizontal stretching of a digital picture of the graph offers a dynamic embodied meaning that links practical drawing of a continuous curve to the formal definition of continuity.

This offers a new embodied insight into the fundamental theorem of calculus (Figure 8).

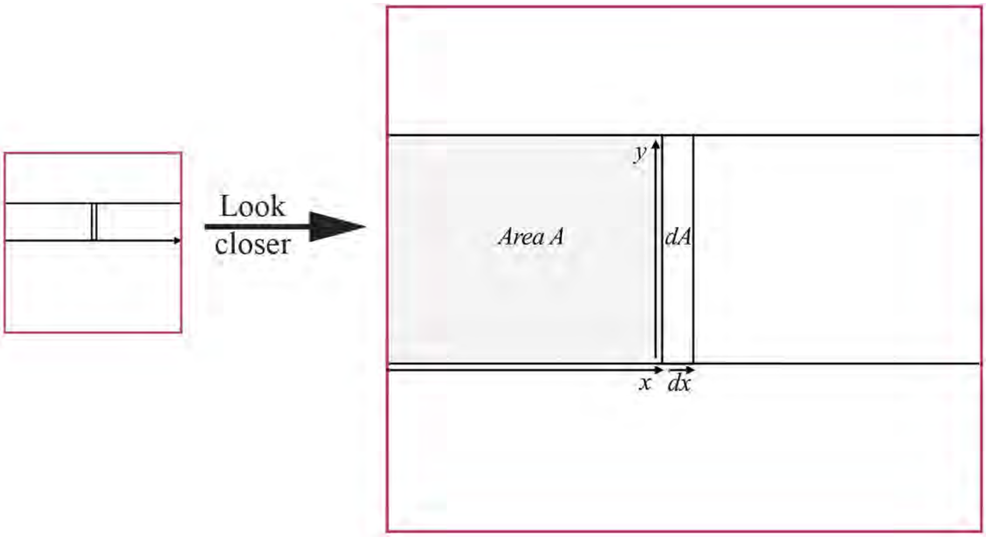

Figure 8: Visualising dA as $y\ dx$

If we look closer at the picture where the graph has been pulled flat in a fixed window, the area A to a point x on the x-axis with y measured up from the x-axis can be imagined to change by an increase dx in x. Then the increase dA in A, calculated as y times dx, is given as $dA = y\ dx$.

This has the useful idea that it is based on a link between the embodied action of drawing and the formal definition of continuity. 🙂 (Wow!)

But on the negative side there are concerns about the meaning. 🙁 (Uhu!)

The horizontal stretching clearly changes the visual area of the strip width dx, height y. But if the product $y\ dx$ is interpreted *as a number* then this remains unchanged. This clearly signals the need to realise that there are (at least) two distinct interpretations of the calculus. One involves the variables as *numbers*, the other involves the variables as *quantities with dimensions*, such as time, length, area, volume, velocity, mass, temperature, density, pressure, national debt, share value, and any other measurable quantities that arise in applications.

In addition, there is still the question about the exact meaning and value of dA being precisely equal to $y\ dx$.

Levels of Sophistication in Calculus and Analysis

To respond to this, we need to look at the bigger picture.

I suggest that **CALCULUS** has two successive levels of sophistication. One is the

> **Practical** level that involves visual representation and symbolic calculation, in which a process *approaches* a limiting value.

Then there is the

> **Theoretical** level that involves informal definition and deduction, focusing on the limit object itself.

ANALYSIS, on the other hand, is based on

> **Formal** mathematics, using set-theoretic definition and formal proof.

Users of Calculus essentially only encounter practical or theoretical mathematics, which involve visual and symbolic aspects. Practical mathematics involves coherent ideas that fit together. Theoretical mathematics introducing more coherent forms of reason, where specific assumptions have necessary consequences.

Human perception and action involve more than visual input. More broadly, they involve a wide range of perceptual senses and forms of action and reason that I term ***Embodiment***. This is physical and mental, visual and gestural, and includes mental thought experiments.

Symbolism involves calculation, compressing processes in time as operational symbols that can be mentally manipulated as objects, where the symbols have multiple meanings.

The Practical limit is *a process that gets as close as is desired*.

The Theoretical limit is the limit *object*.

Embodiment suggests the limit is visually attained. 😮 (Wow!)
Symbolism suggests that the limit is not attained. 🙁 (Uhu!)

Cultural aspects of mathematics

Mathematics arises in different cultural settings. I build my framework on the insight of mathematician Raymond Wilder who used the anthropological term 'culture' to study the evolution of mathematical concepts.

He wrote:

> [A culture is] a collection of customs, rituals, beliefs, tools, mores, and so on, called cultural elements, possessed by a group of people who are related by some associative factor (or factors) such as common membership in a primitive tribe, geographical contiguity, or common occupation.

