Complementing supportive and problematic aspects of mathematics to resolve transgressions in long-term sense making

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In the teaching and learning of mathematics, while it is important to focus on what happens at each stage of development, what matters even more is the cumulative effect of learning over the long-term. As mathematics grows in sophistication, new contexts require new ways of thinking that can act as barriers to progress. Passing through such a barrier may be called a *transgression.* This presentation focuses on aspects of mathematics that remain consistent over several changes in context and contrasts them with others that cause conflict at any given stage. For instance, how we *speak*, and *write* mathematics reveal new insights into making long-term sense of increasingly sophisticated mathematical symbolism in arithmetic and algebra. How the eye tracks a moving object affects how we interpret the notion of variable in the calculus both visually on a number line and symbolically as a variable quantity.

Studying successive changes in mathematics and the positive and negative emotional affects leads to an overall framework for long-term development that applies both to historical evolution and to the individual development of different learners. It offers a practical approach in the classroom and a theoretical framework that brings together widely differing interpretations held by mathematicians, educators, curriculum designers, philosophers, psychologists, neuro-physiologists, and even politicians who currently specify the curriculum.

1. Introduction: the notion of transgression

This paper has been prepared for a plenary at the 4th Interdisciplinary Scientific Conference on Mathematical Transgressions where the term ‘transgress’ has the broad meaning of ‘crossing over’ a limiting boundary to a new, previously untenable context. Here I focus on *mathematical transgressions* in the long-term growth of the individual as more sophisticated mathematics is encountered that requires a reconstruction of earlier knowledge. I will use an analysis of how we humans construct mathematical ideas formulated in my book *How Humans Learn to Think Mathematically* (Tall, 2013) from birth to the full range of adult thinking which has corresponding links to the historical evolution of mathematical thinking.

This will be enhanced by new ideas that have been developed since that book was published, to give a more comprehensive framework for teaching and learning mathematics *meaningfully* over the long term. It does not see difficulties that students encounter as ‘misconceptions’ that need to be corrected. Instead it seeks fundamental ideas in mathematics that link to the natural operation of the biological brain, to focus on thought processes that are *supportive* over the longer term and to contrast these explicitly with *problematic* aspects that require new ways of thinking in new contexts. Supportive connections produce electrochemical changes in the brain that enhance mental activity while problematic links cause conflict that inhibits thinking. The theory has *practical* implications that seek to make sense for a wide range of classroom teachers and learners.

As prepared in advance for ***The Fourth Interdisciplinary Scientific Conference on Mathematical Transgressions***, March 2019, with reflections on the conference added afterwards.
Please note that if this paper is read as a pdf in a web browser, then clicking underlined references will directly open up the link if it is still available on the internet.

2. Transgression, enlightenment and multi-contextual overview

The notion of ‘transgression’ depends on the individual or community that takes a particular viewpoint. If community A disagrees with community B about some fundamental principle or belief, and an individual or subgroup S in community A switches to principles supported by B, then this will be regarded as a *transgression* by those remaining in community A, but as an act of *enlightenment* by those in community B.

Instead of setting up a dispute between different communities, the proposal is that we seek to raise the perspective to a higher level that blends and contrasts ideas, placing the views of different communities within a broader framework. This will require an open mind on the part of the reader to seek to see beyond the boundaries of their accepted practices and to realise that other communities have differing ways of interpreting their own personal and social needs.

Some communities or individuals may only require a practical competency in the subject, some may develop a more theoretical level involving deductive relationships between ideas: a few may move on to the more formal mathematics that evolved at the end of the nineteenth century. In our own life time, phenomenal changes in information technology are offering previously inconceivable ways of thinking mathematically for the wider population. The framework developed here is designed to be sensitive to the full range of possibilities.

This includes the possibility of moving to a higher level where both contexts A and B can coexist and each can be seen to be coherent in its own context. For example, in a religious context it is possible to have multi-faith communities in which each faith respects the beliefs of other faiths while remaining true to its own. In mathematics, changes in context, such as broadening the number systems from counting numbers to fractions, signed numbers, real numbers, complex numbers, introduce new properties which may not hold in other contexts. For example, the context of whole numbers includes the notion of multiples and factors together with related topics of prime numbers and unique factorisation of a whole number into primes. These hold in the context of fractions and signed numbers, provided that the powers involved can be positive or negative, but there are number contexts larger than the whole numbers, such as numbers of the form $a + b\sqrt{15}$ for whole numbers a, b, where factorisation is still possible, but is no longer unique. For example

$$10 = 2 \times 5 = (5 + \sqrt{15})(5 - \sqrt{15})$$

where 2, 5, $5 + \sqrt{15}$, $5 - \sqrt{15}$, are all factors of 10, but none can be factorised further to give a unique factorisation. This led to a new theoretical framework of algebraic number theory, including the search for the proof of Fermat’s last theorem (that there are no solutions of the equation $a^n + b^n = c^n$ where a, b, c, n are whole numbers and $n \geq 3$). It took over three centuries before Andrew Wiles made the transgression into a new theoretical framework where the theorem could be proved (Stewart & Tall, 2013).

At a higher conceptual level, the long-term development of mathematical thinking requires an awareness of the changes in meaning in new contexts, both from the viewpoint of individual experts, and also from the viewpoint of the individual learner.

2.1 Transgressions studied in this presentation

In this presentation I only have space and time to cover a selection of transgressions in the development of mathematical thinking. I will focus on the development of *written and spoken symbolism* in arithmetic and algebra and *visual interpretation of constant and variable quantities*. These build on earlier publications (e.g. Tall 2013, Stewart & Tall 2014, 2015, 2018) which will benefit from being extended in the light of the framework presented here.

3. The Biological Brain

Mathematical thinking takes place in the human brain and depends on how information is

- *input* through the senses,
- *internally processed* (making internal representations and links between them),
- *output* through human action (including gestures, speaking, writing and drawing), and
- *communicated* between individuals.

These need to be considered in terms of:

- *cognitive* aspects of how individuals think about mathematical structures.

It is also essential to consider:

- *affective* aspects that enhance and suppress the making of mental connections.

In particular I propose that general supportive principles can act as a stable foundation for long-term learning and give insight into transgressions caused by problematic changes in meaning.

The operation of the brain is exceedingly complex. Modern technology to study the brain ranges from simple electrodes attached to the scalp to record electrical surface activity to more sophisticated MRI (magnetic resonance image scans) that distinguish the static structure of brain and fMRI (functional MRI) that record electrochemical activity in terms of flow of blood over a period of several seconds. None of these are adequate to study thinking at the neuronal level where connections operate in milliseconds. At best they currently build a broad picture of what parts of the brain perform specific functions.

Even so, by using what we do know about the operation of the brain, it is possible to *see* and *hear* essential features in everyday mathematical activity that offer insight into the changes required to deal with increasingly sophisticated ideas in long-term mathematical development. These include *how we see arrangements of objects in different ways*, *how we read text and interpret mathematical expressions*, and *how we interpret dynamic movement of objects*. We begin by considering some specific examples.

3. 1 Early experience with number

A child's early experience of number begins with a variety of activities, including nursery rhymes, 'one, two, three four five, once I caught a fish alive…', which involve rhythmic repetitions of number names in sequence and early experiences with counting. Even before this, the child has a rudimentary sense of numerosity, recognising the difference between the number of objects in a small set of up to three or four items.

The concept of number develops with experience over time until a child may realise that, when a collection of objects is counted in different ways, the total is always the same. The complexity of this notion, which Piaget described as 'conservation of number', can be seen when counting a collection of six objects (figure 1). (I choose six because it is the smallest whole number that reveals properties of addition *and* multiplication.)

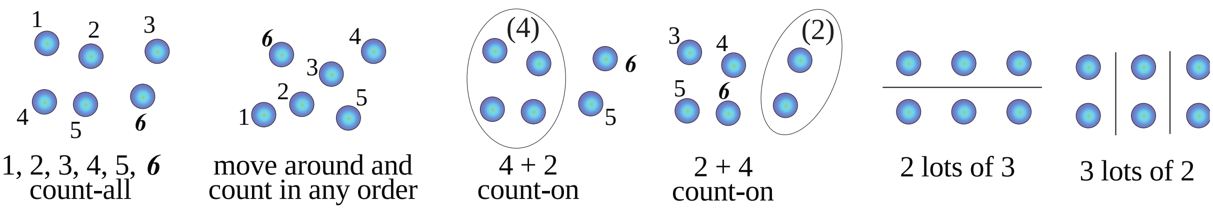

Figure 1: conservation of the number 6

Various ways of proceeding are possible, including:

- count and point once and once only at each object in turn, to ‘count-all’ (1, 2, 3, 4, 5, ***6***),
- move the objects around, and count in any order,
- see a subset of 4 and ‘count on’ 2 after 4 (saying 5, ***6***),
- see a subset of 2 and ‘count on’ 4 after 2 (saying 3, 4, 5, ***6***),
- re-arranging as 2 lots of 3,
- re-arranging as 3 lots of 2.

Notice how the operation of addition involves conceptualising an operation such as 4 + 2 in different ways. Sometimes the symbol 4 may be seen as a process of counting, sometimes as a concept of number. Over the longer term, new ways of interpreting expressions arise as mental connections form which allows operations, such as addition, to become compressed as mental objects in the form of known facts: 4 + 2 *is* 6 and 4 + 2 *is the same as* 2 + 4.

