Foundations For The Future:
The Potential of Multimodal Technologies for Learning Mathematics

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Abstract
We focus on how we expect students in a wide variety of age groups to be learning mathematics in a technology-enhanced environment in the future. This viewpoint will be grounded in our present work and seek to offer a perspective on how teaching and learning in a technology-rich environment might look in the future in the early grades. This is an important age group given the unfulfilled role of technology in elementary schools, and lack of professional development to enable teachers to transform their practices with latest technologies. It is also a critical time for young learners to establish the foundations for their future mathematics learning and motivation to learn.

We will describe how recent developments in multimodal learning environments that utilize haptic and multi-touch technologies create enhanced learning possibilities for more learners to access core mathematical ideas and think mathematically. We will use existing theory on how humans think mathematically (Tall, 2013), particularly the fundamental processes of perception, operation and reason over the longer term, to illustrate various categories of new mathematical activities and how students can learn in a multimodal environment.

OVERVIEW

We are entering a new stage of the digital era where certain technologies are becoming ever more ubiquitous in our lives. Such technologies offer immersive experiences for students and fluid forms of interactivity to enhance engagement. Technology is also capturing the affordances of connectivity to enable users to connect with each other, to share their work and favourite media, and to preserve access to such items as they freely roam around a hot-spotted planet through cloud computing. Yet, it is unclear how such ubiquitous and highly usable forms of technology can and will be used in classrooms in mathematically meaningful ways. The warning of Cuban (2001) about the unfulfilled promise of technology as an agent of transformation is still relevant today and recent national reports in the US describe the challenging student achievement gaps between ethnic groups in mathematics classrooms, especially in urban settings, with technology still not enhancing access for all.

Ten years ago, much of the technology we are referring to was not actually available in mainstream classrooms but now research in various countries describe computers and networks as being widely available in even the poorest of schools (U.S. Department of Education, 2007)—even though these established technologies are seldom used for the purpose of meaningful work (Bretscher, in press). Indeed, some researchers have demonstrated that the main issue now is access to quality professional development (Sinclair et al, 2009). This theme also arises in other countries. We situate our work here in recent research funded by the National Science Foundation that explores the potential benefits of using technologies that embrace multimodal interaction and connectivity in various learning contexts. We outline examples of such technology and use an analytical framework (Tall, 2013) to explain how students can reason and learn mathematically in such environments and discuss the potential impact they can have on mathematics education in the future.

LITERATURE REVIEW & BACKGROUND

Multimodal technologies offer alternative input or combined input/output methods. Common forms of alternative inputs are speech inputs (e.g., voice recognition), touch (e.g., gesture-based interactions) and bodily motion. The latter has rapidly evolved in recent years with the development of multi-touch technologies (e.g., tablet PCs, Interactive Whiteboards, iPads). Combined input/output devices include haptic devices, which integrate visual modes with force feedback loops, offering the user the ability to feel objects or the results of their interactions with the environment.

As a consequence of new modes of interaction, it becomes possible to integrate new ways of thinking mathematically, taking us beyond text books with static pictures and keyboard input as in the days when Logo was first introduced by specifying geometric pictures with typed commands. There is now a growing potential based on successful prior research and falling costs to use multimodal technologies in mathematics classrooms to offer students multiple ways to construct meaning using their natural modalities and bodily experiences.

Multimodal interaction has evolved in various research areas and applications including computer vision/visualization, psychology and artificial intelligence with increasing use in education particularly in early learning and developmental psychology. Jaimes and Sebe (2007) offer a survey of many of these disciplines including face recognition, facial expression analysis, vocal emotion, gesture recognition, human motion analysis, speech recognition and eye tracking. They outline how a multimodal interaction can simply be an environment that responds to inputs in more than one modality or communication channel (e.g., speech, gesture, writing) through perceptual, attentive or enactive interfaces. Dautenhahn (2000) has developed multimodal interactive learning environments as teaching and learning tools for the rehabilitation of children with autism, which establishes a potential importance for their use in special needs education in general.

Multi-touch environments are also evolving and Thompson, Avant and Heller (2011) examined the effectiveness of using TouchMath—a multisensory program that uses key signature points on mathematical objects—with students with physical learning disabilities. Using a multiprobe, multiple baseline design, they discovered all students were successful in reaching the criterion in terms of percentages of correct responses to addition problems.

Recent work in mathematics education explores the mathematical affordances of multi-touch technologies in that it can help develop number sense in part by virtue of the important role that fingers play in counting, but also because of the multimodal feedback that it offers children (Ladel & Kortenkamp, 2011; Jackiw, 2013; Sinclair & SedaghatJou, 2013).

It is of no surprise to us that a lot of work is focused in Special Education and with children with special needs and physical disabilities. A multi-modal approach engages other senses with which to investigate and learn. We believe this approach is relevant to all learners though especially if the technology can increase access to complex mathematical ideas through various forms of interaction.

Over 10 years ago Chris Dede (2000) also foresaw the profound potential of multimodal technologies but found their cost and affordability for education to be the main impediment for full integration into mainstream schools. This is less problematic today as increasingly affordable multi-modal devices flood the market and are being adopted by schools, in particular multi-touch devices that incorporate visual and auditory senses with tactile use. A major challenge lies in the dramatic pace of change in technology, which has been so much faster than the changes that can be incorporated into the curriculum in a reasonable time or through the effective professional development of teachers to rethink their pedagogy with respect to the mathematical affordances and opportunities of such technologies.

In addition to using our senses utilizing our whole body through motion is also another form of mathematizing the world around us and specific studies in mathematics education have found the use of motion detectors and interactive technologies to be important tools as mediators between students' bodily enactments and more complex mathematical representations such as graphs and functions (Brady, 2013; Nemirovsky et al, 2013; Nemirovsky & Borba, 2003; Radford et al, 2003; Radford, Edwards & Arzarello, 2009; Radford, Miranda, & Guzman, 2008; Rasmussen et al, 2004).

These studies reveal how enactive embodiments can offer a foundation for more subtle mathematical ideas, as indicated by Bruner (1966) in terms of his three successive modes of enactive, iconic and symbolic operation. More recent studies (e.g. Tomasello, 1991) have shown how humans extend the ability of other primates by not only imitating what others do, but also to be able to sense and share the intentions of others and to use tools, artefacts and language to build successively more sophisticated levels of thinking. An application of Tomasello's work to mathematics education (Hegedus & Moreno-Armella, 2011) allows us to think about the movement to dynamic, interactive technologies as an illustration of a representational redescription of mathematics in the 21st Century. Such re-descriptions potentially allow more students access to fundamental ways of mathematical thinking although we also appreciate that this may also require new ways of thinking in later contexts that the learner may encounter.

A major question is how these enactive embodiments relate to the more subtle mathematical ideas that develop in more sophisticated contexts. Do they provide an embodied form of meaning that supports later developments or are there aspects that can impede later learning? This requires a theoretical framework that studies the detailed development of mathematical thinking in different individuals as they pass through successive stages of sophistication in mathematics.

Other major and related technological advances in mathematics education include the use of dynamic, interactive representations primarily in the form of software and connectivity through exploiting the use of classroom wireless networks. We offer a brief overview of advances in these domains.

Dynamic interactive mathematics

Dynamic interactive mathematics environments such as The Geometer's Sketchpad® and Cabri-Geometre offer tools to construct and interact with mathematical objects and configuration. Interaction is via the executable representations of these mathematical objects and through this interaction one can *touch* the underlying mathematical structure (Hegedus & Moreno-Armella, 2011). Objects can be selected and dragged by mouse movements in which all user-defined mathematical relationships are preserved. In such environments, students are supported in efforts to formulate conjectures and generalizations by clicking and dragging hotspots on an object, which dynamically re-draw and update information on the screen as the user drags the mouse (Drijvers, Kieran, & Mariotti, 2009). In doing so, the user can explore and efficiently test an entire parameter space of equivalent mathematical constructions.