He used a number of anthropological terms:

- **Cultural stress** involves a need in the community that requires to be resolved
- **Cultural diffusion** refers to moving cultural elements between cultures
- **Cultural lag** acknowledges that diffusion takes time
- **Cultural resistance** occurs when a community opposes diffusion

To these I add a positive heading to balance the negative notion of resistance:

- **Cultural stability** involves maintaining a culture that seems to be working

How do these cultural aspects affect calculus?

In the calculus we have:

- **Stress from different needs in different communities**
- **Stress from fast changing digital technology**
- **Cultural lag in transmitting new ideas to communities that may not even be aware of them**
- **Resistance to new ideas competing with stability**

A major cultural factor is the conflict between cultural stability, and changing need, which underlies the so-called math wars.

Pure mathematicians seek formal proof. Users require theoretical mathematics involving limit objects or practical mathematics involving limit processes. The plan is top-down, the learning is bottom up (Figure 9).

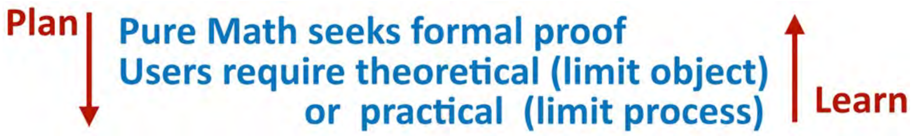

Figure 9: Directions of planning and learning

How does this relate to How Humans Think Mathematically?

To gain insight into how we think mathematically, it is first useful to understand how we interpret what we see (Figure 10).

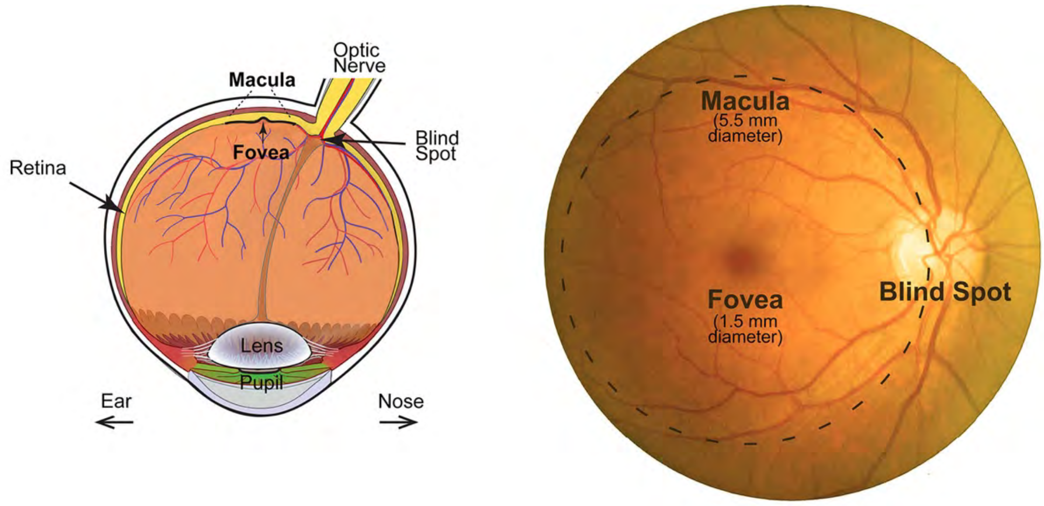

Figure 10: The structure of the eye

On the left is a cross section of the eye looking from above with the pupil at the bottom and the back of the eye at the top. On the right is a view of the central part of the back of the eye as seen through the pupil which includes an area called the ***macula***, about 5.5 mm across. This recognises visual detail.

In the centre of the macula is circular region called the ***fovea***. This recognises high resolution colour and detail through cells called ***cones***. There are around 200,000 cones and the fovea is only about 250 cones across. **We will see that this has a profound effect on how we read symbols and how our eyes follow moving objects.**

When we read text, the eye stops for a brief *fixation* to take in information through the fovea and shifts between fixations in jumps called *saccades*. Read the words on this page to get a sense of how your eye jumps in saccades and stops momentarily in fixations.

Do this now, so that you have a *sense* of what I am talking about …

Reading mathematical expressions

Expressions are read in many languages left to right in chunks.

For instance, this expression

$$\mathbf{1 + 2 \times 3}$$

is read as **one plus two times three**.

The meaning depends on the spoken articulation. If I leave small gaps between symbols, then the meaning can be made more precise. Here I use three dots (an ellipsis …) to denote a brief gap in speech so that "1 + 2 … times 3" means that I say "1 + 2" (which is "3") then, after a gap (…), I say, "times 3". In this case,

"1 + 2 … times 3" is **3 times 3**, which is **9**.

Meanwhile,

"1 + … 2 times 3" is **1 plus 6**, which is **7**.