More generally, an expression combining operations on mathematical concepts is described as a ‘procept’ in Gray & Tall (1994). This paper has the title ‘*Duality, ambiguity and flexibility in mathematical thinking*’. I can now reveal that the title has far deeper implications in how mathematical expressions can be interpreted to *make sense* in different ways. This involves scanning the whole expression to see how sub-expressions can be interpreted flexibly as ‘process’ or ‘concept’, which I will describe in this paper using the terms ‘operation’ or ‘object’. (The term ‘operation’ will be used to represent a sequence of mental actions performed for a specific purpose, considered as a whole. Different operations, such as those represented in figure 1, can represent the same underlying mental object, in this case the number 6 together with all its internal structure and external relationships with other numbers.)

In practice, operations of addition on whole numbers are interspersed with operations of ‘take away’ and limitations arise such as ‘you can’t take a bigger number from a smaller number.’ Such limitations may form a boundary in the context of whole number that will need to be transgressed to move to the context of signed numbers. Another possible difficulty arises with the multiplication of whole numbers, where (except when one of the numbers is 1) the product is bigger than either of them and certainly never less. This can cause an implicit boundary that requires a transgression in moving from whole numbers to fractions.

More generally, subconscious conflict can cause the flooding of the brain with neurotransmitters[1] that enhance or inhibit thinking processes, provoking resilience and determination for some and anxiety and rejection for others. (See also Kahneman, 2010.)

3.2 Input, output, internal processing and communication of text and mathematics

There are significant differences in how text and mathematical expressions are handled in different communities of practice. Speaking and hearing *occurs in time*, and so has a natural direction. Even so, internal processing may involve a change in direction to interpret the meaning. For instance, in German, the 3-digit number 123 is read as ‘ein hundert, drei und zwanzig’ which translates to ‘one hundred, three and twenty’, requiring it to be processed in the order 1, 3, 2:

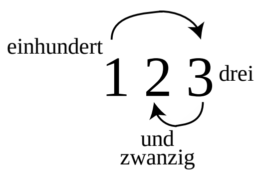

Figure 2: scanning a number in German

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[1] See: https://en.wikipedia.org/wiki/Neurotransmitter

Hindu-Arabic numbers (originally read from right to left in Arabic) need first to be scanned when read in European languages to know that 123456 is ‘one hundred and twenty-three thousand, four hundred and fifty six’, whereas 1234567 is ‘one million, two hundred and thirty four thousand, five hundred and sixty seven’.

Other conventions which may cause confusion include writing symbols next to each other where $2x$ means ‘2 *times* x’, 2½ is ‘2 *plus* ½’ and 24 is ‘2 tens and 4 units’.

In Chinese, characters are written downwards in vertical columns, starting on the right of the page with columns moving successively to the left. Traditional Japanese script (*tategaki* 縦書き) follows the Chinese convention, but Japanese in scientific and mathematical texts follows the European convention, in rows from left to right (*Yokogaki* 横書き).

To simplify the discussion in this presentation I will focus on speaking and hearing text in time, reading and writing a symbol such as 123 + 123456 + 1234567 from left to right. Each individual term, such as 123 or 123456 may require scanning to determine how to say it.

If we place each term in a box and regard each box as a single mental object, then

$$\boxed{123} + \boxed{123456} + \boxed{1234567}$$

can be considered as a sum of three objects read successively from left to right, regardless of the subtleties of scanning the digits within each box.

If the order of terms in such an expression is changed, then we find that

$\boxed{12} + \boxed{4} + \boxed{6} + \boxed{15}$ gives the same answer as $\boxed{4} + \boxed{15} + \boxed{6} + \boxed{12}$.

The same is true for a sequence of additions and subtractions:

$\boxed{7} - \boxed{¾} + \boxed{1.414} + \boxed{-5}$ gives the same answer as $\boxed{1.414} + \boxed{7} + \boxed{-5} - \boxed{¾}$.

3.3 principles of addition, subtraction, multiplication and division

This observation offers an insight that can be expressed as:

> **The General Principle of Addition and Subtraction**: A finite sequence of additions and subtractions of numbers is independent of the order of calculation.

This principle holds good throughout whole numbers, signed numbers, fractions, decimal notation, infinite decimals, real numbers and even complex numbers. It is *a supportive principle* that works throughout school mathematics.

For individuals who attain a more sophisticated level of mathematical thinking, it may lead to a more compact yet profound generalisation:

> **The General Principle of Addition**: The sum of a finite collection of constant or variable quantities is independent of the order of calculation.

There is a corresponding principle for multiplication and division (by a non-zero quantity):

> **The General Principle of Multiplication and Division**: A finite sequence of multiplications and divisions is independent of the order of calculation.

This also has a more sophisticated generalisation to constants and variables at a later stage.

3.4 Giving meaning through spoken articulation

In contrast to these two essentially simple general principles just given, combinations of operations involve problematic aspects with potential transgressions. These may be addressed meaningfully if we pay attention to how we articulate an expression when we say it. Consider the example:

‘What is 2 + 3 × 4?’, spoken as ‘What is two plus three times four?’

If this is spoken in an even tone, without any particular emphasis, then it first states the operation ‘2 + 3’ which is ‘5’, then ‘×4’ which gives **20**.

But children are taught in school to use the convention 'multiplication takes precedence over addition', so one must first perform the second operation '3 × 4' first, to get '12' and then calculate '2 × 12' to get the '*correct*' answer, **14**.

When children are given 'rules without reason', many find arithmetic, and later algebra, mystifying. I suggest that a simple approach is to help learners *make sense* of ideas in a way that appeals to them at the time.

Initially, I propose that different meanings to expressions can be highlighted by different ways in which the expression is spoken by leaving tiny gaps between various words.

I will use three dots (…) (an ellipsis) between symbols to indicate a gap. For example, '2 + 3 … × 4' may be spoken as 'two plus three [gap] times four. This can be interpreted as '2 + 3' (which is 5) times 4, giving 20. Meanwhile, '2 + … 3 × 4' gives '2 + 12', which is 14.

At this point it is helpful to speak the two expressions '2 + 3 … × 4' and '2 + … 3 × 4' out loud to yourself and, if possible, to someone else, to see how these two ways of speaking give two clearly different meanings. *Do this now before proceeding*.

This can be formulated as:

> **The articulation principle**: The meaning of a sequence of operations can be expressed by the manner in which the sequence is *articulated*.

This principle is not like a definition or an axiom in mathematics that can be used to deduce or prove a theorem. It counsels us to think very carefully about how we interpret and communicate mathematical expressions.

I only realised its immense power throughout the whole of mathematics after I had fun talking to my then 11-year-old grandson, Simon, about a humorous video that asked whether 2 + 2 × 2 is 8 or 6 (Tall, Tall & Tall, 2017). He was intrigued about saying expressions in different ways and surprised me the next day by calling me using Facetime on his iPad to ask,

> 'What is the square root of 9 times 9?'

spoken evenly without any implied articulation. I knew that he was familiar with squares of negative numbers, so I replied that the answer could be +9 or –9.' 'No,' he replied, 'it's 27.' I did not his understand his unexpected answer until he explained that he meant

> 'the square root of 9 … times 9'

which gives 27.

At a stroke, this young child had opened up the door to a whole new way of making sense of the order of operations in arithmetic and algebra. Instead of learning arbitrary rules of precedence, he offered a new way of making meaningful sense by focusing on how to *speak* mathematical expressions and to communicate the ideas in ways that other people could *hear*.

3.5 From articulation to the use of brackets and other conventions

Once a learner becomes aware of the role of articulation in giving different meanings to an expression, it becomes possible to use brackets in a meaningful way to distinguish between the meanings. For instance,

> '2 + 3 … × 4' can be written as (2 + 3) × 4,
>
> '2 + … 3 × 4' can be written as 2 + (3 × 4).

The standard curriculum uses brackets to group together sub-expressions that are to be calculated first and then deals with the operations of addition, subtraction, multiplication and division by giving precedence for brackets over multiplication and division (read from left to right) and then assigning lower precedence to addition and subtraction. This gives the convention for precedence as:

Brackets – multiplication and division – addition and subtraction.

Other operations, such as a power (or index), are represented symbolically using positional layout. For example, we write 2^3 to represent

$$2^3 = \underbrace{2 \times 2 \times 2}_{\text{3 times}}$$

To include power (or index) in the hierarchy of precedence, they are placed at a lower level than brackets and a higher level than other operations, giving the precedence:

Brackets – Index – Multiplication and Division – Addition and Subtraction.

Reversing M and D, as their order does not matter, gives the mnemonic

B – I – DM – AS.

which is used in the UK to denote order of precedence. (In the USA, the word 'parenthesis' is used for 'bracket' and 'exponent' for 'index', to give P – E – MD – AS, remembered using the phrase, 'Please Excuse My Dear Aunt Sally'.)

The general principles for addition/subtraction and multiplication/division allow sub-sequences of equal precedence to be written and read in any order.

The notion of 'procept' allows us to interpret an expression such as 2 × 3 in two ways, either as a process (operation) 'two times three' or as a concept (mental object) (the product 2 × 3). By placing an object within a box, these distinctions can be written as:

the operation $\boxed{2} \times \boxed{3}$ or the object $\boxed{2 \times 3}$.

Given a succession of operations in an expression, such as 9 – 2 × 3 + 5, reading the sub-expression with the highest precedence as an object gives the whole expression as

$\boxed{9} - \boxed{2 \times 3} + \boxed{5}$.

The general principle of addition and subtraction allow the terms to be moved around in any order, such as

$\boxed{5} + \boxed{9} - \boxed{3 \times 2}$.