Such environments aim to develop spatial sense and mathematical reasoning by allowing conjectures to be tested, offering "intelligent" tools that constrain users to select, construct or manipulate objects that obey mathematical rules (Mariotti, 2003) alongside well-developed curriculum activities. The core features are construction and manipulation allowing constructs to be dynamically reconfigured. Empirical work states how these features can lead to improvement in student achievement (Battista, 1997; Hollebrands, 2002), student engagement through aesthetic motivation (Sinclair 2001, 2002a, 2002b), student ability to generalize mathematical conjectures (Mariotti, 2000) and students' development of theoretical arguments (Laborde, 2000, 2001; Noss & Hoyles, 1996). Actions of pointing, clicking, grabbing and dragging parts of geometric constructions allows a form of mediation (Falcade, Laborde & Mariotti, 2007) between the object and the user who is attempting to make sense of, or discover some particular attribute of the figure or prove some theorem. This is referred to as *semiotic mediation*, which corresponds to mediation through the use of sign systems and artifacts whose meanings are generated by social construction (Hasan, 1992; Vygotsky, 1980).

Such environments have also been applied to a variety of topics to enable a modeling practices (Jackiw & Sinclair, 2007) ranging from applications in the primary grades (Battista, 1997; Sinclair & Crespo, 2006; Sinclair & Moss, 2012) to applications including analysis (Cuoco & Goldenberg, 1997), trigonometry (Shaffer, 1995), calculus (Gorini, 1997), physics (Olive, 1997), complex analysis (Jackiw, 2003), non-Euclidean geometry (Dwyer & Pfiefer, 1999; Hegedus & Moreno-Armella 2011), data analysis (Flowers, 2002), and Linear Algebra (Gol Tabaghi & Sinclair, 2013).

Dynamic, interactive environments often are representationally rich creating multiple perspectives on mathematical ideas. Simulations are used to explore functional relationships (Falcade, Laborde, & Mariotti, 2007; Yerulshalmy & Naftaliev, 2011), complex systems (Stroup, 2005) and rate and variation (Hegedus & Roschelle, 2013) to name just a few. The affordances of such environment establish a representational infrastructure that provides new ways for students to express, visualize, compute and interact with mathematical objects (Kaput, Hegedus, & Lesh, 2007). Indeed, many of these topics are currently introduced in the secondary or tertiary grades of mathematics education but that does not undermine the potential for using multimodal technologies to maximize the use of dynamic representations, in the early grades as well. The key idea here is that such technologies offer a representational redescription of the core mathematical structures through executable representations. Such representations can link mathematical attributes to modalities such as touch or force feedback. For us it is less about the curriculum that is stated should be introduced at various levels, whether discrete or continuous, but rather the modification of the representational system that such technologies can potentially establish. A graph of a function can be thought of as a static figure and operated on discretely through ordered pairs or represented in tabular format, or it can be re-described as a continuous object that can be smoothly and fluidly examined dynamically and touched; for example, consider the difference between feeling a linear and a quadratic function, how do we sense linear-ness or quadratic-ness? More profoundly, manipulating a graph with the fingers on a tablet can zoom in to *see* how a curved graph (such as a quadratic curve or a circle) magnifies to look 'locally straight'. Tracing the changing slope of the graph offers an embodied meaning linking to the symbolic processes of differentiation, integration, differential equations, the wider aspects of multi-dimensional vector calculus and on to the formal structures of mathematical analysis. Our chapter posits that multi-modal technologies have a role in all grades and their use with young children plays an essential role in the full range of development from elementary to undergraduate classrooms.

Classroom Connectivity

Classroom connectivity (to generally mean networked classroom activities and assessment) has roots in more than a decade of classroom response systems, most notably ClassTalk™ (Abrahamson, 1998, 2000), which enabled instructors to collect, aggregate and display (often as histograms) student responses to questions, and, in so doing, create new levels of interaction in large classes in various domains (Burnstein & Lederman, 2001; Crouch & Mazur, 2001; Dufresne et al., 1996; Hake, 1998; Piazza, 2002) and levels (Hartline, 1997). Roschelle, Abrahamson, and Penuel (2003) show remarkably consistent positive impacts across multiple domains and levels. Some of the new affordances beyond classroom response systems are: (1) Increased mobility of multiple representations of mathematical objects such as functions as reflected in the ability to pass these bi-directionally and flexibly between the teacher and students and among students, using multiple device-types, and (2) Teachers can arrange, organize and analyze, sets of whole-class contributions at once, and students can make sense of their work in a social context, reasoning and generalizing about their contribution with respect to their peers' work. Such affordances transform the communication infrastructure of the classroom (Roschelle, Knudsen and Hegedus, 2010) that extends the normal affordances of social networks by increasing mathematically meaningful participation (Dalton & Hegedus, 2013), supporting a generative activity and investigation space (Stoup, Ares & Hurford, 2005) and the transaction and comparisons between private and public work (Vahey, Tatar and Roschelle, 2007). This can all lead to enhanced engagement and learning due to the collaborative nature of the classroom. Such research has primarily been conducted in secondary grade classrooms but we believe there is a lot of potential for such work in the primary grades that often structure classroom activities around small group work or learning stations. It is important to note that in each of these examples it is not just the technology that is the primary agent of change in these classrooms, but rather the integrated nature of the activity design and the technology that structures learning through enhanced discourse. We will return to this key point later.

In summary, there have been many advances in digital technology in mathematics education, which are situated, or could be situated, in a wide variety of school classrooms. A technology-enhanced curriculum that combines interactive mathematics software, networked classroom connectivity and multimodal interaction has the potential to impact learning and enrich mathematical discourse in the future based on present work and the growing ubiquity of multimodal technologies in society. But we acknowledge that such a claim should be modified by the lack of teacher professional development and preparation that leads to Cuban's warning regarding the unfulfilled role of technology in schools. We return to this issue in our concluding remarks.

We will now outline some applications of these ingredients combined in different ways but primarily multimodal experiences before providing an analytical framework for examining how such technologies can provide a platform for mathematical learning.

NEW MATHEMATICAL ACTIVITIES IN A TECHNOLOGY-ENHANCED LEARNING ENVIRONMENT

We should note that we have not tried to cover all advances in educational technology relevant to mathematics learning as we prefer to situate our position in aspects of development that we believe are most promising in taking a mathematical-activity centered approach vs. a technology-centered approach. Technology should not be just a pedagogical prop or a computational aid in instruction or learning (Moreno-Armella & Hegedus, 2009) but rather a transformative device to enhance the discipline of learning between students and teachers. For example we have chosen not to highlight the major relevance of Computer-Aided instruction or On-line Assessment tutors as a major development in Educational Technology and Computer Science in the past 10 years (see Hegedus & Roschelle, 2012 for further details on such) since our focus is primarily on multimodal technologies that allow students to access the beauty and complexity of mathematics in simple ways that engages many aspects of their biological and social self.

Several examples we offer here have been developed at the Kaput Center with support from the National Science Foundation[i] and others are prototypes based on present technological affordances. We focus on several environments that we have explored utilizing various modalities, representationally-rich interactive software and classroom networks where possible. These include: 1. Force-Feedback in concert with 3D visual shapes and surfaces, and 2. Multi-touch technology exploring mathematical objects and attributes and concepts. The activities serve as exemplars of the types of learning opportunities possible and that we can tap into such technological affordances in mathematically meaningful ways that can be generative in the future.