Once this meaning is understood, the distinction can be written symbolically using brackets as **(1+2)** × **3** and **1** + **(2** × **3)**.

This leads to:

> **The Articulation Principle**: The meaning of a sequence of operations can be expressed by the manner in which the sequence is articulated.

This *Principle of Articulation* can be used at any level to give meaning to the use of brackets. It is not a mathematical definition, but it does lead to meaning for the use of symbolic operations throughout the whole of mathematics.

> I have been developing this idea in recent years in invited papers that are available in draft form from my academic website: (http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html) where they are listed from Tall 2017a to Tall 2020a. For details interpreting operational symbolism, see, for example, Tall, 2020a, pp. 10–13.
> These represent a continual theoretical development with much in common, but each one is written afresh for different audiences. I am currently working on an overall paper devoted to calculus. When completed, it will be added to my academic website downloads page.

How does the eye and brain perceive motion?

To follow a moving object, the mechanism of the eye again works in saccades and fixations. Reading text involves a succession of fixations to read information in chunks separated by quickly jumping saccades. Following a moving object happens in a different way. There is an initial saccade to fix on the object and then the fixation moves smoothly to follow the object.

Place your finger in front of you and move it to the left and right. The eye makes an initial saccade to focus on it, then moves in a smooth fixation to follow it. This will happen if you move your head or keep your head still and just turn your eye.

Try this out for yourself, moving your head or keeping it still, to see how you focus smoothly on your moving finger while the background is blurred.

Do this now to sense how it happens …

Now imagine two points on a number line, one fixed, one moving (Figure 11).

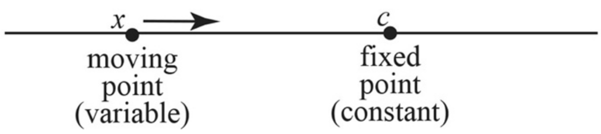

Figure 11: Variables and constants

In the presentation, the moving point is animated to move towards the fixed point, to encourage the viewer to imagine a variable point moving towards a fixed point. This reveals the idea that the brain naturally imagines constants and variables, including quantities that can become arbitrarily small.

The intuitive foundations for calculus are built into the human brain.

Zooming in on a curve on a retina screen

Looking back at the original picture from the 1933 text of Durell and Robson, it becomes evident that looking at this picture gives no sense of the local slope of the graph. Indeed, if one magnifies part of the picture, this simply thickens the lines. Figure 12 shows the original picture magnified in a box where the box remains the same size and the picture is magnified within it.

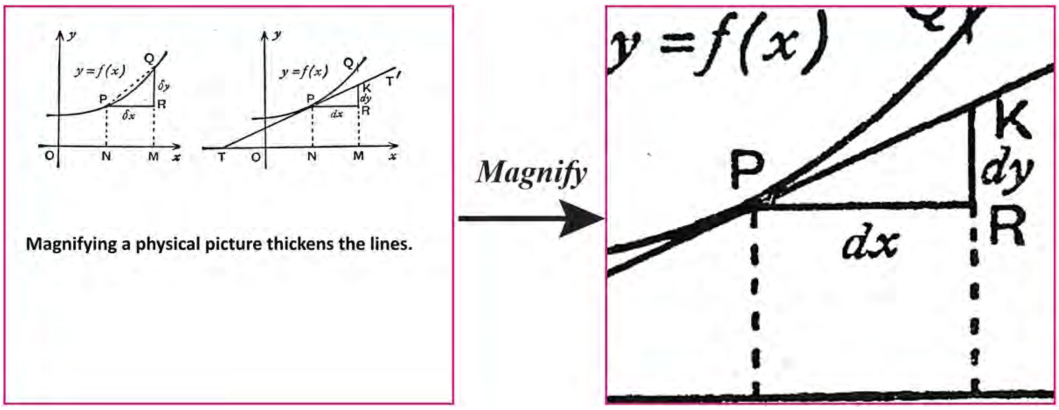

Figure 12: Magnifying a physical picture in a book

Scaling on a retina display can be programmed to maintain the thickness. (Figure 13).