This extends the conservation of number in figure 1 to a more general principle of 'flexible conservation'. How it is interpreted by each individual learner will depend on many factors, including genetic inheritance and the previous social and personal experience of each individual. There are young children with learning difficulties categorised in various ways, including dyslexia, dyscalculia and dyspraxia, with various levels of performance depending on long-term, short-term and working memory capacity. There are also gifted children who may be nurtured in an environment that encourages them to think in more sophisticated ways. Each individual will have a personal concept image of a mathematical idea that changes over time.

If the learner is to make longer-term progress making sense of mathematical symbolism, it is likely that they can make more sense if they realise how the symbolism can be interpreted in different ways depending on how it is spoken and heard, and how they are encouraged to use symbolism flexibly to simplify the manipulation of more sophisticated expressions.

4. Longer-term evolution of ideas

At this point it becomes necessary to shift attention from subtle detail to a higher-level overview of the development of the full range of learners and teachers.

Different approaches to the learning and teaching of mathematics involve many transgressions that act as barriers for some and enlightenment for others. For instance, the shift from arithmetic to algebra may pose a transgression for many because an expression such as 2 + 4 in arithmetic can be calculated to give an answer (6) while an algebraic expression such as $2 + 4x$ cannot be calculated unless x is known. This is known as ‘the lack of closure obstacle’ (Collis, 1974). It may cause a child to perform the addition 2+4 to get 6, then leave x alone (because it is not known) to give the erroneous answer $6x$. On the other hand, some children may sense the statement $2 + 4 = 6$ only as an operation in which $2 + 4$ *makes* 6, whereas $6 = 2 + 4$ has no meaning because 6 does not ‘make’ $2 + 4$. An equation such as $2x + 1 = 7$ may be seen as an operation that can be solved by ‘undoing’ the operation but an equation with expressions on both sides cannot. This phenomenon is termed ‘the didactic cut’ (Filloy & Rojano, 1989) that causes a transgression in the transition from arithmetic to algebra.

Over the longer term, different individuals will be faced with changes in context involving problematic boundaries that some are unable to transgress, while others may see the change in context to shift to a new level of enlightenment.

In *How Humans Learn to Think Mathematically*, I highlighted a sequence of three broad levels of development which I term ‘*Practical*’, ‘*Theoretical*’ and ‘*Formal*’ that involve transgressions to more sophisticated contexts. Most individuals concentrate mainly on practical mathematics involving *input* through the senses, *internal processing* making internal mental links and *output* through gesture, speech, writing and drawing, to *communicate* to others.

> ***Practical mathematics*** refers to human perception and action where ideas fit together in *coherent* ways. For instance, adding two and three gives five, taking three from five gives two. In geometry a triangle with two equal sides has two equal angles. Properties occur coherently *at the same time*.

Some individuals develop a more enlightened theoretical form of mathematics introduced in school, based on definitions and deductions, later enriched in applications of mathematics:

> ***Theoretical mathematics*** relates ideas together so that *if* one property holds, *then* another follows. It involves *consequence*, based on definitions and deductions rooted in natural human perception and action, but increasingly enhanced in human imagination, including non-Euclidean geometries in which the parallel postulate does not hold, or complex numbers where a square number can be negative when visualised as points in the plane.

A third form of mathematics occurs in pure mathematical set-theory and logic, based on formal definition and proof:

> ***Formal mathematics*** is based on specified axioms and definitions from which formal properties may be deduced that must hold in any context that satisfies the particular axioms and definitions.

These three broad levels of mathematical thinking are sequential and have boundaries involving transgressions that need to be traversed. For example, in practical mathematics, an isosceles triangle has two equal sides and two equal angles, and it is not initially self-evident that either property implies the other. Initially the power 2^n means $2 \times 2 \times \ldots \times 2$ with n lots of 2 multiplied together, which has no meaning if n is not a whole number. A transition to theoretical mathematics allows the properties for fractional and negative powers to be deduced from the definition of the power law, $x^m x^n = x^{m+n}$.

The transition from theoretical to formal mathematics involves an even greater possible transgression in an individual’s relationship between mental imagery (concept image) and properties deduced from the formal definition (concept definition) (Tall & Vinner, 1981). Some extend the theoretical level based on the concept image, by *giving* meaning to the concept definition, others *extract* the meaning formally from the concept definition, often involving formal manipulation of complicated multi-quantified set-theoretic definitions (Pinto & Tall, 1999). There are at least two different ways of transitioning to formal proof. ‘Giving meaning’ is a *natural* transition to build from one’s theoretical mental image to formal proof, while ‘extracting meaning’ involves a *formal* transgression to construct meaning based only on quantified axioms and set-theoretic definitions using formal proof.

Even though these three levels develop sequentially, the next level may not be reached at the same time in different situations. At any given time, two or more levels may (and often do) continue to coexist depending on the individual.

Most individuals reach, at best, practical or theoretical ways of thinking mathematically. Those who reach the formal level in a natural or formal way encounter a continuing spiral of sophistication in which some formal theorems, termed *structure theorems*, prove properties that offer more sophisticated forms of visual and symbolic mental representations (Tall, 2013).

If we consider this long-term spiral of sophistication in long-term mathematical thinking, then we will find fundamental differences in the ways that mathematicians, educators, curriculum designers, philosophers, psychologists, neurophysiologists, politicians and others emphasise different aspects. The whole enterprise is too vast for any one community to grasp the total picture. This is why we should strive to raise our sights to a multi-contextual overview rather than limit ourselves to arguing between the views of differing communities of practice.

5 Human sense making

To be able to raise our views to a higher level, as individuals we should be aware of the affective role played by the central limbic region of the brain that may enhance connections based on supportive input and suppress connections that arise from problematic inner conflict. This involves seeking an understanding of how the brain makes sense of increasingly sophisticated ideas in long-term mathematical development.

Earlier we considered the flexible conservation of number in *how we see arrangements of objects in different ways*. Now we consider *how humans read text* and how this affects how we *interpret mathematical expressions*. Then we will consider *how we interpret dynamic movement of objects* and its effect on our ideas in the calculus, particularly in terms of the practical, theoretical and formal interpretations of calculus and analysis.

5.1 How humans read text

When we read text on a page, such as the text on the left of figure 3, then we do not scan the lines smoothly, because the retina in our eye has a small area called the macula which focuses in high detail and the eye jumps along the line of text in successive steps (called ‘saccades’). The view on the right is edited to mimic those parts in high definition and those not in focus.

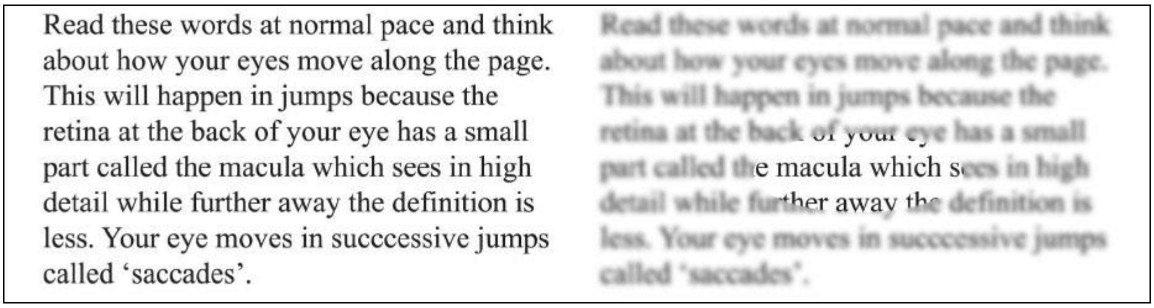

Figure 3: reading text

Read the clear text on the left of figure 3 and sense how your eye jumps along the text taking in successive parts of the text.

Do this now ...

The brain makes sense of successive parts and puts them together to build meaning of the text.

5.2 Interpreting more sophisticated mathematical expressions

If a mathematical expression is written linearly, such as a quadratic equation

$$3x^2 + x - 2 = 0$$

then, treated in the same way as text, it would be read left to right in a sequence of saccades. Longer expressions increase the number of saccades making it more difficult to grasp the structure of the expression.

New techniques are introduced, such as representing the operations spatially. The first example of this is the writing of powers as superscripts. Possibilities proliferate with symbolism for limits, summation, integrals, matrix layouts and so on, which can be written by hand or built up using software templates such as MathType (Figure 4) or specified symbolically using languages such as TeX.

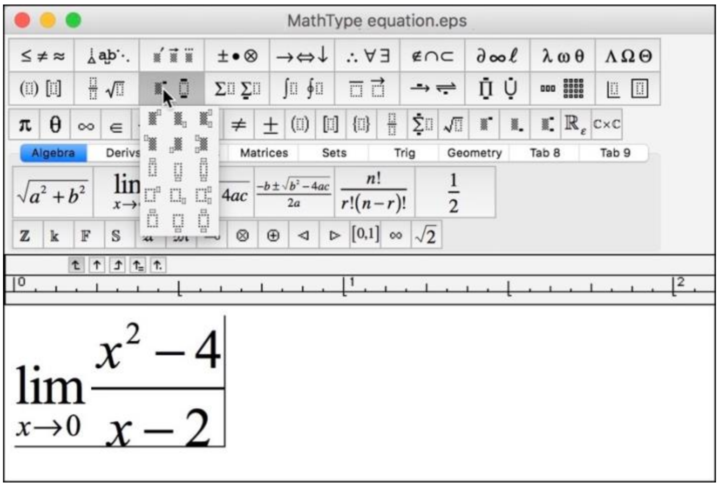

Figure 4: spatial layout of expressions

Reading these expressions involves scanning the spatial layout and attempting to make sense of them. It is possible to extend the procept analysis to distinguish between objects and operations, but now we are not dealing just with binary operations, but with *n*-ary operations in general, such as the unary limit operation above as x tends to zero. Figure 5 shows various ways in which parts of the expression may be seen as a process operating on an object or as an object output by that process.