Force Feedback Technology

Sensable's PHANTOM Omni® (http://www.sensable.com/haptic-phantom-omni.htm)—hereon referred to as Omni—is a desktop haptic device with six degrees of freedom for input (x, y, z, pitch, roll, yaw), and three degrees of output (x, y, z). The Omni's most typical operation is via a stylus-like attachment that includes two buttons (see Figure 24.1). The Omni provides up to 3 forces of feedback for x, y, and z. It is primarily used in research, with a significant presence in dentistry and medicine but growing in mathematics education (Hegedus & Moreno-Armella, 2011). In the environment, models of 2- and 3-dimensional objects and haptic simulations (e.g., magnetism, friction) are used to create a dynamic, visual-haptic scene (Figure 24.2). User interactions with the models within a scene are graphically displayed through the haptic pointer on the computer screen; and physically meditated by the haptic device, by moving the haptic stylus or pressing the buttons on the stylus. For example, one application allows users to move and rotate a cube. In the application, when a user moves the haptic pointer onto the frictional surface of the cube and presses a haptic button, the position and rotation of the cube is synced to those values of the haptic stylus until the button is released.

Figure 24.1. A student operates the PHANTOM Omni® haptic device

Figure 24.2 Students' view of the multi-modal environment

Multi-touch Technology

In collaboration with KCP Technologies, the Kaput Center developed a set of activities for use with *Sketchpad®Explorer* for the iPad (hereon referred to as Explorer), a viewer application of the widely popular The Geometer's Sketchpad® software (hereon referred to as Sketchpad). This application is available in the Apple Store. Activities were constructed in Sketchpad and then transferred to the iPad through email or other forms of file exchange. All activities are pre-configured for students and teachers to use as no construction tools are presently available in this version for the iPad. Students directly interact with objects in the pre-configured activity including geometric objects (e.g., points), iterative counters through flicking, or buttons that had been configured to perform a set of operations (e.g., reflection of an image).

In multi-touch/multi-input platforms such as the iPad, the learner can use multiple modes of input and outputs— their natural modes of seeing and feeling, to make sense of the task. The iPad offers a direct (almost zero-interface) mode to touch and directly manipulate mathematical objects (see figure 24.3), and offer multiple inputs to one mathematical object hitherto impossible on a single-input computer (mouse as pointer and selector) – see figure 24.4. In figure 24.3 you can touch both vertices of the mirror segment at once, which can change the way one thinks about a line of symmetry in a static world. In figure 24.4 both vertices have to be used simultaneously either with two fingers or by two people which can change the mathematical experience.

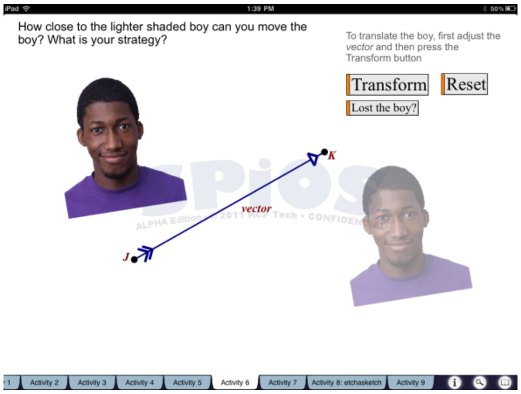

Figure 24.3 Translation as a composition of two reflections

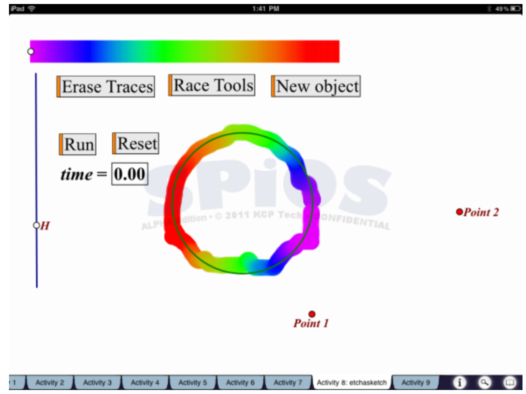

Figure 24.4 Etchasketch – Moving two points simultaneously to obtain a circle

We now present a framework that will enable us to describe how students can use and experience such technologies as a new platform for mathematical learning through a multi-modal approach in the future.

FRAMEWORK: HOW STUDENTS CAN LEARN MATHEMATICS

To formulate a theoretical framework for the development of mathematical thinking and the use of multimodal technologies, we need to consider not only what individuals may or may not learn at a particular stage of development using various modes of *perception and operation*, but also to consider the increasingly sophisticated forms of *reason* that develops in each individual over the longer-term. Contrary to the view that mathematics is a fully coherent system of knowledge, successive mathematical structures involve new meanings that may be supportive in some contexts but problematic in others (Tall, 2013).

For example, the initial stages of counting and number involve physical playing with objects, sorting them and learning the complex act of counting. Once the child realizes that the number does not change if the objects are placed in different ways, it is possible to focus on the idea that a set of six objects can be seen as four plus two or two plus four, or even three lots of two or two lots of three. This perception of the dynamic layout of the objects can lead to more general ideas, such as the observation that the order of addition or multiplication does not affect the final total. This, in turn, gives mental connections between physical and mental perceptions and operations that Tall (2013) refers to as *embodied compression* from the operation of counting to the concept of number.

On the other hand, when focusing on counting without such an overall conception, it is not obvious that 7+2 (a short count-on of 2 that can be performed on one hand) is the same as 2+7 (a longer count-on of 7 that requires the fingers of both hands). Tall (2013) refers to the focus on counting procedures to build the properties of whole number arithmetic as *symbolic compression*.

The term ‘embodiment’ is used with very different meanings in the literature. For example, Dienes (1960) used the term to describe physical materials such as Dienes’ blocks to express the properties of arithmetic of whole numbers, whereas Lakoff & Nunez (2000) use the term more broadly to claim that all human thinking is based on sensori-motor operations that may be expressed metaphorically using language. Here we are interested in the way in which mathematically ideas may be represented and interpreted physically or mentally using multimodal technology.

Embodiments can be supportive and simple in some aspects yet be complicated and even problematic in others. For example, Dienes' blocks are simple embodiments for place value in addition. A unit is a small cube, in base 10, a 'long' is ten cubes glued together to represent 'ten' as a single entity. Adding two collections such as 17 plus 6 is 1 long, 7 units plus 6 units and, combining ten of the units to give a new flat, gives 2 longs and 3 units, symbolized as 23. However, when the same embodiment is used for multiplication, additional features arise in which multiplication by 10 replaces ten units by a 'long' and ten 'longs' by a 10x10 'flat', representing one hundred. So Dienes' blocks are less appropriate for multiplication.

In this case, a new embodiment may offer a more appropriate context for meaning. For instance, in the multiplication of whole numbers, multiplication of decimal numbers by ten can be embodied by physical operations on the symbols themselves. A simple method is to write 3 zeros on a piece of paper and cover them with strips of paper with a single digit on the end of the strip (see Figure 24.5). The number 27 is represented by placing a 7 in the units place and a 2 over the first strip in the tens place to represent 27 as 2 tens and 7 units (or 27 units). Multiplying by 10 is embodied by shifting the number one place to the left to get 2 hundreds and 7 tens, or 27 tens, or 270 units. (Tall, 2013, pp. 135-137).