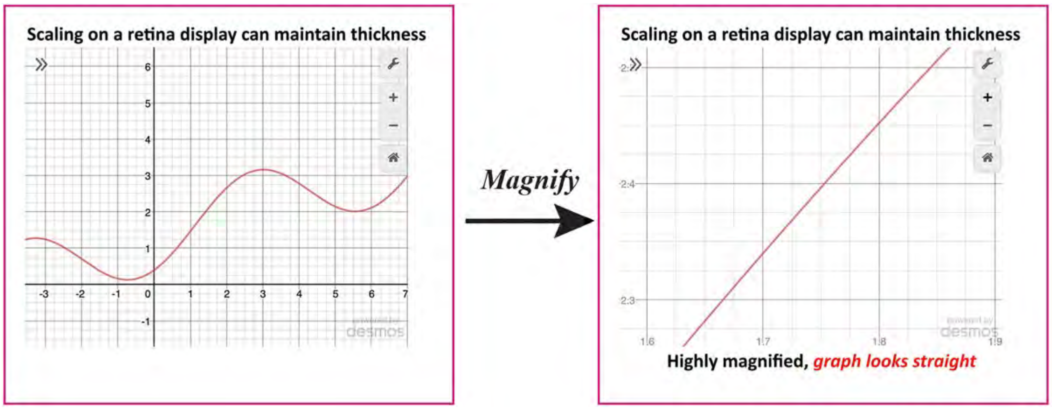

Figure 13: Magnifying a curve on a retina display

Highly magnified the graph looks like a straight line. By maintaining the thickness of the curve in the visual representation, the viewer can now *see* that the graph is locally straight. This changes the focus of attention from the *practical limit* which *approaches* the limit object to the *theoretical limit* which *is* the limit object.

The notion of local straightness relates directly
to the structure and operation of the human eye!

Using a retina display, the practical slope function stabilises on the theoretical slope function. This gives a visual embodiment supporting local straightness which switches attention from

The practical limit as a process that gets as close as is desired

to

The theoretical limit which is the derivative as a limit object.

It gives meaning to the derivative as a theoretical object.

So, we must ask the fundamental question:

Why is local straightness rejected in AP calculus?

Cultural resistance over the centuries still operates today

A reason for the persistence of the traditional approach to the calculus (starting from an informal version of the modern definition) may lie in the desire for cultural stability and its consequent cultural lag and cultural resistance. New forms of technology provide radical changes communication, operation and imagination that cause great cultural stress by changing faster than current communities can cope.

This is a natural feature of the evolution of ideas as new generations build on the developments of earlier generations. Today we still have the vestiges of earlier beliefs that act as obstacles for change. For example, consider the following Greek common notion:

Greek Common Notion
The whole is greater than the part

This is self-evident for counting objects or measuring figures.

But Galileo was found guilty of heresy for suggesting that the whole numbers are in one to one correspondence with the squares of whole numbers.

Cantor realised that this property is the *definition* of an infinite set.

This was met with hostility and cultural rejection and led to his mental breakdown.

I pose the question:

Is this is happening in Calculus today?

The meaning of infinitesimals

Let us consider the formal theorem:

Theorem
A complete ordered field cannot contain infinitesimals

This is self-evident for counting objects or measuring figures.

It implies there are no infinitesimals in the real numbers.

However, it does not imply that there are no infinitesimals on a number line …
… just as there are *no irrationals* on the *rational number line*.

I can prove to you that infinitesimals must occur in *any* ordered extension field of the real numbers.

Let K be any ordered field with $\mathbb{R}$ as an ordered subfield. Define an element x in K to be *finite* if it lies between two real numbers, a and b …

… and to be *infinitesimal* if x is non-zero and lies between $-t$ and t for every positive real number t.

Then we can prove the following structure theorem:

Structure Theorem: Any finite element x in K must either be in $\mathbb{R}$, or of the form $c+\varepsilon$ where c is in $\mathbb{R}$ and ε is an infinitesimal.

The proof is a simple application of the completeness axiom. □

Call elements in K, 'quantities', and elements in $\mathbb{R}$, 'constants'.
Then we have the following Theorem:

Any finite quantity is either a constant or a constant plus an infinitesimal.

For finite x, the unique real c is called *the standard part* of x, written st(x).

To see infinitesimal detail, we magnify the scale by defining the ε-lens pointed at c for any c and non-zero ε to be

$m: K \to K$ such that $m(x) = (x - c)/\varepsilon$.

This magnifies the line by an infinite factor $1/\varepsilon$ focused on the point c.
Define the field of view of m to be the set V such that $(x - c)/\varepsilon$ is finite.
The optical ε-lens pointed at c is μ: $V \to \mathbb{R}$ given by

$$\mu(x) = \mathrm{st}((x - c)/\varepsilon)$$

I introduced the notion of optical lenses in Tall (1982b). I proved the Structure Theorem for Infinitesimals in the form given here for an undergraduate course. I never published it as a mathematics paper because I considered it a simple exercise using the completeness axiom. When Ian Stewart and I wrote new editions of *Foundations* (1977) and *Complex Analysis* (1983) in 2014, 2018 respectively, we incorporated chapters on real and complex infinitesimals using this Structure Theorem. It is now an essential link between formal mathematics and more sophisticated forms of embodiment and symbolism.

This gives a map from the field of view V on the extended number line K to the real number line. When ε is an infinitesimal (in which case the ε -lens is also called an *optical microscope*) the original elements c, $c + \varepsilon$, $c + 2\varepsilon$, …, in K differ by infinitesimal quantities, but their images on the real line are visibly distinct. So, we can now 'see' infinitesimal differences under infinite magnification factor $1/\varepsilon$ (Figure 14).