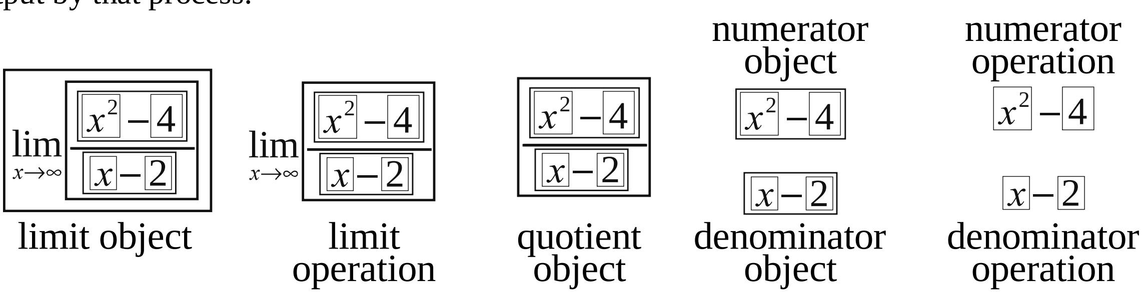

Figure 5: sub-expressions as operation or object

Please understand that I am not advocating this intellectual analysis as a general method to teach all students. My purpose is to show that a rote-learnt mnemonic such as BIDMAS or PEMDAS is not an adequate foundation for long-term learning without giving flexible meaning to the symbols. This involves having a sense of *Duality* (seeing an expression dually as process and concept), *Ambiguity* (realising that the same expression can represent either process or concept to different individuals and even to the same individual at different times) and *Flexibility* (making sense of the symbolism in flexible ways).

As each learner builds up a personal concept image of such expressions over the years, there will be a wide range of interpretations between different individuals. Some may not be able to make sense of the equation at all, some may see it only as an operation that cannot be calculated because *x* is not known, some may see it as a regular algebraic equation to be solved and may know how to solve it by one of several methods. Over the years many research studies have shown a range of 'misconceptions' in manipulating symbols. Now this data should be reconsidered in terms of transgressions involving supporting and problematic aspects in different contexts.

5.3 How we interpret dynamic movements of objects

When we follow the dynamic movement of a moving object, the eye operates in a completely different way from dynamically reading lines of text. You can sense this with a simple experiment. Hold a finger in front of your eye at a comfortable distance and move your finger sideways, keeping your gaze on the moving finger. You will find that there is an initial saccade as your gaze jumps to focus on your finger, but then, as your finger moves smoothly, your gaze also moves smoothly keeping your finger in focus against a moving background. Now imagine a point moving along a number line. The moving point is conceived as being a variable quantity by the natural operation of the human eye (Figure 6).

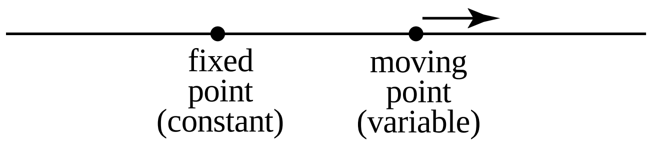

Figure 6: constant and variable points on a line

This reveals the simple fact that the human eye operates in such a way that it is natural for the human brain to imagine the difference between a fixed point that is constant and a moving point that is variable. This has profound implications for the historical and individual development of thinking about variable quantities, including quantities that can become arbitrarily small giving rise to ideas of indivisible and infinitesimal quantities, either as potential never-ending processes or as actual mental objects.

5.4 The limbic system and enhancement and inhibition of mathematical thinking

So far, this presentation has focused on

- *cognitive* aspects of how individuals think about mathematical structures.

To consider the role of transitions and transgressions more fully, it is essential to consider:

- *affective* aspects that enhance and suppress the making of mental connections.

These have profound effects on different individuals in a broad spectrum from a positive willingness and resilience to attack mathematical problems, to anxiety and inability to even think about them.

The limbic system[2] in the centre of the brain is a collection of structures that support a variety of functions, including emotion, behaviour, motivation and links to long-term memory. In particular, it responds to challenge or danger with an immediate 'fight or flight' reaction that suffuses the whole brain with neurotransmitters that excite or inhibit mental connections. Confident students who rise to the challenge are placed on alert ready to tackle the situation. Those who find the mathematics difficult or even impossible are likely to have their mental connections suppressed, causing them to freeze mentally and be unable to respond.

[2] Wikipedia https://en.wikipedia.org/wiki/Limbic_system

6. A strategy to enhance long-term learning by being explicitly aware of transgressions

This conference will feature many different transgressions that exist in various aspects of mathematical thinking, arising in different contexts, in different communities of practice and affecting individuals in different ways at different times in their development. No single approach is likely to solve all difficulties. In this presentation I advocate the need for long-term development based on supportive ideas that make sense at the time yet continue to form a stable basis over several changes in context. For example, in early development of arithmetic and algebra, I suggest the *general principle of addition and subtraction* and the later *general principle of multiplication and division* that build from the general notion of conservation of number. This is then explicitly contrasted with problematic aspects that arise with combinations of operations whose different meaning can be clarified by *the articulation principle*.

Over the longer term, the three broad levels of *practical*, *theoretical* and *formal* mathematics that have been identified involve transgressions to move successively from one level to the next. These transgressions involve changes in meaning that enhance or suppress mental connections between neurons through the activity of the limbic system.

The biological human brain develops from a single cell by successive subdivision to produce symmetric left and right parts where the left side in most people includes Broca's area for hearing speech close to the left ear and Wernicke's area for outputting speech (Figure 7).

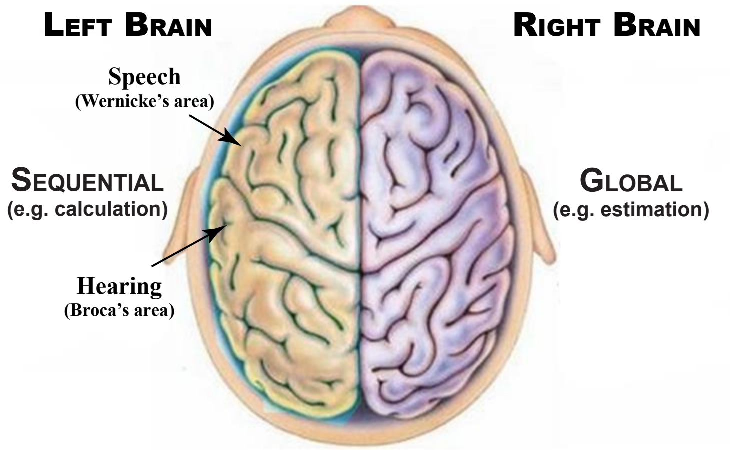

Figure 7: Two sides of the brain

The left brain carries out sequential operations such as counting while the right brain deals with global operations, such as visually estimating size. However, while the literature contains many facts based on research findings, many beliefs do not stand up to scrutiny (Corballis, 2014). For instance, it is true that there is a cross-over in which the left side of the human body is sensed and controlled by the right side of the brain and vice versa. Because most individuals are right-handed, this was taken to imply that the left side of the brain is dominant over the right.

In the 1950s and 60s, Sperry performed experiments on animals and humans in which he split the connections between left and right brains. His research suggested that in most individuals the left brain was responsible for language and the right for emotional and non-verbal functions (Gazzaniga et al. 1965; Sperry, 1982). This led not only to Sperry being awarded the Nobel Prize in 1982 but also to widespread speculation about the radical differences between scientific thought on the left and creativity on the right.

In practice, the two sides of the brain co-operate in subtle ways. FMRI scans can indicate where activities occur over a period of seconds. A different view of how mathematical thinking develops over the longer term can be sought by considering the developmental theories of Piaget, Bruner, Van Hiele, Dienes, Lakoff & Nunez and other pioneers, to seek common themes in long-term development of mathematical sophistication.

Van Hiele theorised that school geometry proceeded through a sequence of levels that have been described and analysed in various ways in the literature. Research has shown that, though these levels are statistically sequential, individuals vary on the level of response on different items (Gutiérrez, et al. 1991). This evidence is consistent with a looser description in terms of broad levels of practical and theoretical geometry in school as a study of the properties of objects. It begins with practical aspects of visual and physical exploration of shape and space (involving a level of *recognition* and one of *description*) and grows in sophistication to theoretical *definition* and *deduction*). I termed this development *conceptual embodiment*.

School arithmetic and algebra is much more complicated as learners encounter successive number contexts involving problematic transitions which give rise to transgressions. It also involves successive compression of knowledge from long counting procedures to flexible use of symbolism as process and concept. Initially I referred to the long-term development as 'proceptual symbolism' to distinguish it from other uses of symbolism (Tall, 2004) but then acknowledged the wide differences between procedural and flexible learning by renaming it as *operational symbolism* to include both possibilities (Tall, 2013). This builds from practical mathematics and grows in sophistication to theoretical mathematics.

I envisaged these two forms of development as one based on objects and their properties, the other based on actions on objects, and thought of them as qualitatively different 'worlds' of mathematics.