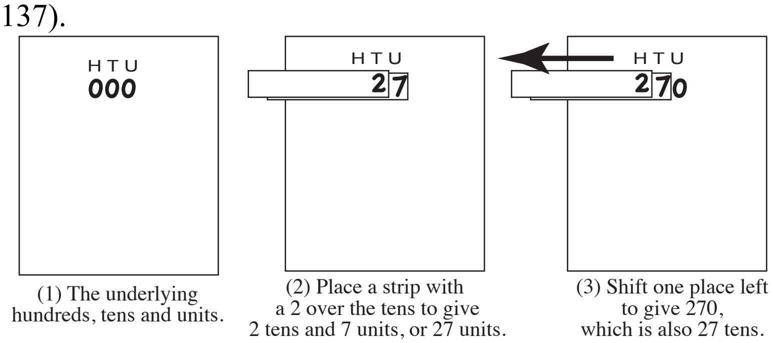

Figure 20.5 Shifting digits physically to the left to multiply by ten

This involves a new physical operation (moving the symbols themselves) that can then be imagined in the mind to 'see' digits moving mentally in a more sophisticated way and so the meaning of the symbolic operation can be supported by more sophisticated embodiment. It also offers a much more flexible meaning for symbolism as a basis for multi-digit multiplication, in which blocks of digits are moved one place to the left to multiply by 10 or one place to the right to divide by 10, including moving the block of numbers over the decimal point to see that 27 divided by 10 is 2.7 which is 2 units and 7 tenths. (Tall, 2013, pp.135-138).

The development of successive levels of sophistication follows the same broad pattern as new contexts are encountered. New embodiments can support certain aspects of the new situation yet be problematic in others. This occurs at successive stages of the curriculum where previous experience can impede new learning, in the shift from whole numbers to fractions, from unsigned numbers to signed numbers, from fractions to infinite decimals, from arithmetic to algebra. At each stage, properties that were supportive in one context (e.g. multiplication gives a bigger result, subtraction produces a smaller result, and so on) become problematic later on.

The development of new multi-media technology changes the paradigm. The multimodal environment offers the learner a way to operate on objects that behave in a predictable way. This provides the opportunity for the learner to gain insight in an intuitive, embodied way prior to developing algorithms for more sophisticated use. However, the particular embodiment may be supportive at one stage but become problematic in a new context, so there remains the longer-term task of how learning at one stage can impact on later learning. This involves not only the development of the individual child making sense of the fundamental mathematical ideas, but also the social interaction with the technology as a critical factor in their collective sense making.

APPLICATION TO HOW CHILDREN USE MULTI-MODAL TECHNOLOGIES

Exemplary Activity 1: Force feedback

In this activity, we offered students two objects to manipulate, a plane and a 3D shape (e.g. a cone, cube, pyramid). First the student can use the Omni as a selection device for clicking-and-dragging the objects around the screen and re-orienting them through twisting the device handle (see figure 24.6).

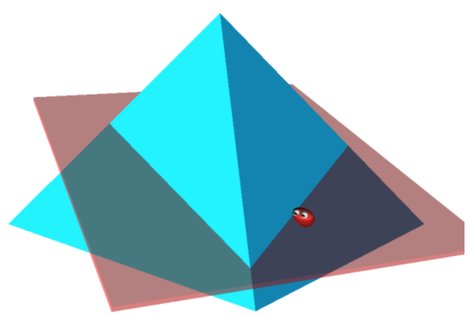

Figure 20.6 Planar intersection with a square-based pyramid

Once the plane has been moved to intersect with the 3D shape the device handle becomes a navigation tool for moving the red bug (see figure 24.6) around the surfaces and intersection providing feedback through continuous forces. Through iterative design we have found that using magnetism is a useful design principle to enable the user to focus on what part of the shapes the bug is located. This experience locks the bug onto the surface and the device begins to provide continuous force and abrupt changes as you move over a specific discontinuity (e.g. a vertex or edge). In particular, the user also feels “locked-on” or “sucked-in” to the intersection joints. Navigation is also driven by 3-dimensional motions of the device so that the user is moving the Omni handle in real space that simulates the experience of feeling the pseudo-3D visual shapes on the screen. This design principle and the use of a bug was discovered to be important for children to coordinate their physical motion with the flat-visual space on the screen. It was a way of calibrating the two modalities so that children might begin to talk about both experiences. This had been problematic in previous editions. As we show below, the use of a red bug was not only fun and engaging for the children but became a useful reference point for the children to talk about the experiences. As we will describe though, children’s discourse moved from talking about the bug explicitly, i.e. the results of the bug’s actions, to focusing on the specific mathematical attributes of the shapes. The bug had a short life in their exploration. Once the child sensed the relationship between the two situations, it became possible to move on from the specifics of the particular embodiment to reasoning about the underlying mathematical relationships.

One variation of this activity was to use a square-based pyramid with the plane. We draw on a case of four 10 year olds exploring this setup to explain how students make sense of such mathematical configurations through our analytical framework. Our primary goal was to examine the types of discourse that students use to make sense of the configurations.

The children were engaged in the exploration within minutes of introducing them to the environment. Initially, the children were focused on perception with statements that were a mixture of mathematical observations, metaphors and behaviours of the elements of the configuration. These are initial observations:

John:	It might be triangle.
Sarah:	Could be.
John:	Could also be a square.
Sarah:	See if it's like a wall.
Peter:	It feels heavy.
Sarah:	Heavy? …
Sarah:	The bug is like sucking on it or something.

As the children continued to explore the interplay between visual and haptic modalities it became more evident where their initial visual forms of perceptions were challenged by physical information back from the Omni device:

Sarah:	A triangle it looks like.
Peter:	*[Continues to trace intersection and repeatedly move bug off and back onto intersection]* When you go up here it's a triangle, *[Referring to the shape made between the front facing edge and one part of the intersection]* because that's how it is. But when you go around here *[Moves bug beneath blue shape]* it kind of feels like a square.

Several of the children began to focus more on the objects and their properties (Operational) as well as attempting to interpret the effect of their interactions (Reason) and how the plane is perceived of in terms of how it cuts particular parts of the pyramid apart:

Megan:	I think the square is cutting it off *[Gestures]* where it's making, because the square it goes like that and it goes like that *[Gestures square with her two hands]*. I think it cuts it off where it goes like that *[Gestures one corner of a square with two hands]*. Because it's cutting – *[Points to screen]* it's just leaving the bottom not the top. *[Screen turns black]* What happened? *[Comes back]* Oh. It's just – it's not cutting the side of it *[Gestures]* where it makes a triangle. It's cutting

in the middle, *[Gestures]* so it makes a square on the bottom just like the square base pyramid.

The children resolve this conflict by counting sides that they feel and turns that they make. They felt the sharp feedback of moving around a vertex or angle whilst being continually "stuck to" the intersection. The children continue to explore though based on an emerging sense of how this configuration can be extended and a potential understanding of the flexibility of this dynamic intersection. This was not prompted by the interviewer:

Sarah:	I think we can get a five-sided shape.
Interviewer:	Sorry, what? I missed that?
Sarah:	When we felt the triangle if we went on the other side of the square {*the base of the pyramid*} we could get a pentagon.
Peter:	Yeah we could.
Interviewer:	Show me how you can do that using this plane? *[Students pass device down to Sarah]*
Sarah:	*[Sarah adjusts plane to intersect with the base] [Peter is drawing]* It's like that.
Interviewer:	Okay.
Sarah:	*[Traces with bug her intersection, counting sides]* Then you have this side and then it has to go like that. And then you have that side and then you have that. And then that.
Interviewer:	So for a pentagon, how many sides do you want to feel?
Sarah:	Five.

This is an example of embodied compression that could potentially lead to forms of symbolic compression with similar explorations of other polyhedra. Children are focusing on the objects involved in this configuration and can flexibly manipulate the intersection through dynamically (visually) editing the slope of the tangent plane. There is evidence that the faces of the shape are important for the children in trying to reason the classification of the intersection (i.e. by moving the plane into the base of the pyramid) but the faces are not being enumerated at this stage.

At this stage, the young students do not express the idea that the planar intersection of a polyhedron of n faces would be at most an n-gon. However, our work with high school and undergraduate students with a similar activity led to such discoveries being made along with similar forms of reasoning. This suggests that the design and use of multimodal technologies in the future can potentially establish learning environments for students at various age levels and needs. This example and subsequent ones outlines an opportunity space for open-ended and semi-structured activities.