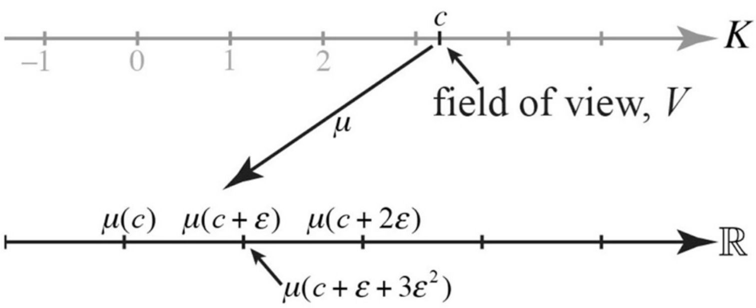

Figure 14: Optical ε-lens

However, $c + \varepsilon$ and $c + \varepsilon + 2\varepsilon^2$ differ by $2\varepsilon^2$ and the optical lens maps them to the same point.

Further details are available in the second edition of *Foundations of Mathematics* (2014) written jointly with Ian Stewart and, in more detail for real and complex analysis in our joint *Complex Analysis* (2018).

Furthermore, the ideas apply in two or more dimensions, simply by using optical lenses on each component. An optical lens in n dimensions can use different magnification factors on each coordinate.
For example, the magnification factor $1/\varepsilon$ can be used on both coordinates to see local straightness of a differentiable function (Figure 15).

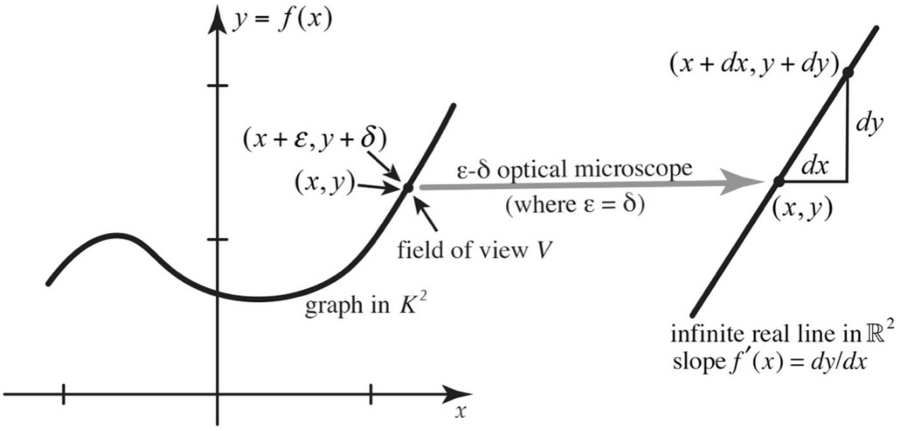

Figure 15: Local straightness of a differentiable function

A horizontal magnification of 1/ε and a vertical magnification of 1 can be used to see local flatness of a continuous function (Figure 16).

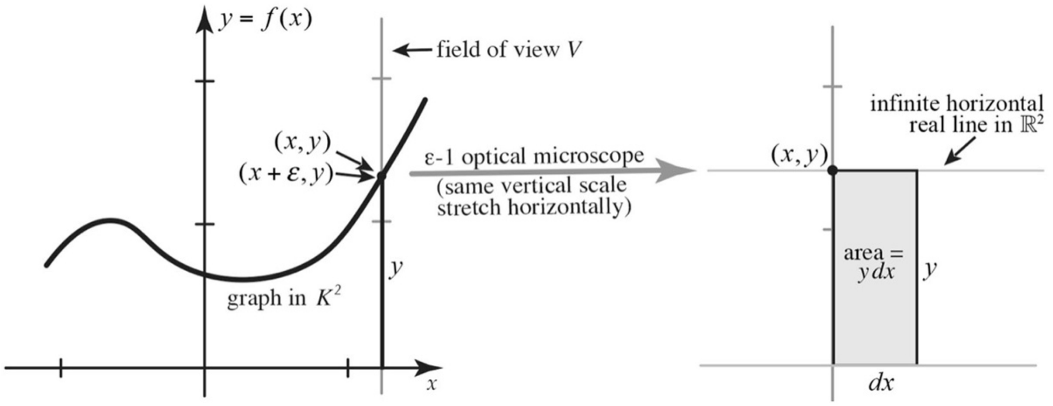

Figure 16: Local flatness of a continuous function

In both this pictures, I have taken the liberty to denote the image μ(*s,t*) for (*s,t*) in *V* by the same symbol (*s,t*) in ℝ, even though the original coordinates *s,t* are in the field of view in *K* and the image coordinates are real numbers. This is a long-standing convention in making maps where we denote the image of, say, New York, on a map by the name 'New York'. In the same way, when we draw a map, we only represent certain level of detail appropriate for the scale of the map.