The term 'world' arose the first time that I heard my doctoral student Anna Poynter explain the way in which vectors can be manipulated as embodied arrows and also as algebraic vectors. Her students had moved a triangle on a table in a given direction by a specific distance to represent a 'free vector' as an operation and one of them said that 'the sum of two free vectors is the single vector that has the same *effect*.' This revealed the parallel between the embodied change in focus from operation to effect corresponded to the symbolic compression from process to concept. I already knew that these two approaches were different again from the formal theory of vector spaces that arises in university mathematics. In a split second I had the inspiration for 'three different worlds of mathematics'. At the time I had not read van Hiele's 1986 book on *Structure and Insight* in which he refers to Popper's philosophical idea of 'three worlds' which has a related, but different, meaning.[3]

The third world of mathematical knowledge arises in the *axiomatic formal* approach to pure mathematics at university level, based on set-theoretic definition and mathematical proof. As explained earlier, this may be interpreted in at least two different ways, one as a natural extension of theoretical mathematics by *giving meaning* to the definition from the concept image, one by *extracting meaning* from the formal concept definition to give full axiomatic proof based solely on the formal definition.

This classifies three fundamentally different worlds of mathematical thinking, one based on *objects* and their properties, one on *symbolizing operations* on objects, the third based on specified *properties* from which all other properties are deduced by mathematical proof.

Long-term growth blends together various aspects of these three worlds, and by shortening their names to *embodied*, *symbolic* and *formal*, the full framework may be represented in a two-dimensional picture (figure 8, adapted from Tall, 2013).

[I have added the term 'axiomatic' to the term 'formal mathematics' to distinguish this from the formal world which includes both theoretical and axiomatic formal definitions and proof.]

[3] See https://en.wikipedia.org/wiki/Popper%27s_three_worlds

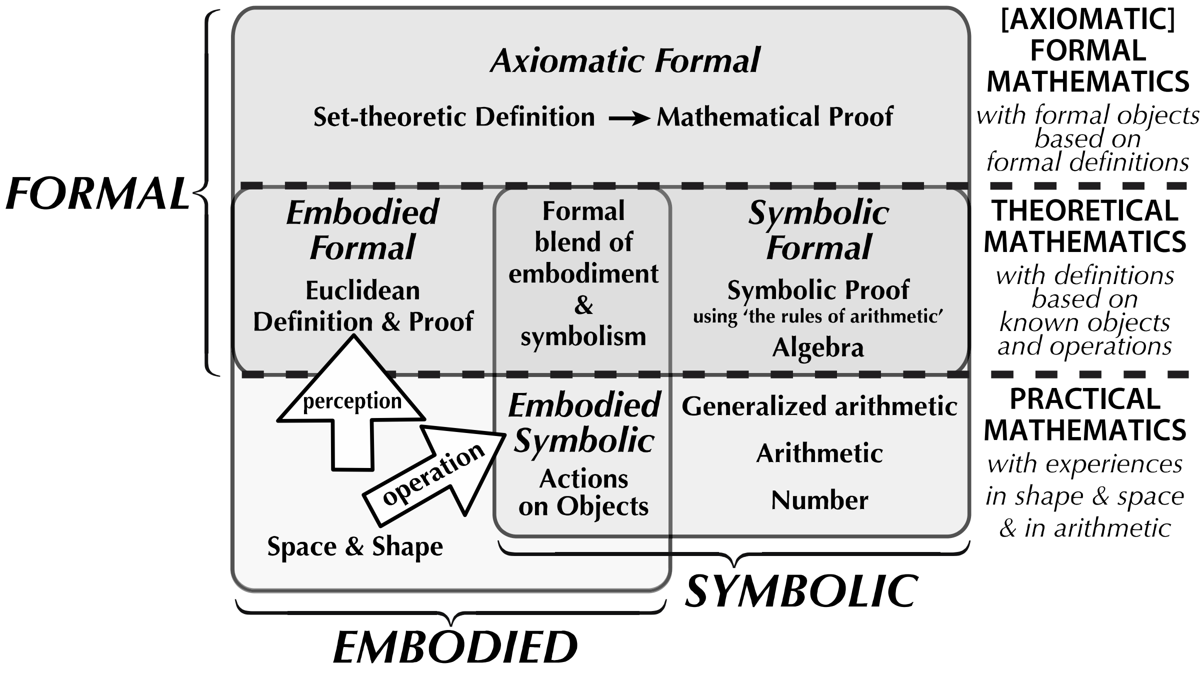

Figure 8: The three worlds of mathematics

This diagram will inevitably be interpreted by different individuals in different ways. A two-dimensional picture cannot represent the whole theory. Not only does it omit the role of the affective aspect of mathematical thinking, it depends on how each individual interprets the diagram in terms of their own personal ways of thinking. My own view is built as the result of a personal journey in which I studied for a doctorate in pure mathematics with Michael Atiyah, who was awarded a Fields medal at the time, and later for a doctorate in the psychology of mathematics education with Richard Skemp, who was a world leader in mathematics education. This allows me to endow the diagram with a rich array of meanings. However, a reader, with completely different experiences and development, may read the diagram in many other ways.

I have been fortunate over a long lifetime to cooperate internationally with other researchers and to supervise PhDs students from many different countries with very different cultures at different levels of education. This gives me the possibility of seeking a multi-contextual overview, although whatever I suggest will be subject to my own personal biases.

7 International comparisons

Current international comparisons between different countries include TIMSS and PISA studies which, in their different ways, seek to collect data on the progress of students using techniques that are intended to be fair to different cultural backgrounds, although their results are often used by politicians to justify educational policies. In PISA 2012, the first seven entries are all East Asian with Shanghai-China a clear leader and Japan seventh (Figure 9).

The Netherlands (10^{th}), Germany (16^{th}) are slightly above average, France (25^{th}) and the UK are average, the USA (36^{th}) scores lower, with a long tail, including Brazil (58^{th} out of 65). In the UK, politicians have taken successive actions to 'raise standards' by studying 'successful' participants such as Shanghai, Singapore and Finland, but this has not led to significant improvements. Indeed, as I write this (August 2018), the front page of the Times newspaper has an article stating that the government intends to legislate against 'exam factory' schools who train their students to learn to pass the exam rather than give them a broad education.

	score	% low	% high
OECD average	494	23.1	12.6
1 Shanghai-China	613	3.8	55.4
2 Singapore	573	8.3	40.0
...			
7 Japan	536	11.1	23.7
8 Liechtenstein	535	14.1	24.8
9 Switzerland	531	12.4	21.4
10 Netherlands	523	14.8	19.3
12 Finland	519	12.3	15.3
16 Germany	514	17.7	17.5
25 France	495	22.4	12.9
26 United Kingdom	494	21.8	11.8
36 United States	481	25.8	8.8
41 Israel	466	33.5	9.4
48 United Arab Emirates	434	46.3	3.5
51 Chile	423	51.5	1.6
58 Brazil	391	67.1	0.8
63 Qatar	376	69.6	2.0
65 Peru	368	74.6	0.6

Figure 9: Selected PISA scores 2012

It has been my privilege over many years not just to read the literature but to co-operate with researchers in different countries to gain invaluable insights into their cultural approaches. This includes supervising doctoral theses with many students from the UK, three university faculty from Brazil, three USA college mathematics professors, five university faculty from Malaysia, and other doctoral students from Taiwan, South Korea, Turkey, Greece and Columbia.

I have also been fortunate to be a visiting consultant in many other countries including the Netherlands, Finland, Germany, New Zealand, Australia, Israel and a long-term project on Lesson Study based in Japan, including 20 communities around the Pacific Rim. This offers the possibility of developing a personal multi-cultural overview of the approaches of different communities in terms of transgression and enlightenment.

At the top end of the PISA scale, Shanghai uses specialist teachers to teach mathematics in the first lesson of each school day and the teachers mark the work immediately to give extra lessons after school to those in difficulty. The Shanghai sample excludes a significant percentage of migrant workers of lower social status which can bias the results. It does, however, seek to focus on relationships in arithmetic that enhance the flexible conservation of number, with an underlying concern that it may prioritise efficiency over problem solving.

Singapore was an active participant in the Lesson Study project in which I was a consultant. Their problem-solving approach follows a similar plan to Lesson Study in which the children are faced with a sequence of lessons in which they are given problems to solve. Early lessons introduce them to ideas that provide alternative ways for the children to solve more sophisticated problems that arise later in the sequence, so the children participate in a well-focused communal lesson to build their own way within an organised curriculum.

My role as a consultant in the Lesson Study project was to introduce the participants to the theoretical development of embodiment and symbolism, with particular attention to the supportive and problematic changes in symbolism caused by changing number contexts and the flexibility of symbolism as process and concept.[4]

[4] Presentations given at various conferences can be found on my downloads page: http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html

As part of this activity I edited (but did not write) the English version of the first three volumes of the Japanese Junior High School mathematics (Isoda & Tall, 2018). These splendid books are designed for large class teaching and build up mathematical ideas by suggesting how different students might interpret them and inviting the class to express their own opinions for individuals to make sense for themselves. The objective is for both teacher and learner to be:

- conversant with the desired sequence of sophistication in mathematics,
- aware of the current thinking of the learners,
- aware of what is necessary to help them make the desired transition.

It was interesting for me to edit these English texts by working carefully through them. They are written by mathematicians, eager to encourage students to think for themselves, informed by research in mathematics education. Sometimes the two views have different priorities. For example, education research shows that students distinguish between 'three twos' and 'two threes', so this is made a feature of the development with the distinction being maintained for some time. This clearly contradicts the principle of seeking proceptual flexibility advocated in this presentation in which the two operations give the same result. Perhaps this will be modified in the next edition. The writing of a curriculum is an ongoing process and Lesson Study continues to refine its approach to help students make sense of new ideas in meaningful ways.