Exemplary Activity 2: Multiple inputs correspond to mathematical variables

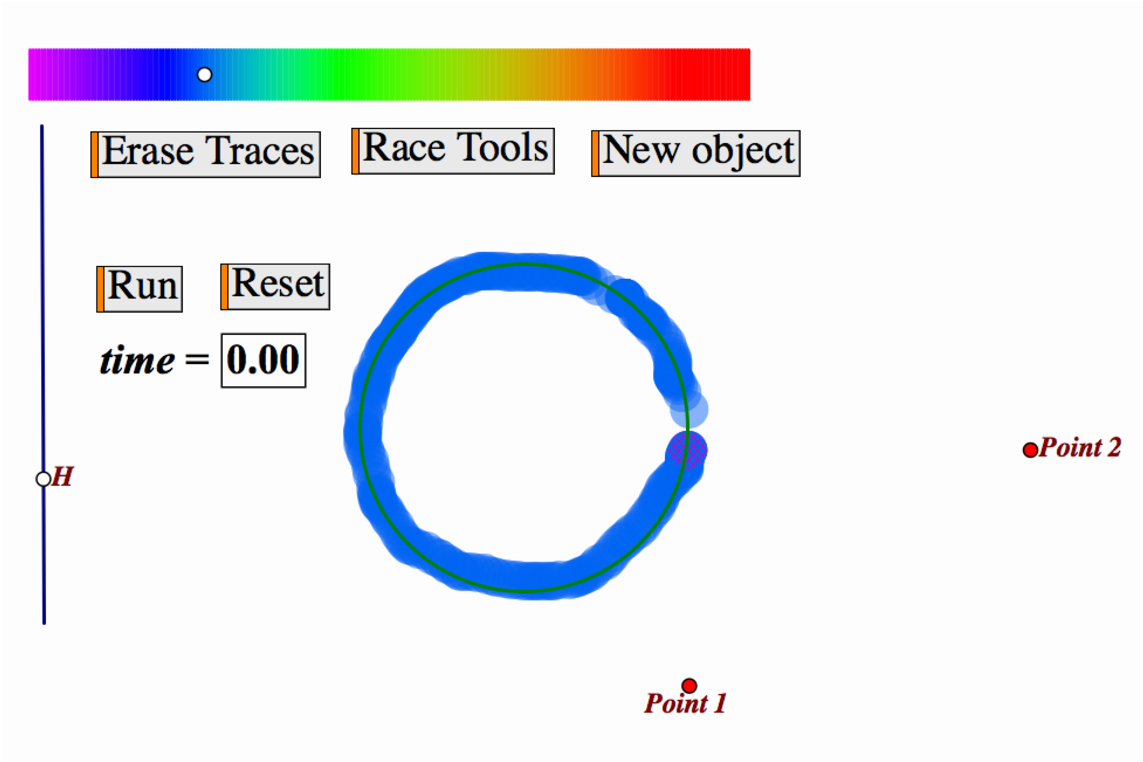

Figure 24.7 Creating a circle from two inputs

In this activity, one student controls the lateral-moving Point 1 and another student controls the vertical-moving Point 2, or one student controls both simultaneously (see Figure 24.7). The goal is for the students to trace a coloured-blob around the fixed circle. A third student (or third finger input) can adjust the colour of the trace, in order to make a rainbow of colour around the circle (point on the spectrum), or the size of the blob (point H).

This activity exemplifies how the technological affordances of multi-touch can be adapted in mathematically meaningful ways hence could be described as mathematical affordances of the technology. Let us offer a simple example to explain what this means. Consider using an iPad to watch a movie. A pinch-gesture can be used to make the movie smaller (or larger if the pinch is reversed outwards). This is a mode of interaction, and an affordance of the operating system and hardware. It allows the user to operate in a particular way. Consider now a geometric shape where the user performs a similar gesture either as a pinch or with two fingers (since the operating system allows such interactions) and the shape is dilated. Here the technological affordances can be made mathematical actions. In a sense, the biological actions are mathematized, putting mathematics at the very tips of students' fingers. Unlike on a computer, here the user can provide two inputs simultaneously to manipulate the sliders (on point 1 and 2). Point 1 is constrained in order to be moved along an invisible horizontal line segment and Point 2 on a vertical line segment orthogonal to the horizontal one. In this construction we have parameterized two input functions using traditional interactive tools. The two sliders are combined to create one output hence the input co-variates create a single visual output, i.e. the blob, which they need to trace around a fixed circle. Such an input methodology can be conducted by one person with two fingers or by two people with single finger-input strokes. We have found the latter to lead to collaborative problem-solving techniques where each person directs the other to create the desired output. The coordination of such discourse moves can establish mathematical arguments regarding the relationship between the input routines (social/individual) and output (computational/visual).

In our studies, elementary school and undergraduate students rapidly move towards moving the point-sliders separately. This has been completed using single-discrete finger movements of point 1 followed by point 2, followed by point 1 and the point 2 to create 4 discrete line segments as the initial trace, i.e. a square. Many groups in our studies have done this both individually and collaboratively. Such initial investigations illustrate perception based on information from actions on base-objects in the learning environment. Following the visual feedback, students realize that they are not meeting the challenge of tracing of a circle and focus more on the effects of their inputs and the properties of these interactive elements controlled by their touch. In one example, two students expressed:

> Rob: We need to move point 1 left and right …
> Sally: And point 2 up and down but with a different speed I think

Given this initial form of reasoning of the essential elements and their potential role in creating a new mathematical object (the blob output point) we highlight this as a form of embodied compression where they focus on the effect of their input operations, in particular the speed of their input as a model for the mathematical variation formerly embedded in this activity. For example, one student in this group described:

> Jared: I think we need to move it faster and slower at some points …

Finally, two students in the group move to a form of reasoning that involves the coordination of both inputs:

> Rob: Point one is kind of a y-axis… And point two is kind of a x-axis
>
> Sally: Like… One person can control … like going up-and-down … like going on the sides of the circle … and the other person can control the … top and bottom of the circle

The combination of such reactions and establishment of properties of the moving inputs and outputs lead to iteratively successful circles. Symbolic compression is at an elementary stage here where the structure of the inputs with respect to the output are informally formulated but the essential properties of such coordination are established: 1. Two inputs create one output, 2. The two inputs need to co-vary, i.e. move together in some fashion, 3. The 2 input-touch motions need to vary in terms of motion to map the circle. This still lacks the formal symbolic clarity of defining how such motions can be described in terms of a phased combination of sinusoidal motions (functions), i.e. $x^2 + y^2 = (\sin t)^2 + (\cos t)^2 = 1$.

With recent prototypes we have explored a new design space that allows students to share their construction and parameterize their contributions in ways that can lead to mathematically rich discussion in terms of comparison, reasoning and deduction. Through the affordances of wireless connectivity, students can share their investigations electronically with the teacher who can display through their device connected to a projector thus providing a public space for discussion. In addition, the activity can be reconfigured at an individual or group level for comparison in this display space, for example, varying the circle diameter or shape, which can lead to a contrast of actions and formulated procedures in a whole-class discussion.

DISCUSSION AND FUTURE PERSPECTIVES

We have looked closely at learners' discourse in terms of their utterances and actions both individually and socially since they have worked in groups. In particular the role of non-scholastic language in making sense of the properties of mathematical objects or concepts, e.g. the planar intersections of solids, categorizing shapes and surfaces, making sense of geometric transformations or composition of transformations through multi-touch to name a few. We have observed in other work (Gucler, Hegedus, Robidoux & Jackiw, 2013) that some forms of non-scholastic statements are mathematically meaningful in scaffolding meaning for the group, and some scholastic language can be used in mathematically inaccurate ways (e.g. drawing on prior knowledge of shapes in incorrect ways). Students sometimes experienced cross-modality where one modality (what they see) conflicts with another modality (what they feel).