The picture of local flatness (figure 16) appears here in this form for the first time. Other details can be found in chapter 13 of *How Humans Learn to Think Mathematically* (Tall, 2014).

Optical lenses arise in Formal Mathematics. They are included here to show formal mathematicians that infinitesimal ideas are a fully valid part of formal mathematics.

> In my presentation, I did not have time to elaborate on important ideas presented in the second edition of *Complex Analysis*, chapter 15, where I put forward the argument that thinking of infinitesimals as 'arbitrarily small variables' is more productive than using formal optical lenses. The reason for this arises in dealing with polar coordinates where small increments *dr* in *r* and *dθ* in *θ* give a small approximate rectangle with side lengths *dr* and *rdθ*. Using optical lenses it is not possible to simultaneously magnify the infinitesimal shape with sides *dr* and *rdθ* while also including the origin, when it is a finite non-zero distance *r* away. However, it is easily possible, *in the mind's eye*, to imagine a picture of the plane with the origin and the tiny shape in the same picture and then to zoom in on the shape to *see* it look like a rectangle. (This could also be represented appropriately on a retina display.) For this reason, applied mathematicians use an embodied human version of the calculus to make sense of arbitrarily small changes.

I also did not have time to respond to the claim that local straightness is inappropriate for calculus based on the idea that, when applied to polar coordinates, it no longer works. This arises from different views of the meaning of a graph in the plane, as to whether it is a two-dimensional object in $\mathbb{R}^2$, or a set of ordered pairs in $\mathbb{R}\times\mathbb{R}$. This confusion has lasted for centuries and it continues today in the distinction between *y* being a function of *x* in $\mathbb{R}\times\mathbb{R}$ and the notion of covariation between variables *x* and *y* in $\mathbb{R}^2$. Both are valid, but with very different meanings. It has taken me many years to understand this, although there are vestiges of the distinction in my published writing. Back in 1977, in the first edition of *Foundations*, in a section written by Ian Stewart, there is an explanation of the notion of a function $f:\mathbb{R}\to\mathbb{R}$ which begins with two copies of the number line $\mathbb{R}$ and then turns the second number line round through 90° with the arrow going from a point on the horizontal *x*-axis to the vertical *y*-axis being redrawn to go vertically from the point *x* to the height *y* and turning through a right angle to end up on the vertical axis at a height *y*. The relationship between *x* and *y* is then represented by the graph of points of the form (x, y) as $(x, f(x))$. Local straightness applies to this interpretation.

The notion of co-variation involves *x* and *y* varying together, with a relationship between the two. This can be considered as a change occurring *in time*. In this interpretation there is a hidden parameter *t.* It doesn't matter how fast *t* changes, *as long as it goes in the same direction*. This is related to Newton's idea of *fluxions* where variables change in time. Leibniz, on the other hand, essentially saw the curve as an object that was made up of an infinite number of infinitesimally short sides.

As individuals with different genetic inheritance and different previous experiences, we all develop different knowledge structures. Mine, as a Grammar School boy and an Oxford undergraduate and graduate, caused me to see the graph as a relationship relating the independent variable *x* to the dependent variable *y*. I saw the relationship as one-way and could not grasp the idea of balanced *co*-variation, because I believed that *y* varies as *x* varies, but not the other way round.

In dealing with implicit functions I was faced with a dilemma. I found the loose way that English text books dealt with a relationship such as $x^2 + y^2 = 1$ to 'differentiate with respect to *x* to get $2x + dy/dx = 0$, and so $dy/dx = -2x$ was, to me, unsatisfactory, because *y* is not a function of *x* in the previously defined sense. So, in designing the calculus curriculum for the School Mathematics Project, I began by introducing a parametrisation, in this case $x(t) = \cos(t)$, $y(t) = \sin(t)$ for $0 \le t \le 2\pi$, to calculate dx/dt, dy/dt, where (dt, dx, dy) is the tangent vector in 3 dimensional (t, x, y) space. This is an alternative way of conceiving *x* and *y* as being co-variant in ordered time *t*, so that co-variation now makes sense to me. Local straightness still applies, but now it is in three dimensions. In fact, it naturally generalises to the multi-dimensional case, but that is another story.

Strangely, I now see Einstein, entering the picture that was imagined by Newton. Einstein lives in four-dimensional spacetime with three-dimensional space and a fourth time dimension that is *directional* and only operates in that one direction. In such a world view, co-variation is a natural concept: *x* and *y* co-vary in a particular direction *in time*. In our imagination we can see time as a video, say of a train travelling in reverse, then we play the video backwards to see the train travelling in a forward direction. Our imagination goes beyond our real-world experience.

Different viewpoints are enormously valuable. They enable us to see possibilities beyond our own personal experience. I have in recent years had the scales removed from my eyes to see such possibilities, but I still sense we need to stand *above* the differences to question our own views and to seek profound underlying patterns.