The Netherlands (in 10th place) scores highly in PISA and is the home of 'realistic mathematics' designed to improve mathematical thinking by encouraging learners to be active participants, solving meaningful problems in imaginative ways (Van den Heuvel-Panhuizen & Drijvers, 2014). This approach spread internationally with widely acclaimed success. Yet in its home country, opinions were divided as some students going to university were less well prepared. Members of the team designing the realistic approach responded:

> … a reflection on the findings of three PhD studies, in the domains of, respectively, subtraction under 100, fractions, and algebra, which independently of each other showed that Dutch students' proficiency fell short of what might be expected of reform in mathematics education aiming at conceptual understanding. In all three cases, the disappointing results appeared to be caused by a deviation from the original intentions of the reform, resulting from the textbooks' focus on individual tasks. It is suggested that this "task propensity", together with a lack of attention for more advanced conceptual mathematical goals, constitutes a general barrier for mathematics education reform.
>
> (Gravemeijer et al., 2016, p. 25.)

This 'general barrier' for mathematics education reform suggests a deep transgression in our mathematical culture: the focus on 'advanced conceptual goals' may be meaningful to an expert, but it may not make sense to students who are attempting to make sense of ideas based on their current development and prefer to focus on the immediate goal of 'passing the test.'

The focus on 'teaching to the test' features not only in one of the most successful Western countries, it also occurs in Brazil which has one of the lowest PISA scores. A Brazilian study of teenagers solving linear and quadratic equations (Tall, Lima & Healy, 2014) did not exhibit the symbolic 'didactic cut' transgression mentioned earlier, nor did students use an embodied solution treating an equation as a balance, instead they were taught to solve equations using rules like 'change sides change signs'. When they moved on to solve quadratics, they were taught to use the standard formula, because it works in all cases, but most students lacked the algebraic fluency to rearrange equations into the form $ax^2 + bx + c = 0$.

At both ends of the PISA scale outside East Asia, the desire to 'teach to the test' may give some immediate success, but in the longer term, it may lead to rote-learning a range of disconnected methods that act as a barrier to the development of more sophisticated learning.

7.1 Can different cultural approaches transgress boundaries?

Given the apparent higher success of East Asian countries, is there any way that their more successful approaches can transgress international boundaries? A project in the Netherlands sought to introduce Lesson Study methods into the more advanced study of calculus. The teachers had initial difficulties in implementing the approach because they were impeded by the Dutch culture of 'following the textbook closely, the strict school guidelines and the pressure for high exam results' (Verhoef & Tall, 2011). It was only in the second year of the study that they began to grasp the students' personal ways of thinking and made sense of using dynamic software to embody practical concepts of differentiation (Verhoef *et al*., 2014).

This exemplifies Stigler and Hiebert's (1999) large-scale research on mathematics teaching approaches between the USA, Germany and Japan that concluded that differences in teachers' competences were dwarfed by the differences in teaching methods that varied greatly across cultures and varied little within cultures.

Attempting to make sense of the long-term development of mathematical thinking in different cultures, with different overall objectives and different variations in individuals is a seemingly impossible task. However, this presentation has suggested unifying features relating human thought and mathematical structure that offer a possible multi-contextual overview.

To encourage confidence in long-term sense-making, I propose that we seek supportive principles that operate through a succession of contexts and contrast them with problematic changes in meaning that can lead to enlightenment rather than transgression. This can be formulated in a three-world context of embodiment, symbolism and formalism that develops through practical, theoretical and formal modes of thinking based on the natural operation of the human brain. It also takes account of the central limbic system that reacts subconsciously and emotionally to inhibit or enhance thinking. It is this latter function that is active in both mathematical transgressions and mathematical enlightenment.

8. A major example: the role of transgressions in the evolution of calculus and analysis

I close this presentation with a brief outline of the evolution of ideas in calculus and analysis, both in terms of the historical development and also of the personal development that is now possible in our new digital age. When I was a teenager, I learned, and loved, calculus from a book with beautiful static pictures (Durell & Robson, 1934) but then found the transition to university analysis more challenging. Now, at the age of 77, I have passed through the most astonishing lifetime transformation of knowledge through the development of digital computing that enables me to control dynamic imagery with my finger on a tablet, supported by the growing power of symbolic manipulation and new forms of artificial intelligence.

8.1 Practical and theoretical embodiment of differentiation

In the mid 1980s, I designed 'Graphic Calculus' software to visualise the changing slope of a graph $y = f(x)$ to represent the slope function $(f(x+h) - f(x))/h$ for variable x and fixed h. This transgressed the norms in which x is first taken to be fixed, h is allowed to tend to zero, and the slope tended to the limit $f'(x)$. I theorised that the notion of limit was a *logical foundation* of formal analysis but not what I termed a *cognitive root*, which I defined as "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" (Tall, 1989).[5] I rejected the notion of limit as a foundation of school calculus and proposed the cognitive root of "local straightness", which simply involves magnifying a portion of a graph on a visual display to see if the graph "looks straight" at high magnification, and so one could *see* the changing slope (figure 10) and plot it as a new graph (figure 11).

[5] See http://homepages.warwick.ac.uk/staff/David.Tall/themes/cognitive-roots.html

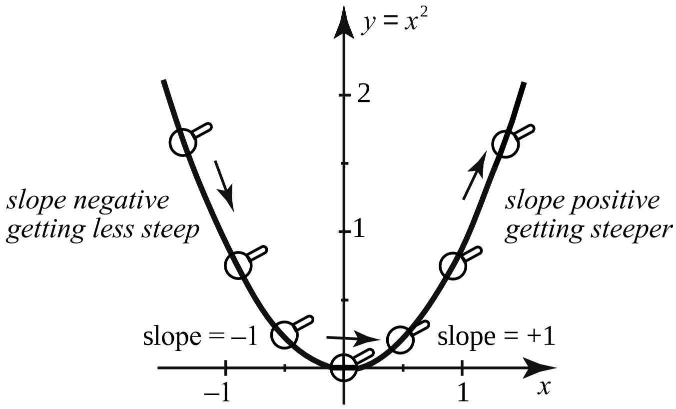

Figure 10: Magnifying the graph of $y = x^2$ to see its changing slope

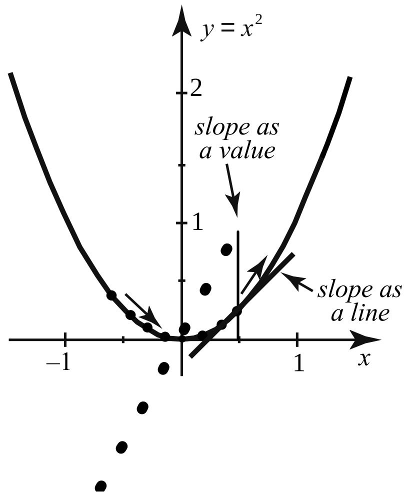

Figure 11: Plotting the value of the slope as points

The corresponding symbolism for calculating the slope of $f(x) = x^2$ involves calculating the slope of the *practical* tangent as

$$(f(x+h) - f(x)) \,/\, h \;=\; ((x + h)^2 - x^2) \,/\, h \;=\; (x^2 + 2xh + h^2 - x^2) \,/\, h \;=\; 2x + h.$$

As h gets small (but not zero), the graph of the practical tangent stabilises so that it looks visually like the *theoretical* tangent $f'(x) = 2x$.

The interesting difference here is that the symbolic version involves a *process* in which h gets small, but is never actually *equal* to zero, but the visual version of the practical slope graph looks to the human eye as if it stabilises on the theoretical derivative.

This is such a powerful idea that one can *look* at the changing slope along a graph and *see* the shape of the slope function. In figure 12, the slope function of sin$x$ (for x in radians) *looks* like cos$x$ and the slope function of cos$x$ looks like the graph of sin$x$ upside down, which gives meaning to the idea that the derivative of cos$x$ is *minus* sin$x$.

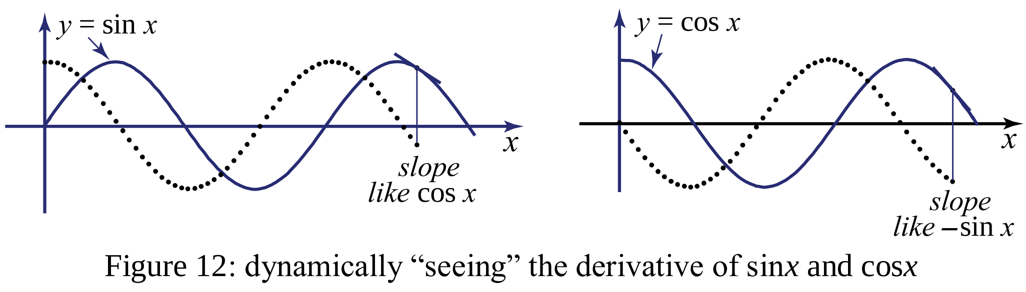

Figure 12: dynamically "seeing" the derivative of sin$x$ and cos$x$

When students used this approach, they found it pleasingly meaningful, but some teachers and 'experts' were uneasy because it transgressed their experience of the formal definition of limit.

Over the years, I extended the notion of local straightness to cover visual representations of differentials, partial derivatives, tangent planes, parametric curves, composite functions, implicit functions, area and integral, numerical methods of solving equations, and so on, summarised in Tall (2013) chapter 11.[6]

This approach still required the student to be able to cope with the algebraic manipulation required to calculate derivatives. Many teachers who had to follow a specified curriculum taught their students (often by rote) how to operate with the rules of calculus to pass tests.