Using embodied compression, children are focusing on objects and the consequences of their fluid interaction with the environment. They are also engaged in co-action (Moreno-Armella & Hegedus, 2009) where they are guiding and being guided by their actions within this dynamic, responsive environment, which potentially allows them to "see through" to the underlying mathematical structures inherent in the figures on the screen and the "invisible" forces of the haptic device (Hegedus & Moreno-Armella, 2011). We see the beginning of this in example 1 where the students realize and rationalize the intersection being a pentagon through visual-haptic arguments and in example 2 in Jared's comments about the dynamics of his movement (finger actions) in that they need to change speed at some points which give rise to the circularity of the output blob. "Seeing through" can be thought of as potential insights, perceptions into, or metaphors regarding the underlying structure. An underlying structure here might be the formal parametric definition of the circle.

We have observed that it is very difficult to parse out the visual from the haptic experience but what the child or children perceive is of particular relevance. The real challenge is developing learning environments with sequences of curriculum activities that enable students to transition to forms of symbolic compression. Without a focus on such students might only focus on the embodiment which could be an impediment to future learning.

These are perennial points though for many popular technologies in mathematics education including Logo, Dynamic Geometry, and Spreadsheets. It might be that any aim of moving towards symbolic compression and comprehension (and even application) through the use of such technologies is at odds with providing access to more students, i.e. you cannot achieve both. But then the practice of mathematical thinking is potentially changing because of the transformative role such multimodal technologies have on the distributed cognitive and communicative activities of the classrooms such that the nature of mathematical discourse evolves. Such technologies can be thought of as *cognitive extensions* of the biological self (i.e. thinking through our fingertips), or the drawing closer of our natural biological senses and the abstraction of mathematical thinking through the mediating effect of multimodal technologies. As Rotman (2000) has suggested:

> … [S]uch a transformation of mathematical practice would have a revolutionary impact on how we conceptualize mathematics, on what we imagine a mathematical object to be, on what we consider ourselves to be doing when we carry out mathematical investigations, and persuade ourselves of certain assertions, certain properties and features of mathematical objects, are to be accepted as "true". Indeed the very rules and protocols that control what is and isn't mathematically meaningful, what consitutes a "theorem", for example, would undergo a sea change (p. 68-69).

The essential point here is that not only do these technologies allow a shift in the mathematical representational infrastructure but can enhance the communication infrastructure of the classroom in the types, and new forms, of mathematical discourse that can arise from such use in the future.

We have explored the affordance of linking mathematical objects or functional relationships through haptic environments. For example, the child can use the haptic device to provide dynamic changes to inputs that are mathematical variables and receive physical feedback that is a dependent variable to the input(s). Here the technological environment co-acts in different ways mathematically to the navigational scenarios described above. In the activity illustrated in Figure 24.8, a child can click and drag points A, B and C (vertices of the triangle) or D (a point to control the translational position of a parallel line which the triangle is constructed on), but the Omni device is programmed to provide resistance (force feedback) that is proportional to the area of the triangle ABC. Moving certain vertices will result in invariant resistance (e.g. C) whereas others will create changing resistance due to the base or height of the triangle being modified.

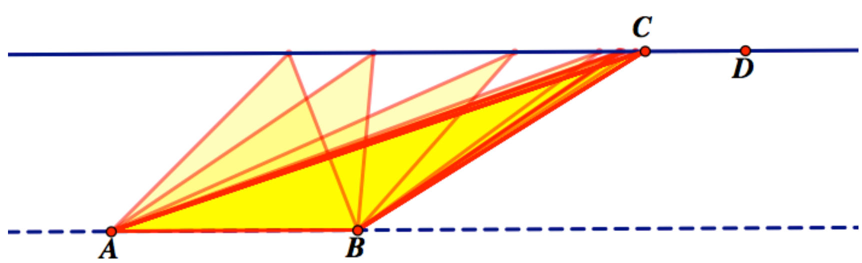

Figure 24.8 Dynamic triangles

In our preliminary work, children can attribute the inputs to mathematical variables that result in the output "feeling" and mathematical representation. Whilst children in our studies have not explained this relationship as Area = ½ * base * height, they have described the Area to be related to changes in the base and height, a form of embodied compression. This is also the beginning of symbolic compression but more work needs to be done to understand and realize how such experiences can establish such forms of symbolizing.

Multi-touch environments such as Explorer allow young learners the interaction with simple configurations that involve mathematically meaningful input and outputs. Such modalities allow children to use various modalities to explore configurations and focus on their actions (to develop their perception) and their operations on objects and their iterative responses based on core interactivity design principles. Such forms of co-actions (between multi-modal iterations and their output which further guide actions) can generate learning environments that enable students to focus on the operations that enable an understanding of the mathematical properties and structures being investigated. Tall (2013) aspires towards the "long-term simplification of mathematical ideas" and such infrastructures can offer such opportunities. This notion of *simplification* is both mathematical (where mathematicians and teachers make sense of the mathematical structures by formulating new concepts that can be manipulated as mental entities) and also personal (where learners make sense of the compressed entities in flexible ways). Such flexible thinking can be encouraged at all levels. For example, by building on the prototypical examples above students could engage in a multi-touch/input approach to collaboratively create an isosceles triangle or some other classification of 2D triangles through group-strategizing. Such actions on the base vertices A or B could establish methods to focus on the steps necessary to create an equi-angular shape or similar magnitudes of sides and develop symbolic compression. In addition, with well-configured sketches, many "parent-child" relationships, which preserve mathematical attributes, could be discovered through a multi-touch approach. A free triangle with no constraints on the vertices would lead to a collaborative strategy (or an individual one) that focuses on the effect of moving one vertex with respect to the other one; or the effects of moving two vertices with respect to the corresponding sides; or the invariance of such actions on some of the properties of the triangle.

In addition, the potential of such devices in a connected environment can help develop the process of symbolic compression as students individually, collaboratively and in comparison, externalize the steps that are necessary in coordinating a workspace of procedures and sense-making of mathematical concepts and attributes. The public workspace could be a collaborative arrangement where each student is developing a variation of a set of configurations, e.g. a family of similar triangles, or an aggregative enterprise where children build particular linked pieces of a constructions knowing how their piece interacts with someone else's product or the whole group construction.

We have attempted to introduce how recent multimodal technologies can create learning environments for students to access the basis for the mathematical properties of shapes, configurations and functional relationships focused on variation and co-variation. This can be achieved, and in the future refined, through the combination of various modalities in particular touch and visual. The primary purpose is to enable access to the embedded mathematical structure in order that a "focus on real-world embodiment can make sense for early learners, giving embodied meaning to symbolic operations." (Tall, 2013, p.173.)

We propose that the long-term development of mathematical thinking can be sustained through the careful use of multimodal technologies and meaningful implementation. Careful because of the potential for causing confusion for students in conflicting modal experiences highlighted earlier and also because of the possibility that formal symbolic compression might not be achieved through such implementation. The opportunity space that we outline here needs to be explored in ways that can create coherent mathematical experiences that not only allow access to mathematical ideas and concepts but also develops into more general symbolic thinking. This might not be possible for the whole spectrum of learners and future works needs to address this critically as more curriculum is developed that integrate such technological affordances in meaningful ways. We have tried to demonstrate here ways of doing this where the physical experience has been pre-programmed to correspond to mathematical attributes or multiple touches (input) correspond to mathematical operations, e.g. transformations. Both of these approaches are demanding on time and should be built and tested in a research environment. The alternative is a form of edu-tainment where the feedback is superfluous to the mathematics of the activity.