We can learn by reflecting deeply on serious data produced by various studies in a range of different areas of expertise. For example, the report on *Insights and Recommendations from the MAA National Study of College Calculus* by Bressoud, Mesa and Rasmussen (2015) can be downloaded from the internet at

https://www.maa.org/sites/default/files/pdf/cspcc/InsightsandRecommendations.pdf

and contains a wealth of detailed information.

Analysing the data from viewpoints in this presentation – including personal brain structure and operation, corporate cultural aspects and the long-term development of mathematical thinking – gives new ways of forming a coherent overview that reflects the realities of how we think mathematically, and how our cultural evolution affects our teaching, learning and thinking about calculus.

The question is: given the cultural pressures to maintain cultural stability in the face of cultural resistance, how is it possible to move ahead? My proposed solution in recent papers is, first to introduce principles such as the Principle of Articulation to give meaning to symbolism while linking embodiment and symbolism. Then I have analysed the changes in context throughout the curriculum to identify those aspects that remain supportive over several changes. These can give a longer-term sense of stability allowing the learner to reflect on those changes that are problematic so that they can be addressed meaningfully at the time that they arise.

But will it happen? This is not in my gift.

Reflections

I have included the notion of optical lenses to show that they justify the notion of infinitesimals on a number line using formal mathematical theory. However, this theory belongs in formal mathematics using the axiom of completeness, not in the practical or theoretical levels of sophistication of calculus.

The idea of optical magnification can be visualized in Practical Embodiment in any program that allows embodied scaling on each coordinate. This is possible in *GeoGebra*.

> The original TI-92 graphic calculator allowed zooming in with differing scale factors, based on my advice to the company, but this is not used in any curriculum as far as I know. The current implementation in GeoGebra allows magnification of the plane by moving finger and thumb apart and has the possibility of stretching horizontally by stretching the horizontal axis. In the version I have seen, it needs reprogramming to allow horizontal stretching when the horizontal axis is not in the picture.
>
> I understand that currently there is competition between *GeoGebra* and the software *Desmos* for the larger market share. The GeoGebra interface operates on the picture itself to stretch it in various ways while Desmos prefers to use sliders to alter parameters. Desmos has the advantage of a broader classroom environment in which all the students work on their own smart phone, iPad or computer, connected using wi-fi. It is then possible to display the screen of any student on the teacher's main display to encourage class discussion. Desmos is therefore smarter in providing an environment for formative assessment of how different students are progressing as compared to the summative multi-choice tests currently used in AP Calculus.

Stretching graphs on a retina display supports the notion of infinitesimals as variable quantities as envisaged in mathematical applications and also as visualised naturally by the human eye and brain. Infinitesimals can be imagined as processes in practical mathematics and embodied as objects in theoretical mathematics. They make sense as 'arbitrarily small quantities' in applied mathematics.

In my presentation I focused on how we think about Calculus, in particular the roles of **Embodiment** and **Symbolism**. Embodiment includes how our brain makes sense of dynamic change, speaking and reading text and mathematical expressions, translating these human activities into operational symbolism. This symbolism involves *embodying* change through operations symbolised as mental objects that can themselves be operated on at a more sophisticated level.

I have identified three significant levels of sophistication:

- Practical limit processes
- Theoretical limit objects
- Formal limit definitions and proof

In the learning and teaching of these ideas, I propose that we build *up* to the limit concept through practical and theoretical experiences, not *down* from a formal definition, as yet unknown to the learner, presented in a manner that focuses on a process that seems unfinished.

In the limited time available, I did not include essential parts of my broader framework which incorporates affective aspects of mental activity, including confidence, anxiety and so on that are included in chapter 5 of *How Humans Learn to Think Mathematically* (Tall, 2013).

> There I built on earlier work of Skemp, based on Freudian psychology where he formulated a theory of *goals* (to be desired) and *anti-goals* (to be avoided).
>
> A goal that one believes is achievable is accompanied by a feeling of confidence, which may change to frustration if it proves subsequently to be difficult to achieve. Frustration sensed by a confident person is likely to act as a positive encouragement to redouble the effort to achieve the goal. Moving towards a goal gives pleasure and moving away from it gives unpleasure—a term used in Freudian analysis to denote the opposite of pleasure.

Coping with an anti-goal is quite different. According to Skemp, an anti-goal that one believes one can avoid gives a sense of security but, when it cannot be avoided, the emotion turns to anxiety. Moving towards a goal instils a sense of fear, while moving away gives relief.

This theory is represented in figure 17 (which I drew in Tall, 2013, p.120) where arrows represent movement to or away from a goal or anti-goal and smiling or frowning icons represent the belief related to the ability to achieve a goal or avoid an anti-goal.