8.2 Addressing initial algebraic difficulties

When I designed the graphic approach to differentiation, the School Mathematics Project Team in the 1980s decided that the algebra for differentiating a power x^n was too difficult. Multiplying out two brackets $(a + b)(c + d)$ already involves coordinating four products, multiplying a by c, a by d, then b by c, b by d and adding them together:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Students often made errors in multiplying three brackets or more, even in the case of calculating $(x + h)^3$. So, in the first year of the SMP curriculum, only the derivative of x^2 was calculated from first principles, with the principle for x^n stated without proof to enable the students to differentiate polynomials symbolically. Now that we have the articulation principle, it offers the potential to deal meaningfully with the ambiguity of notation by using brackets to clarify the manipulation of algebraic symbols expressed verbally.

8.3 Integration and the Fundamental Theorem

In my software, Graphic Calculus, I pictured the fundamental theorem of calculus by stretching the graph of a continuous function horizontally while maintaining the vertical scale. A thin strip height y, width dx is 'pulled flat' to see a visual area $dA = ydx$ that looks like a rectangle. Figure 13 shows a possible picture of this idea where the graph of $y = f(x)$ in the left box with a thin strip width dx is stretched horizontally to fill the right box. The *numerical* value of the area calculated by multiplying height y by width dx has a practical value $dA = ydx$ and its error is contained in the horizontal line of pixels, which can be made as small as required. This allows the learner to visualise a practical interpretation that can be extended theoretically to give the formal fundamental theorem of calculus: $y = dA/dx$.

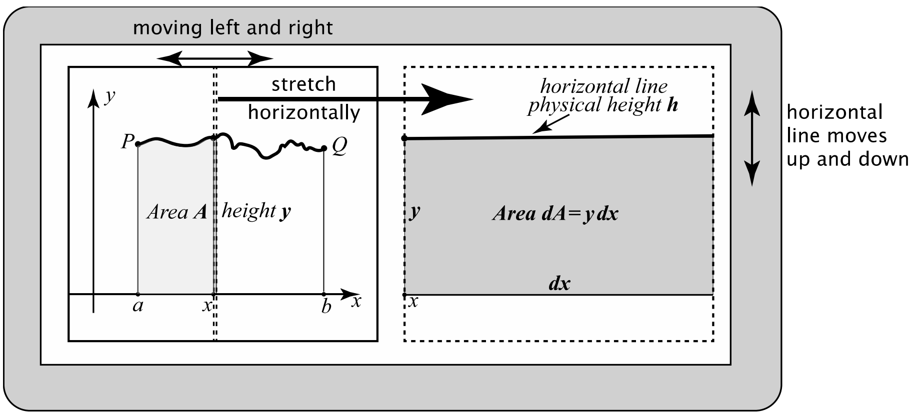

Figure 13: Visual representation of the fundamental theorem

As I write this presentation, my colleague Martin Flashman is programming this dynamically in *Geogebra* (Figure 14).

[6] For overall development, see also http://homepages.warwick.ac.uk/staff/David.Tall/themes/calculus.html.

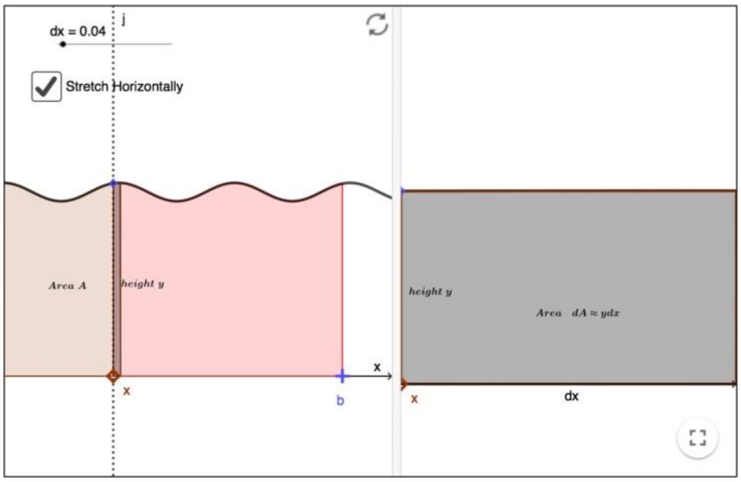

Figure 14: ‘Tall-scope’ visualising horizontally stretched continuous graph

8.4 The transgression from calculus to analysis

The transition from theoretical calculus to formal analysis replaces a visual and symbolic approach by a formal approach using multi-quantified definitions of limits which are far more difficult to manipulate. This gives a major transgression from calculus to analysis.

8.5 The transgression from standard to non-standard analysis

The theory of non-standard analysis was introduced by Robinson in 1966. Even though it involves simpler definitions with fewer quantifiers than standard epsilon-delta analysis, it has a steep initial cost in terms of the definition and construction of an extension field $^*\mathbb{R}$ of the real numbers that also includes infinitesimals.

The definition involves the distinction between *first order logic* where the statements only quantify *elements* of the set concerned, such as

$\forall a,b \in \mathbb{R}: a + b = b + a$

$\exists 0 \in \mathbb{R}: \forall a \in \mathbb{R}: a + 0 = a$

as opposed to statements such as the completeness axiom which quantify subsets:

$\forall S \subseteq \mathbb{R}$: If S is non-empty and bounded above, then it has a least upper bound.

An ordered extension field $^*\mathbb{R}$ of the real numbers $\mathbb{R}$ is said to be a *hyperreal number system* if it satisfies:

> **The transfer principle**: Every statement about the real numbers $\mathbb{R}$ expressed in first order logic is true in the extension field $^*\mathbb{R}$.

The construction of a hyperreal number system involves more logical machinery and the shift from standard analysis to non-standard analysis operates as a transgression for many pure mathematicians who prefer to remain with their experience of standard analysis.

8.6 A new formal approach with a structure theorem to embodiment and symbolism

Recently I introduced a new formal method to build on standard analysis using the real numbers $\mathbb{R}$ to work in *any* ordered extension field K of $\mathbb{R}$. Define an element $x \in K$ to be *finite* if $a < x < b$ for elements $a, b \in \mathbb{R}$ and to be infinitesimal if $-a < x < a$ for all positive elements $a \in \mathbb{R}$. It is a simple application of the completeness of $\mathbb{R}$ to prove the structure theorem that any finite element $x \in K$ is uniquely of the form $x = c + \varepsilon$ where c is real and ε is infinitesimal. The details are given in Tall (2013) and in Stewart & Tall (2014). For finite $x \in K$, the *standard part* of x (written as st(x)) can be defined as the unique real number st(x) = c where $x = c + \varepsilon$.

For d, $k \in K$ where $k \neq 0$, we can now define a *k-lens m pointed at d* to be

$m(x) = (x - d)/k.$

If $m(x)$ is finite, we can define the *optical k-lens o pointed at d* to be

$o(x) = st(m(x)) = st((x - d)/k) \in \mathbb{R}.$

By various choices of d and k, which can be finite, infinite or infinitesimal, this allows us to see detail in K as a *real picture* on the real line $\mathbb{R}$. The same idea extends to n-dimensional space, which may be real or complex (Stewart & Tall, 2018).

The structure theorem allows us to visualise infinite or infinitesimal structure and manipulate the symbolism algebraically, raising us from the formal world to higher levels of embodiment and symbolism. (See Stewart & Tall, 2018, chapter 15.) In particular, the theory may be applied not only to the hyperreals in non-standard analysis, but also to simpler fields generated by power series in an infinitesimal related to the natural imagery of analytic functions of Leibniz and also of the complex analysis of Cauchy (Tall & Katz, 2014).

9. Where do we go from here?

What does this all mean? First it vindicates the increasing sophistication of mathematical structure in the three worlds of embodiment, symbolism and formalism and the role of structure theorems to move from formalism to higher levels of embodiment and symbolism. Then it illustrates the growth of mathematics in school from practical mathematics to theoretical mathematics, and for the small minority going on into pure mathematics to build spirally upwards to more sophisticated levels of formalism, embodiment and symbolism. The historical development also reveals the human mind thinking naturally in terms of variable quantities that can be 'arbitrarily small' or 'infinitesimal', so that different stages in development arise from natural processes of the human brain.

At every level in historical and personal development, we need to be aware of the supportive and problematic aspects that arise as we move into new contexts and, from a human point of view, we need to understand the emotional operation of the limbic system and its capacity to inhibit or enhance mathematical thinking. Moving to previously uncharted territory requires creativity and reflection by teachers, learners, and others involved in mathematical thinking, to be aware of the subtle boundaries that limit thinking in a given context, and to seek enlightenment by building on long-term supportive aspects and confronting problematic concerns. Transgressions to new contexts require an understanding of how the human brain makes sense of mathematical ideas, socially, cognitively and emotionally.

Reflections on the conference added afterwards

The paper as presented here was written in advance of the conference to address the topic of 'transgressions' in mathematics. Owing to ill-health I had not travelled abroad for some time and had mainly stayed at home on my own, reflecting on the framework I had begun in my book on *How Humans Learn to Think Mathematically*. The conference provided me with an opportunity to present new ideas that extend the framework in several new directions (including subsequent papers: Tall, 2019, 2020).

For the actual presentation, it was necessary to produce overheads that were appropriate for a one-hour plenary (with time for questions) focusing on aspects that were essential for the conference. In particular, the plenary extended the notion of transgression by introducing the complementary idea of insight, together with an overview that related contrasting views of different communities of practice (figure 15) and of individual development within a specific community of practice (figure 16). These are potentially relevant to most of the papers presented in the conference.