Future work also needs to address which populations of students are being addressed and what the significance of the research actually means. In the studies reported here we have worked with traditional populations but we also see the need to work with multimodal technologies for specific categories of learners, such as those with special educational needs (as outlined at the start of this chapter). Future work needs to focus on pedagogical strategies and the preparation of teachers to implement such strategies.

One of the long-term critical issues for effective implementation of technologies in schools is the professional development of teachers not only to understand how to use the new tools but also the pedagogical implications. Depending on the stage of their careers, teachers might be faced with re-thinking how such technologies can enhance the learning environment or even transform the very nature of the classroom in terms of how mathematics is re-described. If teachers are later in their career this might involve a shift in mindset to how content is conceived of and how it might be effectively introduced through such new technologies. Hennessy, Ruthven and Brindley (2005) build on Cuban's message by reporting on use of ICT in England. They report that teachers in their study might affirm that their practices are changing but they are not participating actively in technology-based curriculum design and implementation in their schools. They propose five specific aims for successful integration of ICT at the level of departmental policy and planning: (a) Providing teachers with "opportunities for long-term collegial interaction involving critical reflection, sharing ideas, and research concerning the use of ICT" (p. 187); (b) Integrating ICTs into a structure of work in alignment with the national prescribed curriculum and help in meeting learning objectives with critical and appropriate use; (c) Taking into account and enhancing students and teachers' levels of technological expertise; (d) Systematically evaluating the affordances of ICT for attainment of specific learning goals and (e) Offering a balance of complementary ICT-based and other learning activities. They claim organizational change processes are important in generating strategies to successful implementation of technology-based instruction and emphasize "shared ownership of plans" (p. 186) at organizational level as a requisite to implement technology into the school instructional practices with a significant impact on student learning. Our work and proposed claims for future development and implementation need to address such aims to have any widespread significance in mathematics classrooms in the future.

In conclusion, the central focus here is on the fundamentals of mathematics and how the latest advances of multi-modal technologies can promote learning through the development of mathematically meaningful transformations of technological affordances. This should potentially impact curriculum design, how we teach and how our children learn in the future.

REFERENCES

Abrahamson, A. L. (1998). An overview of teaching and learning research with classroom communication systems, *Paper presented at the Samos International Conference on the Teaching of Mathematics, Village of Pythagorion*. Samos, Greece.

Abrahamson, A. L. (2000). A brief history of ClassTalk, *Paper presented at the Teachers Teaching with Technology International Conference*. Dallas, TX.

Battista, M. T. (1997). *Shape makers: Developing geometric reasoning with the Geometer's Sketchpad*. Berkeley, CA: Key Curriculum Press.

Brady, C.E (2013). *Perpectives in Motion*. Unpublished Doctoral Dissertation, University of Massachusetts Dartmouth. USA

Bretscher, N. (in press). Exploring the quantitative and qualitative gap between expectations and implantation: A survey of English mathematics teachers' use of ICT. In A. Clark-Wilson, O. Robutti, N. Sinclair (Eds.). *The mathematics teacher in the digital age*. Springer.

Bruner, J. S. (1966). *Toward a Theory of Instruction*. Harvard University Press.

Burnstein, R., & Lederman, L. M. (2001). Using wireless keypads in lecture classes. *Physics Teacher, 39*(8), 8-11.

Crouch, C. H., & Mazur, E. (2001). Peer instruction: Ten years of experience and results. *The Physics Teacher, 69*(9), 970-977.

Cuban, L. (2001). *Oversold and underused: Computers in the classroom*. Cambridge, MA: Harvard University Press.

Cuoco, A., & Goldenberg, P. (1997). Dynamic geometry as a bridge from Euclidean Geometry to analysis. In J. R. King & D. Schattschneider (Eds.), *Geometry turned on!: Dynamic software in learning, teaching and research* (pp. 69-70). Washington, DC: Mathematical Association of America.

Dalton, S. & Hegedus, S. (2013). Learning and Motivation in High Schools classrooms**.** In S. Hegedus & J. Roschelle, (Eds) (2013). *Democratizing Access to Important Mathematics through Dynamic Representations: Contributions and Visions from the SimCalc Research Program*. Berlin: Springer.

Dautenhahn, K. (2000). Design issues on interactive environments for children with autism. In *Proceedings of the 3rd International Conference on Disability, Virtual Reality and Associated Technologies, ICDVRAT* (pp. 153-161). Alghero, Sardinia, Italy.

Dede, C. (2000). Emerging influences of information technology on school curriculum. *Journal of Curriculum Studies, 32*(2), 281-303.

Dede, C., Salzman, M. C., Loftin, R. B., & Sprague, D. (1999). Multisensory immersion as a modeling environment for learning complex scientific concepts. In N. Roberts, W. Feurzeig & B. Hunter (Eds.), *Computer modeling and simulation in science education* (pp. 282-319). New York: Springer-Verlag.
Dienes, Z. P. (1960). *Building Up Mathematics*. London: Hutchinson.
Drijvers, P., Kieran, C., & Mariotti, M-A. (2009). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J-B. Lagrange. (Eds.), *Mathematics education and technology—rethinking the terrain* (pp. 89-132). New York: Springer
Dufresne, R. J., Gerace, W. J., Leonard, W. J., Mestre, J. P., & Wenk, L. (1996). Classtalk: A classroom communication system for active learning. *Journal for Computing in Higher Education, 7*(2), 3-47.
Dwyer, M. C., & Pfeifer, R. E. (1999). Exploring hyperbolic geometry with The Geometer's Sketchpad. *Mathematics Teacher, 92*(7), 632-637.
Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66*(3), 317-333.
Flowers, L. A. (2002). Developing purposes in college: Differences between freshman and seniors. *College Student Journal, 36*, 478-484.
Gol Tabaghi, S. & Sinclair, N. (2013). Using Dynamic Geometry Software to Explore Eigenvectors: The Emergence of Dynamic-Synthetic-Geometric Thinking. *Technology, Knowledge and Learning 18*(3), 149-164.
Gorini, C. A. (1997). Dynamic visualization in calculus. In J. R. King & D. Schattschneider (Eds.), *Geometry turned on!: Dynamic software in learning, teaching and research* (pp. 89-94). Washington, DC: The Mathematical Association of America.
Güçler, B., Hegedus, S., Robidoux, R. and Jackiw, N. (2013). Investigating the mathematical discourse of young learners involved in multi-modal mathematical investigations: The case of haptic technologies. In D. Martinovic, Freiman, V. and Karadag, Z. (Eds.), Visual Mathematics Cyberlearning (97-118). NY: Springer
Hake, R. R. (1998). Interactive-engagement vs. traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics, 66*, 64-74.
Hartline, F. (1997). *Analysis of 1st semester of Classtalk use at MacIntosh elementary school*. Yorktown, VA: Better Education, Inc.
Hassan, R. (1992). Speech genre, semiotic mediation and the development of higher mental functions. *Language Sciences*, *14*(4), 489–528.
Hegedus, S.J. & Roschelle, J. (2012). Highly Adaptive, Interactive Instruction: Insights for the Networked Classroom. In C. Dede & J. Richards (Eds.), *Digital teaching platforms* (pp. 103-115). New York: Teachers College Press.
Hegedus, S. & Roschelle, J. (2013). (Eds). *Democratizing Access to Important Mathematics through Dynamic Representations: Contributions and Visions from the SimCalc Research Program*. Berlin: Springer.
Hegedus, S., & Moreno-Armella, L. (2011). The emergence of mathematical structures. *Educational Studies in Mathematics, 77*(2&3), 369-388.