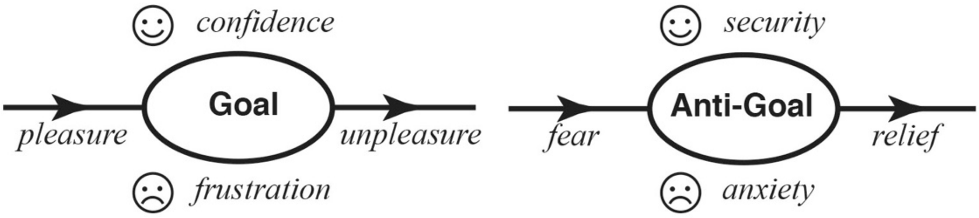

Figure 17: Emotional reactions to goals and anti-goals

This reveals the vast difference between positive emotions of confidence and pleasure relating to goals that are considered achievable and emotions relating to anti-goals which offer, at best, a sense of security and relief and, at worst, a sense of anxiety and fear.

Since 2013, I have studied the literature on neurophysiology more closely, comparing its studies of fMRI scans which only note changes over a period of two seconds or so, as compared with what can be observed in a classroom by an observer familiar with the structural operation of the brain. In particular, I have taken great interest in the operation of the limbic system which links to automatic body regulation and operates quickly before the frontal cortex can make a considered decision. Here the mathematician and neuroscientist Stanislaus Dehaene has provided some very interesting data recording the parts of the brain that respond to various forms of mathematical thinking and suggests that the brain recruits areas related to handling of number and shape and space to think mathematically that are not related to areas involved with language.

Without language there would be no sophisticated mathematics, for it is in *naming* ideas that we can begin to think about them and communicate them in increasingly sophisticated ways. However, as human beings our brains develop specialised functions that we pass on to later generations, accumulating mathematical knowledge. This relates to our natural abilities to repeat sequences of actions and to name operations using operational symbolism, leading to a compression of knowledge where the symbols become mental objects that can themselves be operated upon.

The question of meaningful and rote learning can now be considered in a different light. If rote-learning is prioritised over making meaningful links, then, in the long term, the compression of knowledge is less likely to occur in ways that enable more sophisticated levels of mathematical thinking.

Cultural stability operates to preserve those aspects which are considered to be important. But does it serve us well in the long-term?

I have proposed the need to think about cultural resistance that may impede our progress.

This requires a reflection on the subtle aspects that underpin our personal thinking. I seek a deeper simplicity in my life. I have a few more years on this earth and I wish to spend it in quiet contemplation and have no desire to become involved in emotional disputes. I do welcome any comments that cause me to rethink my position as this can help me improve my understanding.

Meanwhile, I encourage the participants to enjoy the conference, but do not forget to dream!

We have new tools to make more sense in the future, so use them wisely. In a time when we have a wealth of technology to support our human thinking, remember the words of Richard Buckminster Fuller:

> You never change things by fighting the existing reality. To change something, build a new model that makes the existing model obsolete.

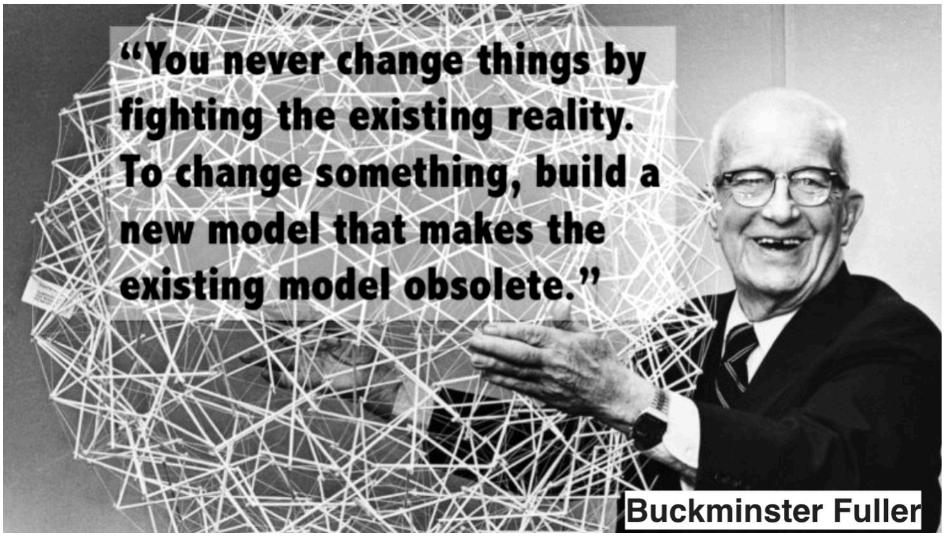

Figure 18: Build a new model that makes the existing model obsolete

Today we have technology for computation, symbol manipulation, interactive retina graphics and a verbal interface using a version of artificial intelligence. The evolution of ideas goes on into the future.

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