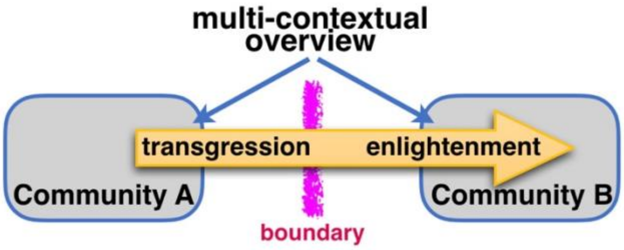

Figure 14: Transition between communities across a boundary

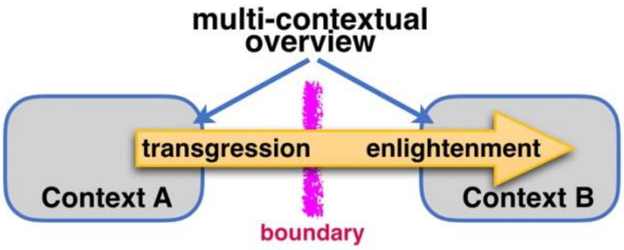

Figure 15: Transition by an individual across a boundary

In the case of differences between one community and another, the possibilities are:

Transgression: crossing *out of* a previous community over a boundary

Enlightenment: crossing *into* a new community over a boundary

Overview: encouraging communication *between* communities.

Examples include differences between communities of pure and applied mathematicians, between mathematicians and mathematics educators, between different levels of teaching in early learning, primary, secondary, university, different forms of expertise in mathematics.

In the case of an individual seeking to make a change in context within a single community, there are three possibilities:

Transgression: *inability* to change context

Enlightenment: *ability* to change context

Overview: ability to switch *between* contexts.

Examples include generalising number systems from counting numbers to fractions, to signed numbers, to rationals, reals, complex numbers, from arithmetic to algebra, from practical drawing to Euclidean proof, through various van Hiele levels, from school mathematics to university, and so on.

The keynote of Rina Zaskis 'On interplay between research, mathematics and pedagogy' offers valuable evidence of the reality she has encountered in student teachers' difficulties with mathematics, revealing the genuine practicalities that face teacher educators, complementing the broad brush-strokes of my own presentation.

Anna Sfard's keynote, on 'discursive gaps in mathematics in mathematics classroom: the invisible pitfalls of routines', uses the metaphor 'mind the gap' to refer to a picture of the gap between a stationary train and the platform. This has a structure similar to figures 14 and 15, highlighting the many instances where the language of expert and learner may cause significant difficulties with one well-chosen example. The difference between the Sfard (1991) dualist-framework between operation and structure and the Tall (2013) three-world framework goes back to Autumn 1990 when we worked together for six weeks at Warwick University and we found a significant difference between the Sfard notion of 'structure' and Bourbaki's set-theoretic definition of mathematical structure. This caused me to see distinct categories of embodied structure of objects (physical or imagined) and structure of axiomatic formal theories, much later extending the two categories to three (Tall et al., 2000).

There is neurophysiological evidence (Amalric & Dehaene, 2016) that 'high level mathematical thinking makes minimal use of language areas and instead recruits circuits initially involved in space and number.' Language is clearly essential for the formulation and communication of sophisticated theoretical ideas but it is also important to be able to make coherent links with spatial and operational meanings. In a teaching experiment in linear algebra, Hannah, Stewart & Thomas (2016) encouraged students to explain the links between geometric, symbolic and formal ideas in everyday language. The students highly rated the course and, though they found verbal explanation difficult to achieve, it gave them an opportunity to give meaning to symbolic calculations and formal relationships that was reflected in improved performances in examinations.

The question about the balance between the role of language and other aspects of mathematical thinking remains open. What matters even more is to provide practical and theoretical ways of helping teachers and students to operate in ways appropriate for their personal needs and for their role in society.

The fourth keynote by Gerald Goldin on 'Beauty, Power, and Connection: The Conative Dimension of Mathematical Engagement' enriches the connections between cognition and affect by highlighting the aesthetic role of mathematics. This can be related to brain activity mentioned in a paper (Tall, 2020) extending the opening keynote where affective aspects of brain activity arise in the central limbic system that flood the whole brain with neurotransmitters that enhance or suppress mental activity in an immediate unconscious activity before conscious decision-making occurs (Figure 16).

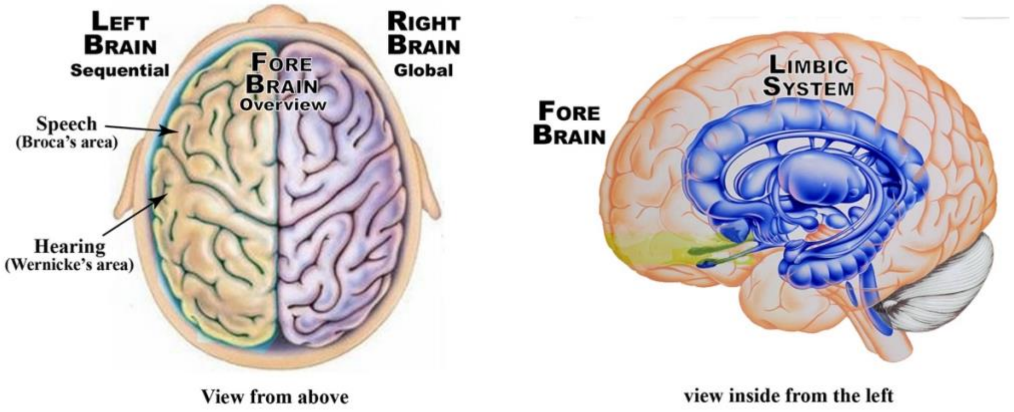

Figure 16: The brain and the limbic system

The interrelationship between the different parts of the limbic system is still not fully understood.[7] However, we already know that different neuro-transmitters are gender-related and are generated by experiences of pleasure and fear in which beauty, power and connection play a vital affective role. These are related to the three-world cognitive framework of embodiment, symbolism and formalism and the social frameworks of different communities.

Many other presentations had links to my opening keynote. I mention only those from academics from the host institution.

Miroslawa Sajka showed videos of her eye-tracking research using the latest portable equipment with spectacular demonstrations of the changing focus of attention as an individual looks at pictures, reads information and makes decisions. This offers me new personal insight into the thinking of mathematicians and students, enhancing the framework I proposed.

Piotr Blaszczyk presented a link between the historical work of Euler and modern standard and non-standard analysis that resolved a problematic aspect that had remained unsolved for over thirty years since it arose in the *Foundations* book I wrote with Ian Stewart. This concerned whether the epsilon in the definition of a Cauchy sequence should be real or rational. Knowing what we do know about real numbers, this hardly seems to matter as there are always real and rational numbers between two distinct reals. But what if the 'real numbers' we are talking about are not related to our embodied imagination of a number line? What if they were only given formally in terms of the axioms of an ordered field? What if the field were non-Archimedean?

The resolution given by Błaszczyk (2016) solved this problem by considering the limit of $(1/n)$ as n tends to infinity using a definition that distinguishes between properties in Archimedean and non-Archimedean fields. Making sense of this result requires a knowledge structure that can grasp its meaning. It now makes sense to me, but it may be problematic for you, the reader. This possibility emphasizes the main point of my presentation: that we must be willing to question our own understandings to pass through an impenetrable boundary to find insight.

My final comment is to thank Basia Pieronkiewicz for inviting me to the conference and sharing her publications on transgressions and the origins of the theory (Pieronkiewicz, 2015, Kozielecki, 1987, 1997). Not only did she make all the complicated arrangements for my attendance, she also thoughtfully presented me with a copy of Tomasello's *Cultural Origins of Human Cognition*. This has turned another corner in my journey to understand the evolution of mathematical thinking, not only in the millenia of evolutionary time, but also in historical time within the life of an individual.

In the last half century or so, we have passed through a remarkable period of conceptual change with the invention and development of digital technology. Not only does this offer us new tools to carry out numerical and symbolic calculations and draw dynamic pictures controlled by the movement of a finger, it also gives us new ways of imagining the amazing shift in meaning from practical drawing to mathematical imagination. The drawing of a line with a pencil does not allow us to *see* the difference between a rational number and an irrational. It only allows us to draw a *continuum* in which the mark made by the pencil leaves a trace that does not reveal fine detail. Using a smart phone, it is possible to zoom in on a graph and continually redraw it without changing its thickness. This enables us to zoom in to *see* the notion of local straightness that gives rise to differentiability and also to use a 'Tall-scope' (as in figure 14) to stretch a graph horizontally to 'pull it flat' and give meaning to the fundamental theorem of calculus.

[7] https://en.wikipedia.org/wiki/Limbic_system

Half a life-time ago, I gave a dynamic visual proof a function ('the blancmange') that is continuous everywhere, but differentiable nowhere, (Tall, 1982). Using the fundamental theorem, its integral is differentiable once and is everywhere continuous. Repeat this integration several times gives a C^n function that is differentiable *n* times with a continuous *n*th derivative. The theory of embodied meaning is not just a naïve beginning for apprentices encountering mathematics at an early stage. It has a profound meaning that takes us higher and higher in aspiration that extends to ever greater mathematical sophistication. For the very first time in history, we are at a threshold that takes us to a new level of sense-making in mathematics in general and in calculus in particular. We are very privileged to live in this amazing period of intellectual advancement.

I end this reflection with the first slide I presented in the plenary (figure 17).

Figure 17

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