Hennessy, S., Ruthven, K., & Brindley, S. (2005). Teacher perspectives on integrating ICT into subject teaching: commitments, constrains, caution, and change. *Journal of Curriculum Studies, 37*(2), 155-192.

Hollebrands, K. (2002). The role of a dynamic software program for geometry in high school students' understandings of geometric transformations. In D. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant & K. Nooney (Eds.), *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 695-705). Columbus, OH: ERIC Clearinghouse on Science, Mathematics and Environmental Education.

Jackiw, N. (2003). Visualizing complex functions with the Geometer's Sketchpad. In T. Triandafillidis & K. Hatzikiriakou (Eds.), *Proceedings of the 6th International Conference on Technology in Mathematics Teaching* (pp. 291-299). Volos, Greece: University of Thessaly.

Jackiw, N. (2013). Touch & Multitouch in dynamic geometry : Sketchpad Explorere and "digital" mathematics. Proceedings of ICTMT12 (pp. 149-155), Bari, Italy.

Jackiw, N., & Sinclair, N. (2007). Dynamic geometry activity design for elementary school mathematics. In C. Hoyles, J.-B. Lagrange, L. H. Sun, & N. Sinclair (Eds.), *Proceedings of the Seventeenth ICMI Study Conference "Technology Revisited"* (pp. 236-245). Paris, France: Hanoi Institute of Technology and Didirem University.

Jaimes, A., & Sebe, N. (2007). Multimodal human computer interaction: A survey. *Computer Vision and Image Understanding, 108*(1-2), 116-134.

Kaput, J., Hegedus, S., & Lesh, R. (2007). Technology becoming infrastructural in mathematics education. In R. Lesh, E. Hamilton & J. Kaput (Eds.), *Foundations for the future in mathematics and science* (pp. 172-192). Mahwah, NJ: Lawrence Erlbaum Associates.

Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics (Special edition on proof in dynamic geometry environments), 44*(1-3), 151-161.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometre. *International Journal of Computers for Mathematical Learning, 6*, 283-317.

Lakoff, G., & Nuñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*: Basic books.

Ladel, S., & Kortenkamp, U. (2011). Implementation of a multi-touch-environment supporting finger symbol sets. In M. Pytlak, T. Rowland, E. Swoboda, eds. Proceedings of the seventh Congress of the European Society for Research in Maths Education, University of Rzeszów, Poland, pp. 1792-1801.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44*(1-2), 25-53.

Mariotti, M. A. (2003). The influence of technological advances on students' mathematics learning. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 695-723). Mahwah, NJ: Lawrence Erlbaum AssociatesMoreno-Armella, L., Hegedus, S., & Kaput, J. (2008). From static to

dynamic mathematics: Historical and representational perspectives. *Educational Studies in Mathematics, 68*(2), 99-111.
Moreno-Armella, L., & Hegedus, S. (2009). Co-action with digital technologies. *ZDM: The International Journal on Mathematics Education: Transforming Mathematics Education through the Use of Dynamic Mathematics Technologies, 41*(4), 505-519.
Nemirovsky, R. & Borba, M. (2003). Perceptuo-motor activity and imagination in mathematics learning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th International Conference for the Psychology of Mathematics* (Vol. 1, pp. 105-136). Program Committee: Honolulu, Hawai'i.
Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. Journal for Research in Mathematics Education, 44(2), 372-415.
Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
Olive, J. (1997). Creating airfoils from circles: The Jouowski transformation. In J. R. King & D. Schattschneider (Eds.), *Geometry turned on!: Dynamic software in learning, teaching and research* (pp. 169-177). Washington, DC: Mathematical Association of America.
Piazza, S. (2002). Peer instruction using an electronic response system in large lecture classes. Presentation document presented at the Pennsylvania State University, Center for Educational Technology Services "Teaching with Technology" Series: Departments of Kinesiology, Mechanical Engineering, and Orthopedics and Rehabilitation.
Radford, L., Demers, S., Guzmán, J., & Cerulli, M. (2003). Calculators, graphs, gestures and the production of meaning. In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 55-62). Honolulu, HI: University of Hawaii.
Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics, 70*(2), 91-95.
Radford, L., Miranda, I., & Guzmán, J. (2008). *Relative motion, graphs and the heteroglossic transformation of meanings: A semiotic analysis.* Paper presented at the proceedings of the 32nd conference of the International Group for the Psychology of Mathematics Education, Morelia, México.
Rasmussen, C., Nemirovsky, R., Olszewski, J., Dost, K., & Johnson, J. L. (2004). On forms of knowing: The role of bodily activity and tools in mathematical learning. *Educational Studies in Mathematics, 57*(3), 313-316.
Roschelle, J., Abrahamson, A. L., & Penuel, W. (2003). Catalyst: Toward scientific studies of the pedagogical integration of learning theory and classroom networks.
Roschelle, J., Knudsen, J., & Hegedus, S. (2010). From new technological infrastructures to curricular activity systems: Advanced designs for teaching and learning. In M. J. Jacobsen & P. Reimann (Eds.), *Designs for learning environments of the future* (pp. 233-262). New York: Springer.
Rotman, B. (2000). *Mathematics as sign*. Palo Alto, CA: Stanford University Press.

Shaffer, D. W. (1995). *Exploring trigonometry with the Geometer's Sketchpad*. Berkeley, CA: Key Curriculum Press.

Sinclair, N. (2001). The aesthetic is relevant. *For the Learning of Mathematics, 21*(1), 25-33.

Sinclair, N. (2002a). Re-constructing a painting with geometric eyes. *For the Learning of Mathematics, 22*(3), 19-23.

Sinclair, N. (2002b). The kissing triangles: The aesthetics of mathematical discovery. *International Journal of Computers for Mathematical Learning, 7*(1), 45-63.

Sinclair, N., & Crespo, S. (2006). Learning mathematics with dynamic computer environments. *Teaching Children Mathematics, 12*(9), 436-444.

Sinclair, N., Arzarello, F., Trigueros Gaisman, & M. Dolores Lozano, M. (2009). Implementing digital technologies at a national scale. In C. Hoyles and J. Lagrange (Eds.), *Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain*. Springer.

Sinclair, N. & Moss, J. (2012). The more it changes, the more it becomes the same: The development of the routine of shape identification in dynamic geometry environments. International Journal of Education Research, 51&52, 28-44.

Sinclair, N. & Sedaghat Jou, M. (2013). Finger counting and adding with TouchCounts. CERME 8. Early Mathematics Working Group. Antalya, Turkey.

Stroup, W. (2005). Learning the basics with calculus. *Journal of Computers in Mathematics and Science Teaching, 24*(2), 179-196.

Stroup, W. M., Ares, N. M., & Hurford, A. (2005). A dialectic analysis of generativity: Issues of network supported design in mathematics and science. *Journal of Mathematical Thinking and Learning, 7*(3), 181-206.

Tall, D. O. (2013). *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*. Cambridge University Press.

Thompson Avant, M. J. & Heller, K. W. (2011). Examining the effectiveness of TouchMath with students with physical disabilities. *Remedial and Special Education, 32*(4), 309-321.

Tomasello, M. (1999). *The Cultural Evolution of Human Cognition*. Harvard University Press.

U.S. Department of Education. (2007). *The nation's report card mathematics 2007: National assessment of educational progress at grades 4 and 8*. Washington, DC.

Vahey, P., Roschelle, J., & Tatar, D. (2007). Using handhelds to link private cognition and public interaction. *Educational Technology, 47(3),* 13–16.

Vygotsky, L. (1980). Mind in society: The development of the higher psychological processes. Cambridge, MA: Harvard University Press.

Yerushalmy, M. & Naftaliev, E. (2011). Design of interactive diagrams structured upon generic animations. *Technology, Knowledge and Learning*, *16*(3), 221-246.

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